Herbstschule HEP 2026 – Student talk

Electroweak four-particle decays of the W boson

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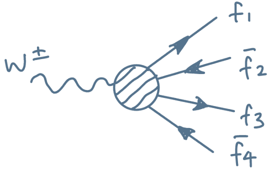
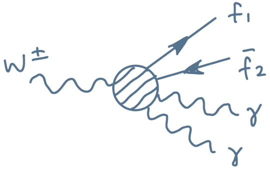
09.09.2026





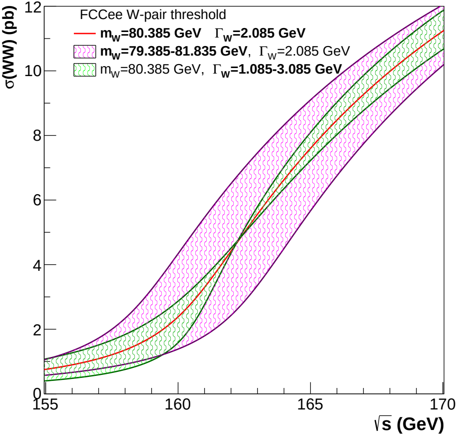
Measurements of the W decay width

- PDG 2026:
  $\Gamma_W = 2.14(5)\,\text{GeV}$

- Projected uncertainties at FCC-*ee*:

$$\sigma^{(\text{stat})}_{\Gamma_W} \approx 0.27\,\text{MeV},$$
$$\sigma^{(\text{syst})}_{\Gamma_W} \approx 0.2\,\text{MeV},$$

[FCC Feasibility Study, 2025, arXiv:2505.00272]



Standard Model prediction of the W decay width

Current status:

LO:	✓		
NLO:	α_s ✓	α ✓	
NNLO:	α_s^2 ✓	$\alpha\alpha_s$ ✓	α^2 ✗
N^{3}LO:	α_s^3 ✓		$\cdots$ ✗
N^{4}LO:	α_s^4 ✓		$\cdots$ ✗
$\vdots$			

Our Work:

Contribution to α^2 correction from $\underbrace{W \to 4f}_{\text{complete}}$ and $\underbrace{W \to 2f\,2\gamma}_{\text{in progress}}$ (tree level),

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N^3LO:	α_s^3	✓			$\cdots$	✗
N^4LO:	α_s^4	✓			$\cdots$	✗
$\vdots$						

Our Work:

Contribution to α^2 correction from $\underbrace{W \to 4f}_{\text{complete}}$ and $\underbrace{W \to 2f\,2\gamma}_{\text{in progress}}$ (tree level),

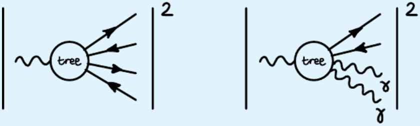
$$\Delta\Gamma_W^{(\alpha^2,\ \text{double-real})} = \sum_{\mathrm{FS}} \frac{1}{2M_W} \int \mathrm{d}\Phi_4\, |\mathcal{M}|^2 .$$

- Massless fermions except top quark
- Analytical evaluation of infrared divergence structure in $D = 4 - 2\epsilon$ dimensions

Kinoshita–Lee–Nauenberg theorem

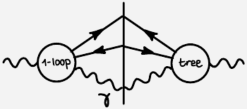
α^2 correction contains:

double-real



$+$

real-virtual



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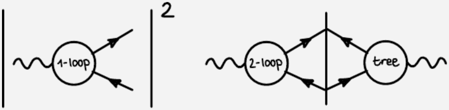
double-virtual

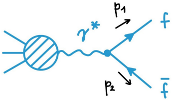


$=$ infrared-finite

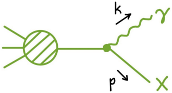
individually infrared-divergent

Infrared divergences in real-emission corrections

NLO real corrections:



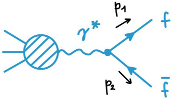
Collinear singularity for $\sphericalangle(\vec{p}_1, \vec{p}_2) \to 0$ if $m_f = 0$



Collinear singularity for $\sphericalangle(\vec{p}, \vec{k}) \to 0$ if $m_X = 0$
Soft singularity for $E_\gamma \to 0$

Infrared divergences in real-emission corrections

NLO real corrections:



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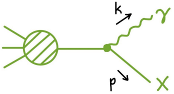
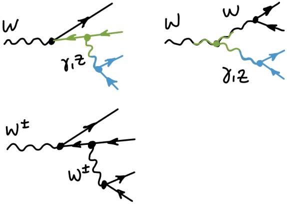


Collinear singularity for $\sphericalangle(\vec{p}, \vec{k}) \to 0$ if $m_X = 0$

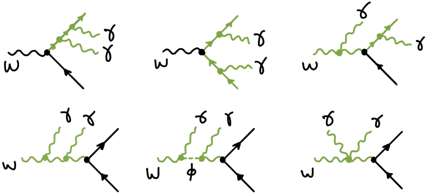
Soft singularity for $E_\gamma \to 0$

NNLO massless fermions (except top); 't Hooft–Feynman gauge ($\xi = 1$)

$W \to 4f$



$W \to 2f\,2\gamma$



Four-particle phase-space parameterization

Integrate over final-state momenta $p_1^\mu, \ldots, p_4^\mu$ in $D = 4-2\epsilon$ dimensions

In rest frame of decaying W: 5 non-trivial integrations

Four-particle phase-space parameterization

Integrate over final-state momenta $p_1^\mu, \ldots, p_4^\mu$ in $D = 4-2\epsilon$ dimensions

In rest frame of decaying W: 5 non-trivial integrations

Parameterization from [G. Heinrich, 2008, arXiv:0803.4177]

$$\int \mathrm{d}\Phi_4(p_1, \ldots, p_4) \propto \int_0^1 \mathrm{d}t_1 \cdots \int_0^1 \mathrm{d}t_5 \; {t_1}^{-\epsilon}(1-t_1)^{1-2\epsilon}\, {t_2}^{-\epsilon}(1-t_2)^{-\epsilon}$$
$$\times \; {t_3}^{-1/2-\epsilon}(1-t_3)^{-1/2-\epsilon}\, {t_4}^{1-2\epsilon}(1-t_4)^{1-2\epsilon}\, {t_5}^{-\epsilon}(1-t_5)^{-\epsilon}$$

Simple form for most invariants $s_{ij} = (p_i + p_j)^2$:

$$s_{34} = t_4\, t_1\, M_W^2, \qquad s_{14} = t_4(1-t_1)(1-t_2)\, M_W^2,$$
$$s_{13} = t_4(1-t_1)t_2\, M_W^2, \qquad s_{12} = (1-t_4)(1-t_5)(1-t_1)\, M_W^2.$$

But not all: $s_{23} = (1-t_4)\left[t_1(1-t_2)(1-t_5) + t_2 t_5 + (2t_3-1)\sqrt{\ldots}\right] M_W^2.$

Four-particle phase-space integration

$$I = \int_0^1 \mathrm{d}t_1 \cdots \int_0^1 \mathrm{d}t_5 \; t_1^{-\epsilon}(1-t_1)^{1-2\epsilon}\, t_2^{-\epsilon}(1-t_2)^{-\epsilon}\, t_3^{-1/2-\epsilon}(1-t_3)^{-1/2-\epsilon}$$
$$\times\; t_4^{1-2\epsilon}(1-t_4)^{1-2\epsilon}\, t_5^{-\epsilon}(1-t_5)^{-\epsilon}\; \frac{f(t_1,\ldots,t_5;\epsilon)}{g(t_1,\ldots,t_5)}$$

f, g polynomials

$$I \;=\; \underbrace{\frac{I^{(-n)}}{\epsilon^{n}} \;+\; \ldots \;+\; \frac{I^{(-1)}}{\epsilon}}_{\text{analytical eval.}} \;+\; \underbrace{I^{(0)}}_{\text{numerical eval.}} \;+\; \mathcal{O}(\epsilon)$$

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f, g polynomials

$$\int_0^1 \mathrm{d}z_1 \cdots \int_0^1 \mathrm{d}z_5 \; z_1^{-n_1-a_1\epsilon} \dots z_5^{-n_5-a_5\epsilon} \underbrace{h(z_1,\dots,z_5;\epsilon)}_{\text{regular}}$$

pole extraction

$$I = \underbrace{\frac{I^{(-n)}}{\epsilon^n} + \ldots + \frac{I^{(-1)}}{\epsilon}}_{\text{analytical eval.}} + \underbrace{I^{(0)}}_{\text{numerical eval.}} + \mathcal{O}(\epsilon)$$

Extraction of poles

Extract poles via **endpoint subtraction**. Simplest example:

$$
\begin{aligned}
I &= \int_0^1 \mathrm{d}x\, x^{-1-a\epsilon} f(x) \\
&= f(0) \int_0^1 \mathrm{d}x\, x^{-1-a\epsilon} + \int_0^1 \mathrm{d}x\, x^{-1-a\epsilon} \underbrace{[f(x) - f(0)]}_{= x f'(0) + \mathcal{O}(x^2)} \\
&= -\frac{f(0)}{a\epsilon} + \text{finite}.
\end{aligned}
$$

$\rightarrow$ Recursive application...

Four-particle phase-space integration

$$I = \int_0^1 \mathrm{d}t_1 \cdots \int_0^1 \mathrm{d}t_5 \; t_1^{-\epsilon}(1-t_1)^{1-2\epsilon}\, t_2^{-\epsilon}(1-t_2)^{-\epsilon}\, t_3^{-1/2-\epsilon}(1-t_3)^{-1/2-\epsilon}$$
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$$\int_0^1 \mathrm{d}z_1 \cdots \int_0^1 \mathrm{d}z_5 \; z_1^{-n_1-a_1\epsilon} \ldots z_5^{-n_5-a_5\epsilon}\, \underbrace{h(z_1,\ldots,z_5;\epsilon)}_{\text{regular}}$$

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f, g polynomials

sector decomposition

$$\int_0^1 \mathrm{d}z_1 \cdots \int_0^1 \mathrm{d}z_5 \; z_1^{-n_1-a_1\epsilon} \ldots z_5^{-n_5-a_5\epsilon} \underbrace{h(z_1,\ldots,z_5;\epsilon)}_{\text{regular}}$$

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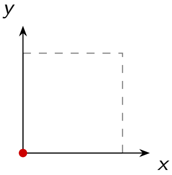
Sector decomposition

$$I = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \; \frac{x^{-\epsilon} y^{-\epsilon} f(x,y)}{(x+y)^2}$$

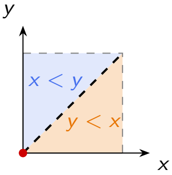


overlapping singularity
for $x, y \to 0$

Sector decomposition

$$I = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \frac{x^{-\epsilon} y^{-\epsilon} f(x,y)}{(x+y)^2}$$

$$\overset{\text{split}}{=} \int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x \, \frac{x^{-\epsilon} y^{-\epsilon} f(x,y)}{(x+y)^2} + \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \, \frac{x^{-\epsilon} y^{-\epsilon} f(x,y)}{(x+y)^2}$$



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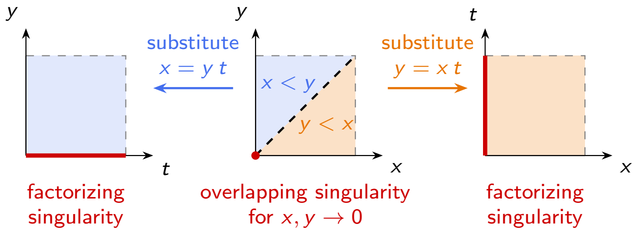
$$\overset{\text{subs.}}{=} \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}t \, \frac{y^{-2\epsilon}\, t^{-\epsilon} f(yt,y)}{y\,(1+t)^2} + \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}t \, \frac{x^{-2\epsilon}\, t^{-\epsilon} f(x,xt)}{x\,(1+t)^2}$$



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f, g polynomials

sector decomposition

$$I = \boxed{\sum} \int_0^1 \mathrm{d}z_1 \cdots \int_0^1 \mathrm{d}z_5 \; z_1^{-n_1-a_1\epsilon} \ldots z_5^{-n_5-a_5\epsilon} \underbrace{h(z_1,\ldots,z_5;\epsilon)}_{\text{regular}}$$

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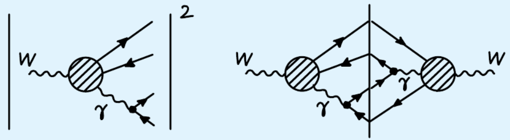
Results for $W \to 4f$

Infrared-divergent contributions



↓

analytical treatment of divergence structure

The rest:

infrared-finite

↓

numerical integration with custom Monte-Carlo program

Input parameters:

$$\begin{aligned} \alpha(M_Z)^{-1} &= 128.93, \\ M_W &= 80.385\ \text{GeV}, \\ M_Z &= 91.1876\ \text{GeV}, \\ m_t &= 173.34\ \text{GeV}. \end{aligned}$$

Result:

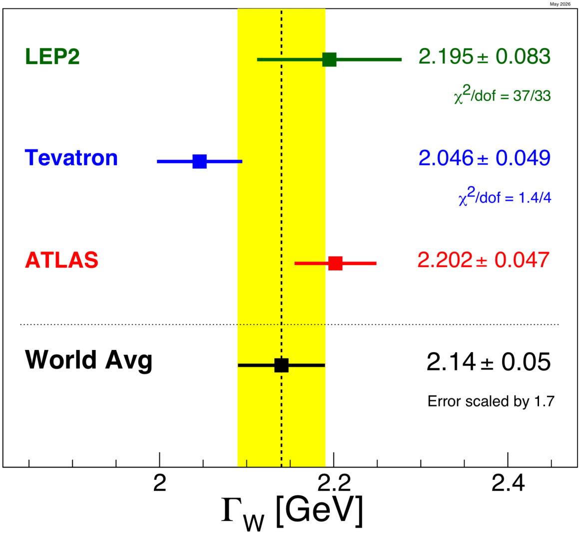
$$\frac{\Delta\Gamma^{(\text{NNLO, ew})}_{W\to 4f}}{\Gamma^{(\text{LO})}_W} = -\frac{7\alpha^2}{36\pi^2}\left(\frac{4\pi\mu^2}{M_W^2}\right)^{2\epsilon}\Gamma(1+\epsilon)^2\left[\frac{1}{\epsilon^3}+\frac{188}{27\epsilon^2}+\frac{197881-9762\pi^2+2760\zeta(3)}{4536\epsilon}\right]$$
$$-\ 4.28674\times 10^{-5},$$

Conclusion and outlook

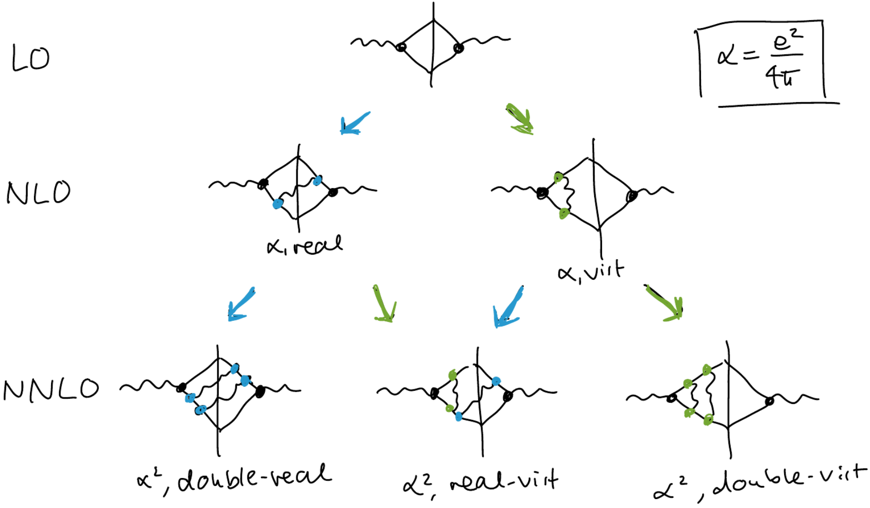
- EW NNLO correction to W decay width needed to contend with expected precision at future (circular) e^+e^- collider.
- Analytical evaluation of divergence structure provides full control over cancellation of divergences.
- Computed double-real NNLO EW correction from $W \to 4f$.
  $W \to 2f\,2\gamma$ in progress.

Backup

World average for the W decay width – PDG 2026



LO, NLO α, NNLO α^2 – example diagrams

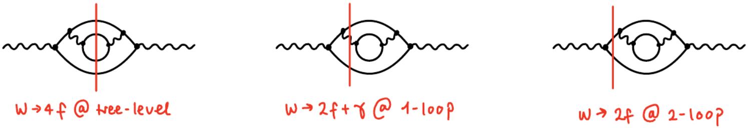


Kinoshita–Lee–Nauenberg theorem

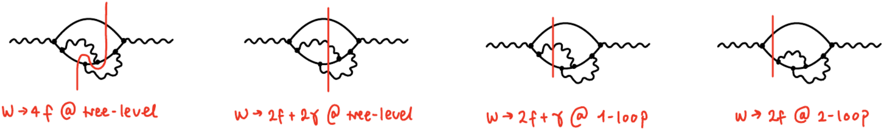
Infrared divergences cancel in observables that are sufficiently inclusive over degenerate initial and final states.

Explicit mechanism: cancellation between interference diagrams that are the same up to the placement of the unitarity cut.

Example 1



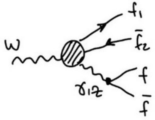
Example 2

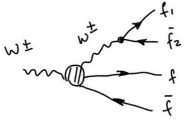


Contributions to the amplitude for $W \to 4f$

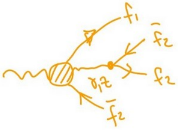
consider $W \to f_1 \bar{f}_2 f \bar{f}$, $f \neq f_1, f_2$

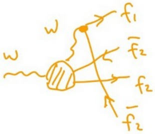
Contributions to amplitude:





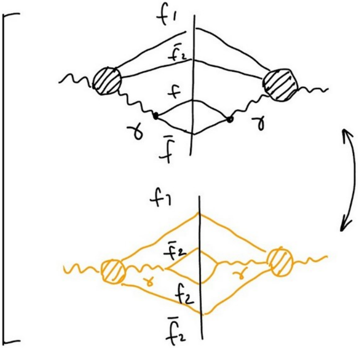
Additional diagrams for $W \to f_1 \bar{f}_2 f_2 \bar{f}_2$:





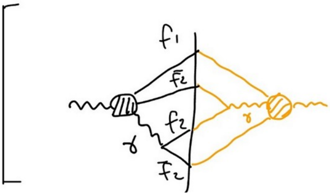
Infrared-divergent interference contributions

two closed fermion lines: "diagonal" contributions



Final-state permutation

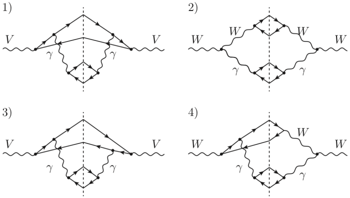
one closed fermion line: "off-diagonal" contributions



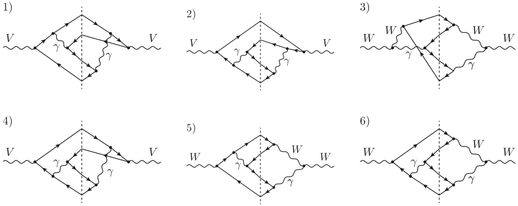
+ compl. conj.

Divergent contributions to $W \to 4f$

Two closed fermion loops

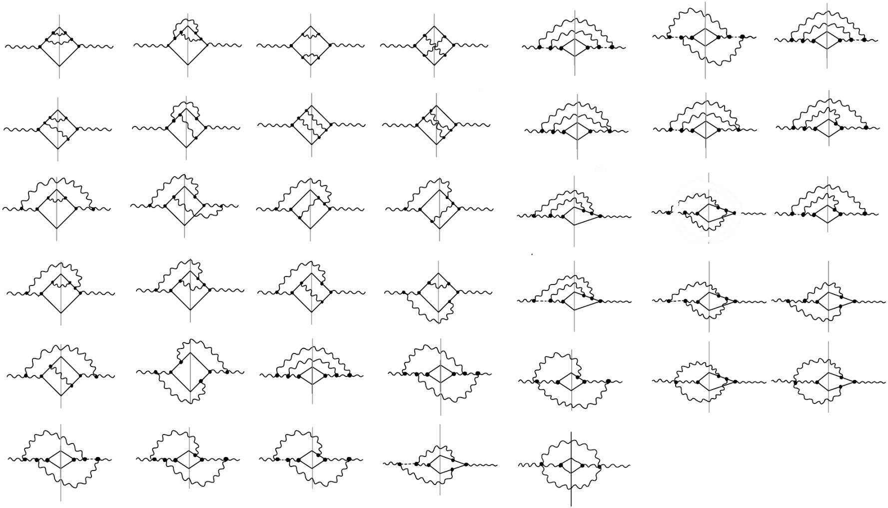


One closed fermion loop (appears only for final states with identical particles)



Contributions to $W \to 2f\,2\gamma$

40 topologically distinct interference terms



Four-particle phase-space parameterization

$$s_{12} = y_1 Q^2, \quad s_{13} = y_2 Q^2, \quad s_{23} = y_3 Q^2, \quad s_{14} = y_4 Q^2, \quad s_{24} = y_5 Q^2, \quad s_{34} = y_6 Q^2.$$

$$\begin{aligned} \mathrm{d}\Phi(p_1, p_2, p_3, p_4; Q) = {} & \frac{\mu^{3(4-D)}(Q^2)^{3D/2-4}}{2^{2D-1}(2\pi)^{3D-4}} (-\hat{\Delta}_4)^{\frac{D-5}{2}} \prod_{i=1}^{6} \mathrm{d}y_i\, \Theta(y_i) \\ & \times \mathrm{d}^{D-2}\Omega_1\, \mathrm{d}^{D-3}\Omega_{12}\, \mathrm{d}^{D-4}\Omega_{13}\ \Theta(-\hat{\Delta}_4)\, \delta\Big(1 - \sum_{i=1}^{6} y_i\Big), \end{aligned}$$

Gram determinant:

$$\hat{\Delta}_4 = \lambda(y_1 y_6,\ y_2 y_5,\ y_3 y_4), \qquad \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx.$$

Why 5 non-trivial integrations?

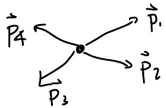
$$d\Phi_4(p_1, \dots, p_4; Q)$$

$$= \frac{\mu^{3(4-D)}}{(2\pi)^{3D-4}} \int d^D p_1 \, \delta(p_1^2 - m_1^2) \, \dots \int d^D p_4 \, \delta(p_4^2 - m_4^2)$$

$$\times \, \delta^{(D)}\left(Q - p_1 - p_2 - p_3 - p_4\right)$$

$$\text{Total \# integrations} = 4 \cdot (D - 1) - D = 3D - 4 \overset{D=4}{\downarrow}{=} 8$$

Rest frame of W:



2 trivial angular integration due to symmetry under simultaneous rotation of all momenta

+ 1 trivial angular integration due to symmetry under simultaneous rotation of three momenta around the direction of the fourth.

Four-particle phase-space parameterization

$$
\begin{aligned}
y_6 &= t_4\, t_1, \\
y_2 &= t_4(1-t_1)t_2, \\
y_4 &= t_4(1-t_1)(1-t_2), \\
y_1 &= (1-t_4)(1-t_5)(1-t_1), \\
y_3 &= y_3^- + t_3(y_3^+ - y_3^-), \\
y_3^\pm &= (1-t_4)\left[t_1(1-t_2)(1-t_5) + t_2 t_5 \pm 2\sqrt{t_1(1-t_2)t_2(1-t_5)t_5}\right].
\end{aligned}
$$

$$
\begin{aligned}
\int \mathrm{d}\Phi(p_1,p_2,p_3,p_4;Q) = \frac{2^7 \mu^{3(4-D)}(Q^2)^{3D/2-4}}{(2\pi)^{3D-4}} &\int_0^1 \mathrm{d}t_1\,\ldots\,\mathrm{d}t_5\;\left[(1-t_1)t_4(1-t_4)\right]^{D-3} \\
&\times\;\left[t_3(1-t_3)\right]^{\frac{D-5}{2}}\left[t_1 t_2(1-t_2)t_5(1-t_5)\right]^{\frac{D-4}{2}} \\
&\times\;\int \mathrm{d}^{D-2}\Omega_1 \int \mathrm{d}^{D-3}\Omega_{12} \int \mathrm{d}^{D-4}\Omega_{13},
\end{aligned}
$$

Extraction of higher-order poles

$$I = \int_0^1 \mathrm{d}x \; x^{-n-a\epsilon} f(x)$$

Taylor expansion:

$$f(x) = f(0) + f'(0)x + \cdots = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(0)\, x^k + \underbrace{R(x)}_{\in \mathcal{O}(x^n)}$$

$$I = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \int_0^1 \mathrm{d}x \; x^{k-n-a\epsilon} + \underbrace{\int_0^1 \mathrm{d}x \; x^{-n-a\epsilon} R(x)}_{\text{regular for } x, \epsilon \to 0}$$

Trivial integration:

$$\int_0^1 \mathrm{d}x \; x^{m-a\epsilon} = \frac{1}{m+1-a\epsilon} = \begin{cases} -\dfrac{1}{a\epsilon}, & m+1=0, \\ \displaystyle\sum_{k=0}^{\infty} \frac{(a\epsilon)^k}{(m+1)^{k+1}}, & m+1<0, \end{cases} \qquad m \in \{-n, \ldots, -1\}.$$