Infrared Subtraction from NLO to N^{k}LO

From soft and collinear limits to nested and all-order structures

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57th Herbstschule HEP
Bad Honnef, September 2026

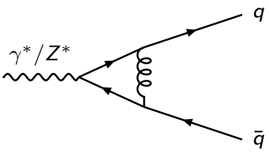
Roadmap: the cancellation problem at increasing complexity

1. **Origin of the poles:** one unresolved gluon can be soft, collinear, or both.
2. **NLO—the primitive:** subtract the singular limit locally and add back its analytic integral.
3. **NNLO—the nesting problem:** several unresolved limits overlap and require inclusion–exclusion across multiplicities.
4. **N^{3}LO and N^{k}LO—the integrated structure:** reverse unitarity, connected cumulants, and anomalous dimensions organize the pole cancellation.

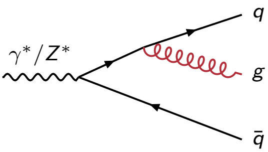
Two type of diagrams have contributions to the IR divergence

At Born level, $e^+e^- \to \gamma^*/Z^* \to q\bar{q}$. At order α_s, two qualitatively different contributions appear:

Virtual: two-parton kinematics



Real: three-parton kinematics



$$V \sim -\frac{2}{\epsilon_{\mathrm{IR}}^2} - \frac{3}{\epsilon_{\mathrm{IR}}} + \text{finite.}$$

$$\int R \to \infty \quad \text{for} \quad E_g \to 0 \text{ or } p_g \parallel p_q, p_{\bar{q}}.$$

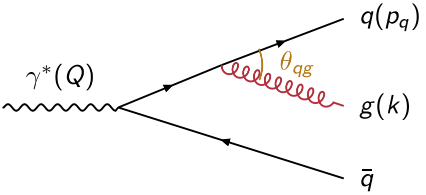
The physical NLO correction is finite, which requires the sum of divergent diagrams to be finite.

Ellis–Ross–Terrano, Nucl. Phys. B178 (1981) 421; Catani, Phys. Lett. B427 (1998) 161.

Soft and collinear poles come from unresolved gluon emission

At NLO the real channel contains $e^+e^- \to q\bar{q}g$. If the gluon momentum is k, its emission probes

$$p_q \cdot k = E_q E_g \big(1 - \cos\theta_{qg}\big), \qquad 1 - x_{\bar{q}} = x_q \frac{E_g}{\sqrt{s}} \big(1 - \cos\theta_{qg}\big), \quad x_i \equiv \frac{2E_i}{\sqrt{s}}.$$



Emission factors contain inverse powers of $p_q \cdot k$ (and likewise $p_{\bar{q}} \cdot k$).

Soft limit

$E_g \to 0$: the gluon carries vanishing energy and cannot be resolved.

Collinear limit

$\theta_{qg} \to 0$: the quark and gluon become parallel and cannot be resolved as two separate partons.

The soft-collinear overlap produces a double pole

$$\int_0 \frac{\mathrm{d}E_g}{E_g^{1+2\epsilon}} \int_0 \frac{\mathrm{d}\theta_{qg}^2}{(\theta_{qg}^2)^{1+\epsilon}} \sim \frac{1}{\epsilon^2}:$$ the soft–collinear overlap produces a double pole.

Catani–Seymour, Nucl. Phys. B485 (1997) 291 [hep-ph/9605323].

Real and virtual poles cancel in the total cross section in NLO

After integrating the real emission over the full three-particle phase space (with the common factor $c_\Gamma(\epsilon, s, \mu)$ factored out),

$$\frac{\sigma_V}{\sigma_0} = \frac{\alpha_s C_F}{2\pi}\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \pi^2 - 8\right],$$

$$\frac{\sigma_R}{\sigma_0} = \frac{\alpha_s C_F}{2\pi}\left[+\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \pi^2 + \frac{19}{2}\right].$$

	$1/\epsilon^2$	$1/\epsilon$	finite coefficient
virtual	-2	-3	$\pi^2 - 8$
integrated real	$+2$	$+3$	$19/2 - \pi^2$
sum	0	0	$3/2$

Hence, for $C_F = 4/3$,

$$\frac{\sigma_{\mathrm{NLO}}}{\sigma_0} = 1 + \frac{\alpha_s C_F}{2\pi}\frac{3}{2} = 1 + \frac{\alpha_s}{\pi}.$$

Catani–Seymour, Nucl. Phys. B485 (1997) 291 [hep-ph/9605323].

NLO subtraction makes the cancellation local

R is singular at unresolved phase-space boundaries, while V contains explicit loop poles. Add and subtract a local approximation S to R:

$$\boxed{\sigma^{\mathrm{NLO}} = \sigma_0 + \int_{\Phi_{m+1}} \left[R - S\right]_{d=4} + \int_{\Phi_m} \left[V + \mathcal{I}_1[S]\right]_{d=4} .}$$

local limit
$S \to R$ in every soft/collinear region

→

momentum map
$\Phi_{m+1} \to \widetilde{\Phi}_m$

→

analytic integral
$\mathcal{I}_1[S] = \int \mathrm{d}\Phi_{\mathrm{unres}}\, S$

→

two finite four-dimensional terms

What $\mathrm{d}\Phi_{\mathrm{unres}}$ means

It is the full unresolved one-particle phase-space factor in $d = 4 - 2\epsilon$ dimensions—including energy fractions and angles, not a one-dimensional integration variable.

Catani–Seymour, Nucl. Phys. B485 (1997) 291 [hep-ph/9605323].

The explicit example becomes a sum of two finite numbers

The two-parton contribution is finite point by point in Born phase space:

$$\frac{V + \mathcal{I}_1[S]}{B} = \frac{\alpha_s C_F}{2\pi}\left[\cancel{-\frac{2}{\epsilon^2}} + \cancel{\frac{2}{\epsilon^2}} + \cancel{-\frac{3}{\epsilon}} + \cancel{\frac{3}{\epsilon}} + (\pi^2 - 8) + (10 - \pi^2)\right] = \frac{\alpha_s C_F}{2\pi}\,2.$$

For the inclusive measurement $J = 1$, the subtracted three-parton integral gives

$$\frac{1}{B}\int_{\Phi_3}(R - S) = \frac{\alpha_s C_F}{2\pi}\left(-\frac{1}{2}\right).$$

Therefore

$$\frac{\sigma_{\rm NLO} - \sigma_0}{\sigma_0} = \frac{\alpha_s C_F}{2\pi}\left(2 - \frac{1}{2}\right) = \frac{\alpha_s C_F}{2\pi}\frac{3}{2} = \frac{\alpha_s}{\pi}.$$

Subtracted pieces need not be positive or unique; only their finite sum is physical.

Catani–Seymour, Nucl. Phys. B485 (1997) 291 [hep-ph/9605323].

NNLO changes the cancellation geometry

double real: RR
tree-level
$q\bar{q} + 2$ partons
Φ_4; single/double unresolved

real–virtual: RV
one-loop $q\bar{q}g$
Φ_3; explicit poles
$+$ one unresolved

double virtual: VV
two-loop $+$
one-loop^2 $q\bar{q}$
Φ_2; poles through $1/\epsilon^4$

At NLO

$$\Phi_3 \longleftrightarrow \Phi_2, \qquad \text{one unresolved limit.}$$

At NNLO

$$\Phi_4 \longleftrightarrow \Phi_3 \longleftrightarrow \Phi_2, \qquad \text{nested limits.}$$

The new problem

A four-parton configuration can approach either a three-parton or a two-parton event, and these unresolved regions overlap.

Gehrmann–De Ridder, Gehrmann and Glover, Nucl. Phys. B691 (2004) 195 [hep-ph/0403057].

A single NLO-like counterterm oversubtracts the overlap

S_0: cover the complete two-jet singular region

- maps $\Phi_4 \to \widetilde{\Phi}_2$;
- captures genuinely double-unresolved limits;
- also extends onto single-unresolved boundaries.

S_1: restore the three-jet boundary

- maps $\Phi_4 \to \widetilde{\Phi}_3$;
- reproduces the singly-unresolved overlap;
- is added back to repair the oversubtraction.

$$\boxed{S = S_0 - S_1, \qquad \mathrm{RR} - S = \mathrm{RR} - S_0 + S_1.}$$

Why S_0 removes too much

On a single-unresolved boundary, the physical reduced event has three partons. S_0 has already projected that configuration all the way to Φ_2.

Gehrmann–De Ridder, Gehrmann and Glover, Nucl. Phys. B691 (2004) 195 [hep-ph/0403057].

Different maps move different singular information downward

S_0: tripole map

$$\begin{aligned}\Phi_4 &\to \widetilde{\Phi}_2\\ \mathrm{d}\Phi_4 &= \mathrm{d}\Phi_2\,\mathrm{d}\Phi_T\\ \mathcal{I}_2[S_0] &= \int \mathrm{d}\Phi_T\, S_0\end{aligned}$$

S_1: dipole map

$$\begin{aligned}\Phi_4 &\to \widetilde{\Phi}_3\\ \mathrm{d}\Phi_4 &= \mathrm{d}\Phi_3\,\mathrm{d}\Phi_D\\ \mathcal{I}_1[S_1] &= \int \mathrm{d}\Phi_D\, S_1\end{aligned}$$

VS_1: one-loop dipole

$$\begin{aligned}\Phi_3 &\to \widetilde{\Phi}_2\\ \mathrm{d}\Phi_3 &= \mathrm{d}\Phi_2\,\mathrm{d}\Phi_D\\ \mathcal{I}_1[VS_1] &= \int \mathrm{d}\Phi_D\, VS_1\end{aligned}$$

$$\Phi_4 \xrightarrow{S_1,\mathcal{I}_1} \Phi_3 \xrightarrow{VS_1,\mathcal{I}_1} \Phi_2, \qquad \Phi_4 \xrightarrow{S_0,\mathcal{I}_2} \Phi_2.$$

How to read the factorized phase spaces

Φ_D and Φ_T are multi-dimensional unresolved phase spaces. The labels $\mathcal{I}_1$ and $\mathcal{I}_2$ count unresolved partons, not integration dimensions.

Notation

VS_1 is the name of the one-loop single-unresolved counterterm; it does not mean $V \times S_1$.

Gehrmann–De Ridder, Gehrmann and Glover, Nucl. Phys. B691 (2004) 195 [hep-ph/0403057].

The NNLO coefficient becomes three separately finite channels

Φ_4 :	$\mathrm{RR} - S_0 + S_1$	finite double-real weight,
Φ_3 :	$\mathrm{RV} - VS_1 - \mathcal{I}_1[S_1]$	finite real–virtual weight,
Φ_2 :	$\mathrm{VV} + \mathcal{I}_1[VS_1] + \mathcal{I}_2[S_0]$	finite double-virtual weight.

Local bookkeeping

A counterterm first removes a singular boundary on the higher-multiplicity phase space.

Integrated bookkeeping

The same counterterm reappears at lower multiplicity after its unresolved variables have been integrated.

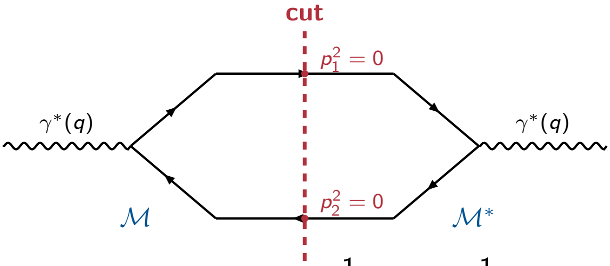
Role of Catani factorization

It predicts the integrated virtual pole skeleton that this ledger must cancel; it is not, by itself, a prescription for the local momentum maps.

Gehrmann–De Ridder et al., NPB691 (2004) 195; Catani, Phys. Lett. B427 (1998) 161.

Total cross section: Reverse unitarity turns phase space into a cut diagram

A cut marks the final-state lines that are forced on shell.



$$\delta_+(p^2) \quad \Longleftrightarrow \quad \frac{1}{p^2 - \mathrm{i}\varepsilon} - \frac{1}{p^2 + \mathrm{i}\varepsilon}$$

Phase space
on-shell delta functions

Cut graph
the same lines are cut propagators

Payoff
real and virtual pieces share one integral language

It changes the integral language; it does not itself cancel the infrared poles.

Cutkosky, J. Math. Phys. 1 (1960) 429; Anastasiou–Melnikov, NPB646 (2002) 220.

At N^{3}LO, all four layers enter one four-loop integral family

More real emissions give more cut lines; more virtual corrections give fewer.

layer	content	cut lines
RRR	3 real + 0 virtual	5
VRR	2 real + 1 virtual	4
VVR	1 real + 2 virtual	3
VVV	0 real + 3 virtual	2

four real–virtual layers → one four-loop cut-integral family → IBP relations → master integrals

Different particle numbers become different cuts of the same forward graph.

The poles cancel only after UV renormalization and summing all four layers.

Anastasiou–Melnikov, NPB646 (2002) 220; Anastasiou et al., PRL114 (2015) 212001.

At N^{k}LO, connected webs organize what is genuinely new

Every contribution is labelled by the number of real emissions n and loops ℓ:

$$n + \ell = k, \qquad \text{there are } k+1 \text{ real–virtual layers.}$$

sum all layers into an inclusive object G → $\log G$ isolates connected radiation webs → KLN + anomalous dimensions constrain pole towers → integrated target for higher-order subtraction

What exponentiation gives

A compact organization of leading, and part of the subleading, integrated infrared singularities.

What it does not yet give

A local differential counterterm or the momentum maps needed in every overlapping unresolved region.

Gardi et al., JHEP 11 (2010) 155; Almelid–Duhr–Gardi, PRL117 (2016) 172002.

Summary: construct locally, validate inclusively

NLO

one emission
local term +
map + integral

$\longrightarrow$

NNLO

overlapping limits
nested inclusion–
exclusion

$\longrightarrow$

N^{k}LO

$k+1$ layers
inclusive
pole target

local counterterms and physical maps $\xrightarrow{\int \mathrm{d}\Phi_{\mathrm{unres}}}$ **universal inclusive poles**

Take-home message

A valid N^{k}LO scheme must be local in every unresolved limit and reproduce the KLN- and anomalous-dimension constraints after integration.

Backup derivations for discussion

Reverse unitarity, IBP, connected cumulants, and pole cancellation

Reverse unitarity turns an on-shell constraint into a cut denominator

Start from an m-particle phase-space integral:

$$\int \mathrm{d}\Phi_m\, F = \int \prod_{i=1}^{m} \frac{\mathrm{d}^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2)\,(2\pi)^d \delta^{(d)}\left(q - \sum_i p_i\right) F(\{p_i\}).$$

For every final-state line use the discontinuity identity

$$2\pi i\,\delta_+(p^2) = \frac{1}{p^2 - i0} - \frac{1}{p^2 + i0} \equiv \left[\frac{1}{p^2}\right]_c .$$

Momentum conservation removes one p_i; the remaining $m-1$ momenta are integrated like loop momenta:

$$\int \mathrm{d}\Phi_m\, F \longrightarrow \int \prod_{j=1}^{m-1} \mathrm{d}^d k_j\, \frac{N(k_j, q)}{D_1^{\nu_1} \cdots D_s^{\nu_s}} \prod_{c \in \text{final state}} \left[\frac{1}{D_c^{\nu_c}}\right]_c .$$

The cut is not an optional propagator

A cut line must remain on shell, so a reduction obeys $[D_c^{,0}]_c = 0$: any integral in which a required cut denominator is pinched vanishes.

Anastasiou–Melnikov, NPB646 (2002) 220; Anastasiou et al., NPB570 (2000) 176.

The apparent mismatch of particle numbers disappears after cutting

At N^{k}LO, label a layer by

$$n = \text{real emissions beyond Born}, \qquad \ell = \text{total virtual order}, \qquad n + \ell = k.$$

For a $1 \to 2$ Born process, the cut has $m = n + 2$ final-state lines and the phase space supplies $m - 1$ independent loop integrations. Therefore

$$\boxed{L_{\text{forward}} = (m-1) + \ell = (n+1) + (k-n) = k + 1.}$$

$$\mathrm{RRR} : (3, 0; 5), \qquad \mathrm{VRR} : (2, 1; 4), \qquad \mathrm{VVR} : (1, 2; 3), \qquad \mathrm{VVV} : (0, 3; 2),$$

where each triple is $(n, \ell; \text{number of cut lines})$ and every case has $L_{\text{forward}} = 4$.

Resolution of the mismatch

The layers still have different cuts, but they now belong to integral families of the same four-loop two-point function. IBP can therefore reduce every layer to cut master integrals in one common loop-integral framework.

Anastasiou–Melnikov, NPB646 (2002) 220.

IBP identities are total derivatives that vanish in dimensional regularization

For $I(\boldsymbol{\nu}) = \int \prod_i \mathrm{d}^d k_i \prod_a D_a^{-\nu_a}$, dimensional regularization gives no boundary term for a total derivative:

$$0 = \int \prod_i \mathrm{d}^d k_i \, \frac{\partial}{\partial k_j^\mu} \left(v^\mu \prod_a D_a^{-\nu_a} \right), \qquad v^\mu \in \{k_1, \ldots, k_L, q_1, \ldots\}.$$

Expanding the derivative gives a linear relation among integrals with shifted powers:

$$0 = (\partial_{k_j} \cdot v) I(\boldsymbol{\nu}) - \sum_a \nu_a \int \prod_i \mathrm{d}^d k_i \, \frac{v \cdot \partial_{k_j} D_a}{D_a^{\nu_a+1}} \prod_{b \neq a} D_b^{-\nu_b}.$$

Rewrite numerator scalar products in terms of D_a. Repeating the identity produces a linear system whose independent solutions are the master integrals.

What changes for reverse unitarity?

The derivative algebra is unchanged. Only the boundary condition is new: if a required cut power reaches zero, that term is set to zero rather than becoming an uncut topology.

Chetyrkin–Tkachov, NPB192 (1981) 159; Laporta, IJMPA15 (2000) 5087.

A one-loop IBP identity shows how powers are shifted

Define

$$D_1 = k^2, \qquad D_2 = (q-k)^2, \qquad I(a,b) = \int \frac{\mathrm{d}^d k}{D_1^a D_2^b}.$$

Choose the total derivative $\partial_{k^\mu}(k^\mu / D_1^a D_2^b)$:

$$0 = \int \mathrm{d}^d k \, \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{D_1^a D_2^b} \right)$$

$$= (d - 2a - b) I(a,b) - b\, I(a-1, b+1) + b q^2 I(a, b+1).$$

The only algebraic input is

$$k \cdot \partial_k D_1 = 2 D_1, \qquad k \cdot \partial_k D_2 = D_1 + D_2 - q^2.$$

If D_1 is cut

The term $I(a-1, b+1)$ is allowed while $a-1>0$, but vanishes when the shift would remove the cut: $I_c(0, b+1) = 0$. This is how the same IBP system respects a fixed final-state cut.

Chetyrkin–Tkachov, NPB192 (1981) 159.

The r–κ relation is ordinary moment–cumulant algebra

Expand the generating function and its logarithm:

$$G(x,a) = 1 + r_{1,0}x + r_{0,1}a + r_{2,0}x^2 + r_{1,1}xa + r_{3,0}x^3 + \cdots,$$
$$K(x,a) = \log G(x,a) = \sum \kappa_{n,\ell}x^n a^\ell.$$

Using $\log(1+u) = u - u^2/2 + u^3/3 - \cdots$ gives

$$\begin{aligned}
\kappa_{1,0} &= r_{1,0}, & \kappa_{0,1} &= r_{0,1},\\
\kappa_{2,0} &= r_{2,0} - \tfrac{1}{2}r_{1,0}^2, & \kappa_{1,1} &= r_{1,1} - r_{1,0}r_{0,1},\\
\kappa_{3,0} &= r_{3,0} - r_{1,0}r_{2,0} + \tfrac{1}{3}r_{1,0}^3.
\end{aligned}$$

Conversely, $G = e^K$ reconstructs the rates:

$$r_{2,0} = \kappa_{2,0} + \frac{1}{2}\kappa_{1,0}^2, \qquad r_{1,1} = \kappa_{1,1} + \kappa_{1,0}\kappa_{0,1},$$

Interpretation

r contains connected radiation and every possible product of lower-order radiation; κ retains only the new connected contribution.

Gatheral, Phys. Lett. B133 (1983) 90; Frenkel–Taylor, NPB246 (1984) 231.

Anomalous dimensions encode the scale response of infrared radiation

Separate a renormalized amplitude into an infrared factor and a finite hard part:

$$|\mathcal{M}(\{p\},\mu,\epsilon)\rangle = \mathbf{Z}(\{p\},\mu,\epsilon)\,|\mathcal{H}(\{p\},\mu)\rangle, \qquad \frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}\ln\mu} = -\mathbf{Z}\,\Gamma.$$

A common convention has the schematic form

$$\Gamma = \sum_{i<j} \mathbf{T}_i\cdot\mathbf{T}_j\,\Gamma_{\mathrm{cusp}}(\alpha_s)\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma_i(\alpha_s) + \Delta_{\mathrm{multi\text{–}leg}}.$$

- Γ_{cusp}: response to a change of Wilson-line direction; it controls soft–collinear double logarithms and the deepest poles.
- γ_i: endpoint response of a quark or gluon; it controls hard-collinear single logarithms and enters one pole lower.
- $\Delta_{\mathrm{multi\text{–}leg}}$: genuinely correlated wide-angle soft radiation; it carries nontrivial multi-leg color information.

They do not perform the phase-space integration: they predict the scale and color structure that the integrated layers must reproduce.

Catani, PLB427 (1998) 161; Becher–Neubert, PRL102 (2009) 162001.

At N^{3}LO, the connected deepest poles cancel on one diagonal

For a color dipole in representation R, the connected deepest-pole residues on the $n+\ell=3$ diagonal are

layer	(n,ℓ)	$\kappa_{n,\ell}$
RRR	$(3,0)$	$\frac{4}{3}C_R C_A^2$
VRR	$(2,1)$	$-\frac{20}{9}C_R C_A^2$
VVR	$(1,2)$	$\frac{8}{9}C_R C_A^2$
VVV	$(0,3)$	0

Therefore

$$\sum_{n+\ell=3}\kappa_{n,\ell}=\left(\frac{12-20+8}{9}\right)C_R C_A^2=0.$$

Why $\kappa_{0,3}=0$ does not mean the three-loop form factor has no pole

The pure-virtual deepest poles are products generated by exponentiating the one-loop cumulant $\kappa_{0,1}=-4C_R$. They are present in $r_{0,3}$ but are removed from the new connected quantity $\kappa_{0,3}$.

KLN theorem: Kinoshita, JMP3 (1962) 650; Lee–Nauenberg, PR133 (1964) B1549.

The pole cancellation occurs after renormalization and summing all cuts

Reverse unitarity and IBP produce the Laurent series of each cut. Apply coupling and operator renormalization before testing infrared cancellation:

$$a_s^{\mathrm{bare}} = S_\epsilon\, a_s\, Z_{a_s}, \qquad Z_{a_s} = 1 - \frac{\beta_0}{\epsilon} a_s + \mathcal{O}(a_s^2).$$

After renormalization, write the N^{3}LO coefficient as

$$\sigma^{(3)}_{\mathrm{ren}}(\epsilon) = \sum_{n+\ell=3} \sum_{j=-6}^{\infty} c_j^{(n,\ell)} \epsilon^j.$$

KLN finiteness requires six independent cancellations,

$$\boxed{\sum_{n+\ell=3} c_j^{(n,\ell)} = 0, \qquad j = -6, -5, \ldots, -1.}$$

$$\textbf{bare cuts} \xrightarrow{\mathrm{IBP}} \textbf{Laurent series} \xrightarrow{\mathrm{UV\ ren.}} \textbf{renormalized layers} \xrightarrow{\sum \mathrm{cuts}} \textbf{finite rate}$$

Diagnostic: a leftover pole signals a missing channel, a wrong symmetry/color factor, insufficient ϵ depth, or an incorrect renormalization counterterm.

Catani, PLB427 (1998) 161; Anastasiou et al., PRL114 (2015) 212001.