

Infrared Subtraction from NLO to $N^k\text{LO}$ From soft and collinear limits to nested and all-order structures

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Perturbative QCD predictions beyond leading order contain infrared singularities from unresolved parton radiation. Real-emission terms become singular when a gluon is soft or collinear, while virtual corrections contain explicit poles in the dimensional regulator ϵ . Infrared-safe observables are finite only after degenerate real and virtual configurations are combined. Subtraction reorganizes this cancellation into contributions that can be evaluated separately in four dimensions.

At NLO, the real contribution R is singular on the boundaries of the $(m+1)$ -particle phase space, whereas the virtual contribution V has explicit loop poles on the m -particle phase space. A local counterterm S reproduces every single-unresolved limit of R . A physical momentum map reduces the event to m -particle kinematics, and S is analytically integrated over the unresolved phase space. Thus $R - S$ is locally finite and the integrated counterterm cancels the poles of V . The NLO building block is therefore a local approximation, a momentum map and an analytic integral.

At NNLO, double-real, real-virtual and double-virtual terms introduce overlapping single- and double-unresolved regions. A counterterm covering the double-unresolved region also reaches single-unresolved boundaries and therefore oversubtracts them. Further terms restore this overlap through an inclusion-exclusion structure and different maps between the $(m+2)$ -, $(m+1)$ - and m -particle phase spaces. Their unresolved integrals reappear at lower multiplicities and cancel the explicit virtual poles, leaving three separately finite channels.

At $N^3\text{LO}$, the RRR, VRR, VVR and VVV layers satisfy $n + \ell = 3$, where n counts real emissions and ℓ the virtual order. Reverse unitarity replaces on-shell delta functions by cut propagators, so all four layers become different cuts of one four-loop forward correlator. Integration-by-parts identities then reduce them to common master integrals. This unifies the integral language but does not remove the divergences: after ultraviolet renormalization, poles through $1/\epsilon^6$ cancel only in the sum of all layers.

At $N^k\text{LO}$ there are $k+1$ real-virtual layers with $n + \ell = k$. Their inclusive sum defines a generating object G , while $\log G$ isolates connected radiation webs, analogous to cumulants. KLN cancellation and universal anomalous dimensions constrain the scale, colour and pole structure that integrated counterterms must reproduce. These relations provide exact consistency conditions, but do not construct the local counterterms or momentum maps needed for differential observables.

The central lesson is to construct locally and validate inclusively: every unresolved singularity must be removed point by point, while the integrated counterterms must reproduce the universal pole structure predicted by factorization, KLN cancellation and anomalous dimensions.