Standard Model Precision Physics

Stefan Weinzierl

Institut für Physik, Universität Mainz

57. Herbstschule für Hochenergiephysik Maria Laach 2026

Section 3

Leading-order calculations

The master formula for the calculation of observables

$$\langle O_N \rangle = \underbrace{\sum_{a,b} \int dx_1 \; f_a(x_1) \int dx_2 \; f_b(x_2)}_{\text{pdf's}} \; \underbrace{\frac{1}{2K(\hat{s})}}_{\text{flux factor}} \; \underbrace{\frac{1}{n_a^{\text{spin}} n_b^{\text{spin}} n_a^{\text{colour}} n_b^{\text{colour}}}}_{\text{average over initial spins and colours}}$$

$$\times \sum_{\mathbf{n} \geq \mathbf{N}} \underbrace{\int d\phi_n}_{\text{integral over phase space}} \; \underbrace{O(p_1, ..., p_n)}_{\text{observable}} \; \underbrace{\left| \mathcal{A}_{\mathbf{n+2}} \right|^2}_{\text{amplitude}}$$

At leading order:

$$\mathbf{n} = \mathbf{N} \qquad \text{and} \qquad \mathcal{A}_{\mathbf{n+2}} \approx \mathbf{g}^{\mathbf{n}} \mathcal{A}^{(0)}_{\mathbf{n+2}}$$

The amplitude squared

Let us consider the Born amplitude squared:

$$\left| \mathcal{A}_{n+2}^{(0)} \right|^2$$

Implicitly understood is a sum over the internal degrees of freedom, e.g.

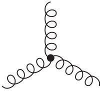
- spins/helicity
- colours

Brute force

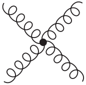
Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

2	4
3	25
4	220
5	2485
6	34300
7	559405
8	10525900

Feynman rules:



$$= -gf^{abc}\left[g^{\mu\nu}\left(q_1^{\rho}-q_2^{\rho}\right)+g^{\nu\rho}\left(q_2^{\mu}-q_3^{\mu}\right)+g^{\rho\mu}\left(q_3^{\nu}-q_1^{\nu}\right)\right]$$



$$= -ig^2\left[f^{abe}f^{ecd}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma}\right)+f^{bce}f^{ead}\left(g^{\nu\mu}g^{\rho\sigma}-g^{\rho\mu}g^{\nu\sigma}\right)+f^{cae}f^{ebd}\left(g^{\rho\nu}g^{\mu\sigma}-g^{\mu\nu}g^{\rho\sigma}\right)\right]$$

Feynman diagrams are not the method of choice !

Efficiency improvements

- How do we compute $|\mathcal{A}_n^{(0)}|^2$ efficiently?
  1. Spinor-helicity method
  2. Colour decomposition
  3. Recurrence relations
  4. Monte Carlo techniques

Efficiency improvements

- How do we compute $|\mathcal{A}_n^{(0)}|^2$ efficiently?
  1. Spinor-helicity method
  2. Colour decomposition
  3. Recurrence relations
  4. Monte Carlo techniques

- Comments:
  - We will take $gg \to n \cdot g$ as our running example.
  - Colour decomposition applies to any unbroken non-abelian gauge symmetry. In the Standard Model there is only $\mathrm{SU}(3)$.
  - All other techniques apply with minor modifications to the full Standard Model.

Spinor-helicity method

Helicity amplitudes

Suppose that an amplitude is given as the sum of N Feynman diagrams.

To calculate the **amplitude squared** à la Bjorken-Drell:
Sum over all spins and use

$$\begin{aligned}
\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) &= \not{p} + m, \\
\sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) &= \not{p} - m, \\
\sum_\lambda \varepsilon_\mu^*(k,\lambda)\varepsilon_\nu(k,\lambda) &= -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{kn} - n^2 \frac{k_\mu k_\nu}{(kn)^2}.
\end{aligned}$$

This gives of the order N^2 terms.

Better: For each spin configuration evaluate the amplitude to a complex number.
Taking the norm of a complex number is a cheap operation.

Dirac matrices

The **Weyl representation** of the Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbf{0} \end{pmatrix}$$

Here, the 4-dimensional σ^μ-matrices are defined by

$$\sigma^\mu_{A\dot{B}} = \left(\mathbf{1}, -\vec{\sigma}\right), \qquad \bar{\sigma}^{\mu \dot{A} B} = \left(\mathbf{1}, \vec{\sigma}\right),$$

and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the standard Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The massless Dirac equation

- We first consider **massless** fermions.
- The spin states are described by **spinors**, which are solutions of the Dirac equation:

$$\not{p}\, u(p) \;=\; 0$$

- For massless fermions, four-component Dirac spinors decompose into two two-component **Weyl spinors**:

$$u(p) \;=\; \begin{pmatrix} |p+\rangle \\ |p-\rangle \end{pmatrix} = \begin{pmatrix} |p\rangle \\ |p] \end{pmatrix} = \begin{pmatrix} p_A \\ p^{\dot{B}} \end{pmatrix} = \begin{pmatrix} u_+(p) \\ u_-(p) \end{pmatrix}.$$

$|p+\rangle$ and $|p-\rangle$ give the two independent solutions of the Dirac equation.

- Bra-spinors are given by

$$\overline{u}(p) \;=\; \left(\langle p-|,\, \langle p+| \right) = \left(\langle p|,\, [p| \right) = \left(p^A,\, p_{\dot{B}} \right) = \left(\bar{u}_-(p),\, \bar{u}_+(p) \right).$$

Solving the massless Dirac equation

- Express the four-vector $p^\mu = (p^0, p^1, p^2, p^3)$ in terms of **light-cone coordinates**:

$$p^+ = p^0 + p^3, \quad p^- = p^0 - p^3, \quad p^\perp = p^1 + ip^2, \quad p^{\perp *} = p^1 - ip^2.$$

Solving the massless Dirac equation

- Express the four-vector $p^\mu = (p^0, p^1, p^2, p^3)$ in terms of **light-cone coordinates**:

$$p^+ = p^0 + p^3, \quad p^- = p^0 - p^3, \quad p^\perp = p^1 + ip^2, \quad p^{\perp *} = p^1 - ip^2.$$

- Then the equation for the ket-spinors becomes

$$\begin{pmatrix} p^- & -p^{\perp *} \\ -p^\perp & p^+ \end{pmatrix} |p+\rangle = 0, \qquad \begin{pmatrix} p^+ & p^{\perp *} \\ p^\perp & p^- \end{pmatrix} |p-\rangle = 0,$$

Solving the massless Dirac equation

- Express the four-vector $p^\mu = (p^0, p^1, p^2, p^3)$ in terms of **light-cone coordinates**:

$$p^+ = p^0 + p^3, \quad p^- = p^0 - p^3, \quad p^\perp = p^1 + ip^2, \quad p^{\perp *} = p^1 - ip^2.$$

- Then the equation for the ket-spinors becomes

$$\left(\begin{array}{cc} p^- & -p^{\perp *} \\ -p^\perp & p^+ \end{array} \right) |p+\rangle = 0, \qquad \left(\begin{array}{cc} p^+ & p^{\perp *} \\ p^\perp & p^- \end{array} \right) |p-\rangle = 0,$$

- Solutions are

$$|p+\rangle = \frac{1}{\sqrt{p^-}} \left(\begin{array}{c} p^{\perp *} \\ p^- \end{array} \right), \qquad |p-\rangle = \frac{1}{\sqrt{p^-}} \left(\begin{array}{c} p^- \\ -p^\perp \end{array} \right).$$

Solving the massless Dirac equation

- Express the four-vector $p^\mu = (p^0, p^1, p^2, p^3)$ in terms of **light-cone coordinates**:

$$p^+ = p^0 + p^3, \quad p^- = p^0 - p^3, \quad p^\perp = p^1 + ip^2, \quad p^{\perp *} = p^1 - ip^2.$$

- Then the equation for the ket-spinors becomes

$$\begin{pmatrix} p^- & -p^{\perp *} \\ -p^\perp & p^+ \end{pmatrix} |p+\rangle = 0, \qquad \begin{pmatrix} p^+ & p^{\perp *} \\ p^\perp & p^- \end{pmatrix} |p-\rangle = 0,$$

- Solutions are

$$|p+\rangle = \frac{1}{\sqrt{p^-}} \begin{pmatrix} p^{\perp *} \\ p^- \end{pmatrix}, \qquad |p-\rangle = \frac{1}{\sqrt{p^-}} \begin{pmatrix} p^- \\ -p^\perp \end{pmatrix}.$$

- Similar for the bra-spinors:

$$\langle p+| = \frac{1}{\sqrt{p^-}} \left(p^\perp, p^- \right), \qquad \langle p-| = \frac{1}{\sqrt{p^-}} \left(p^-, -p^{\perp *} \right).$$

Spinor products

Spinor products are defined by

$$\begin{aligned} \langle pq \rangle &= \langle p - | q + \rangle = \frac{1}{\sqrt{p^-}\sqrt{q^-}} \left(p^- q^{\perp *} - q^- p^{\perp *} \right), \\ [qp] &= \langle q + | p - \rangle = \frac{1}{\sqrt{p^-}\sqrt{q^-}} \left(p^- q^{\perp} - q^- p^{\perp} \right). \end{aligned}$$

The spinor products are **anti-symmetric**.

The spinor helicity method

Gluon polarisation vectors:

$$\varepsilon_\mu^+(p,q) = \frac{\langle p+|\bar{\sigma}_\mu|q+\rangle}{\sqrt{2}\langle q-|p+\rangle}, \qquad \varepsilon_\mu^-(p,q) = \frac{\langle p-|\sigma_\mu|q-\rangle}{\sqrt{2}\langle p+|q-\rangle}.$$

q is an arbitrary light-like **reference momentum**. Dependency on q drops out in gauge invariant quantities.

The polarisation vectors satisfy:

$$\begin{aligned} \varepsilon_\mu^\pm(p,q)\, p^\mu &= 0, \\ \varepsilon_\mu^\pm(p,q)\, q^\mu &= 0. \end{aligned}$$

$$\begin{aligned} \varepsilon^+ \cdot \left(\varepsilon^+\right)^* &= \varepsilon^- \cdot \left(\varepsilon^-\right)^* = -1, \\ \varepsilon^+ \cdot \left(\varepsilon^-\right)^* &= 0. \end{aligned}$$

$$\left(\varepsilon_\mu^+\right)^* = \varepsilon_\mu^- .$$

Helicity amplitudes

Example ($gg \to 7g$)

For $gg \to 7g$ we have $N = 559405$ Born diagrams.

- Helicity amplitudes reduce the complexity from

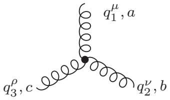
$$\begin{array}{rcll} N^2 & = & 312933954025 \text{ terms} & \approx \mathcal{O}\left(10^{11}\right) \text{ terms} \\ \text{to} \quad 2^n \cdot N & = & 512 \cdot 559405 \text{ terms} & \approx \mathcal{O}\left(10^{8}\right) \text{ terms.} \end{array}$$

- Factor $2^n = 2^9 = 512$ from sum over all helicities.
- By the end of this lecture:
  - Replace sum over helicities by Monte Carlo integration over helicity angles. **Monte Carlo error is independent of the number of dimensions**, this removes the factor 2^n.
  - Recurrence relations reduce the cost for calculating the amplitude from $N = 559405$ to $n^3 = 729$.

Colour decomposition

Colour decomposition

Each QCD Feynman rule has a colour part and a kinematical part:

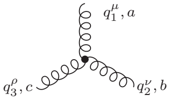


$$= \quad -g \underbrace{f^{abc}}_{\text{colour}} \left[\underbrace{g^{\mu\nu}\left(q_1^\rho - q_2^\rho\right) + g^{\nu\rho}\left(q_2^\mu - q_3^\mu\right) + g^{\rho\mu}\left(q_3^\nu - q_1^\nu\right)}_{\text{kinematic}} \right]$$

In an amplitude **collect all terms with the same colour structure**.

Colour decomposition

Each QCD Feynman rule has a colour part and a kinematical part:



$$= \quad -g \underbrace{f^{abc}}_{\text{colour}} \underbrace{\left[g^{\mu\nu}\left(q_1^\rho - q_2^\rho\right) + g^{\nu\rho}\left(q_2^\mu - q_3^\mu\right) + g^{\rho\mu}\left(q_3^\nu - q_1^\nu\right) \right]}_{\text{kinematic}}$$

In an amplitude **collect all terms with the same colour structure**.

Example: The *n*-gluon amplitude:

$$\mathcal{A}_n^{(0)}(g_1, g_2, ..., g_n) \quad = \quad g^{n-2} \sum_{\sigma \in S_n / Z_n} \underbrace{2 \, \mathrm{Tr}\left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}\right)}_{\text{colour factors}} \underbrace{A_n^{(0)}\left(g_{\sigma(1)}, ..., g_{\sigma(n)}\right)}_{\textbf{partial amplitudes}}.$$

The partial amplitudes do not contain any colour information and **are gauge-invariant**.
Each partial amplitude has a **fixed cyclic order** of the external legs.

Colour decomposition

Lie algebra of $SU(N)$:

$$\left[T^a, T^b\right] = i f^{abc} T^c, \qquad \text{Tr}\,\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$$

Multiply commutator relation by T^d and take the trace:

$$\mathbf{i f^{abc}} \;=\; \mathbf{2}\text{Tr}\,\left(\mathbf{T^a T^b T^c}\right) - \mathbf{2}\text{Tr}\,\left(\mathbf{T^b T^a T^c}\right)$$

Fierz identities for $SU(N)$:

$$\begin{aligned}
\text{Tr}\left(T^a X\right)\text{Tr}\left(T^a Y\right) &= \frac{1}{2}\left[\text{Tr}\left(XY\right) - \frac{1}{N}\text{Tr}\left(X\right)\text{Tr}\left(Y\right)\right], \\
\text{Tr}\left(T^a X T^a Y\right) &= \frac{1}{2}\left[\text{Tr}\left(X\right)\text{Tr}\left(Y\right) - \frac{1}{N}\text{Tr}\left(XY\right)\right].
\end{aligned}$$

Fierz identities for $SU(N)$

$$T^a_{ij} T^a_{kl} \quad = \quad \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).$$

Fierz identities for $SU(N)$

$$T^a_{ij} T^a_{kl} = \frac{1}{2} \left(\delta_{il}\delta_{jk} - \frac{1}{N} \delta_{ij}\delta_{kl} \right).$$

The unit matrix $\mathbf{1}$ and the T^a's form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix A can be written as

$$A = c_0 \mathbf{1} + c_a T^a.$$

Fierz identities for $SU(N)$

$$T^a_{ij} T^a_{kl} = \frac{1}{2} \left(\delta_{il}\delta_{jk} - \frac{1}{N} \delta_{ij}\delta_{kl} \right).$$

The unit matrix $\mathbf{1}$ and the T^a's form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix A can be written as

$$A = c_0 \mathbf{1} + c_a T^a.$$

Using the fact that the T^a are traceless and the normalisation condition:

$$c_0 = \frac{1}{N} \mathrm{Tr}\left(A\right), \qquad c_a = 2\mathrm{Tr}\left(T^a A\right).$$

Fierz identities for $SU(N)$

$$T^a_{ij} T^a_{kl} \quad = \quad \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right).$$

The unit matrix **1** and the T^a's form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix A can be written as

$$A \quad = \quad c_0 \mathbf{1} + c_a T^a.$$

Using the fact that the T^a are traceless and the normalisation condition:

$$c_0 = \frac{1}{N} \mathrm{Tr}\left(A\right), \qquad c_a = 2\mathrm{Tr}\left(T^a A\right).$$

Therefore

$$A_{lk}\left(2T^a_{ij} T^a_{kl} + \frac{1}{N}\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk} \right) \quad = \quad 0.$$

This has to hold for an arbitrary A, therefore the Fierz identity follows.

Improvement due to the colour decomposition

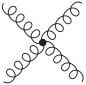
Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

n	2	3	4	5	6	7	8
unordered	4	25	220	2485	34300	559405	10525900
cyclic ordered	**3**	**10**	**38**	**154**	**654**	**2871**	**12925**

Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

n	2	3	4	5	6	7	8
unordered	4	25	220	2485	34300	559405	10525900
cyclic ordered	**3**	**10**	**38**	**154**	**654**	**2871**	**12925**

Feynman rules: Four-gluon vertex



Traditional (unordered):

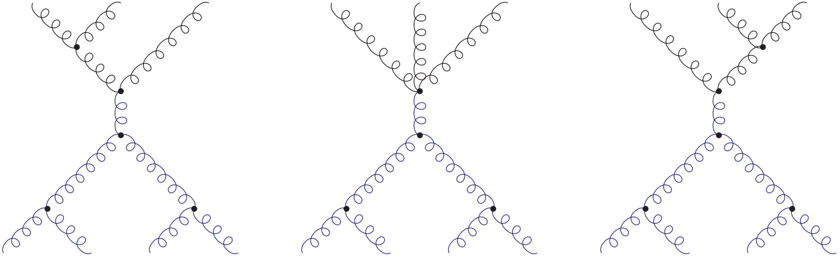
$$-ig^2\left[f^{abe}f^{ecd}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma}\right)+f^{bce}f^{ead}\left(g^{\nu\mu}g^{\rho\sigma}-g^{\rho\mu}g^{\nu\sigma}\right)+f^{cae}f^{ebd}\left(g^{\rho\nu}g^{\mu\sigma}-g^{\mu\nu}g^{\rho\sigma}\right)\right]$$

Colour-stripped and cyclic ordered:

$$i\left(2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)$$

Recurrence relations

How to avoid to compute the same sub-expression again and again

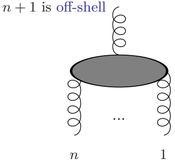


Lower part identical in all three diagrams.

Strategy: Compute this sub-expression once and store the result.

Off-shell currents

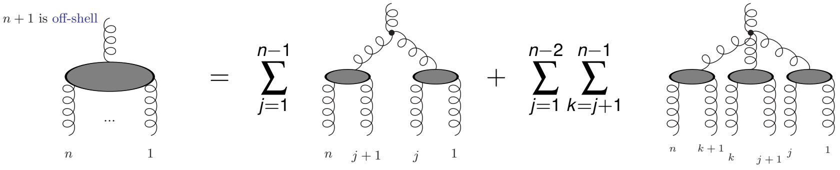
We introduce an off-shell current $J^{\mu}(g_1, ..., g_n)$



- The blob stands for the sum of all possible diagrams.
- Momentum conservation: $p_{n+1} = p_1 + p_2 + ... + p_n$.
- Legs 1 to n are on-shell: $p_j^2 = m_j^2$.
- Legs $(n+1)$ is allowed to be **off-shell**.
- $J^{\mu}(g_1, ..., g_n)$ is **gauge-dependent**!

Recurrence relations

Off-shell currents $J^\mu(g_1, ..., g_n)$ provide an efficient way to calculate amplitudes:



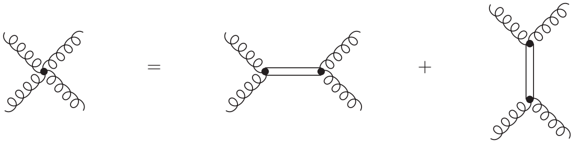
Recursion start: $J^\mu(g_1) = \varepsilon_1^\mu$.

No Feynman diagrams are calculated in this approach !

Computational costs

Born amplitudes with n particles and three- and four-valent vertices scale as n^4.

Can replace four-gluon vertex by a tensor particle, obtain only three-valent vertices:



Scaling reduced to n^3.

Technical details

Introduce an auxiliary tensor field $B_{[\mu\nu]}$ with the Lagrange density ($A_\mu = \frac{g}{i} T^a A_\mu^a$)

$$\mathcal{L}_{\mathrm{aux}} = \frac{2}{g^2} \mathrm{Tr} \left(\frac{1}{2} B_{[\mu\nu]} B^{[\mu\nu]} - \frac{i}{\sqrt{2}} B^{[\mu\nu]} [A_\mu, A_\nu] \right).$$

The auxiliary field $B_{[\mu\nu]}$ occurs at the most quadratically and can be integrated out

$$\int \mathcal{D}B_{[\mu\nu]} \exp\left(i \int d^D x \, \mathcal{L}_{\mathrm{aux}} \right) = \mathcal{N} \exp\left(i \int d^D x \, \mathcal{L}_{\mathrm{gggg}} \right),$$

where the (irrelevant) prefactor $\mathcal{N}$ is given by

$$\mathcal{N} = \int \mathcal{D}B_{[\mu\nu]} \exp\left(\frac{i}{g^2} \int d^D x \, \mathrm{Tr} \, B_{[\mu\nu]} B^{[\mu\nu]} \right).$$

From the Lagrangian $\mathcal{L}_{\mathrm{aux}}$ we may derive the Feynman rules for this tensor field.

Monte Carlo techniques

Integration over the phase space

Recall the master formula at leading order:

$$\langle O_n \rangle = \sum_{a,b} \int dx_1 \; f_a(x_1) \int dx_2 \; f_b(x_2) \; \frac{1}{2K(\hat{s})} \; \frac{1}{n_a^{\text{spin}} n_b^{\text{spin}} n_a^{\text{colour}} n_b^{\text{colour}}} \int \mathbf{d\phi_n O(p_1, ..., p_n)} \; \left| \mathcal{A}_{n+2}^{(0)} \right|^2$$

The phase space measure:

$$d\phi_n \quad = \quad \frac{1}{n!} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \, (2\pi)^4 \, \delta^4 \left(p_a + p_b - \sum_{i=1}^{n} p_i \right)$$

Integration over $(3n - 4)$ dimensions.

Goal: Once we have calculated the matrix element squared $|\mathcal{A}_{n+2}^{(0)}|^2$ for a given process, we would like to get predictions for **different observables** related to this process.

Solution: **Numerical integration over the phase space** with **Monte Carlo techniques**.

Monte Carlo integration

We are interested in **multi-dimensional integrals**:

$$I = \int\limits_{[0,1]^N} d^N u \, f(u_1, ..., u_N)$$

Evaluate the integrand at N_{eval} random points $\vec{u}_j = (u_{j,1}, ..., u_{j,N})$.

$$I = \frac{1}{N_{\mathrm{eval}}} \sum_{j=1}^{N_{\mathrm{eval}}} f(\vec{u}_j)$$

Error scales as

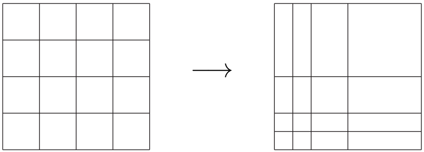
$$\sigma \sim \frac{1}{\sqrt{N_{\mathrm{eval}}}}$$

Integration over the phase space

Change integration variables such that the phase space integral becomes an integral over the unit hypercube $[0,1]^{(3n-4)}$.

This is a **high dimensional integral**. For numerical integration in more than two dimensions Monte Carlo techniques are usually better suited then quadrature rules. Error estimate of Monte Carlo integration is independent of the number of dimensions, **scales like** $1/\sqrt{N_{\mathrm{eval}}}$.

Can use variance-reducing techniques like importance sampling.
VEGAS: Integration over the unit hypercube with importance sampling.



Adapts where the integrand is largest and "learns" about the integrand as it proceeds.

Histograms

Suppose we would like to calculate for the process $pp \rightarrow 2$ jets

- the cross section,
- the $p_\perp$-distributions of the jets,
- the rapidity distributions of the jets.

Can do it in one Monte Carlo run:

- Set up the calculation for the cross section
- For every event put the corresponding weight in the appropriate bin of the histogram.

Attention: When using Vegas, we need to know the Vegas weight.

Phase space generators

The **concrete form** of the change of variables from $d\phi_n$ to the unit hypercube **affects the error of the Monte Carlo integration**.

Several possibilities:

- Sequential generator: Based on successive two-body decays. Phase space weight varies from event to event.
- RAMBO: Constant weight for every phase space point.
- Generators based on QCD radiation pattern: Peak structure of the matrix element squared absorbed into the phase space weight.
- Generators mapping a Breit-Wigner shape, essential for electro-weak physics.

Control variates

How to improve the efficiency of the Monte Carlo integration?

$$I = \int\limits_{[0,1]^N} d^N u \, f(u)$$

Assume that

- $f(u)$ is **expensive** to evaluate.
- $g(u)$ is a **cheap** approximation to $f(u)$.
  (Could be a neural network approximation.)

Then

$$\int\limits_{[0,1]^N} d^N u \, f(u) = \underbrace{\int\limits_{[0,1]^N} d^N u \, [f(u) - g(u)]}_{\textit{small}} + \underbrace{\int\limits_{[0,1]^N} d^N u \, g(u)}_{\textit{cheap}}$$

Symmetric phase space integration

Example ($q\bar{q} \to ng$)

Colour decomposition:

$$\mathcal{A}^{(0)}_{n+2}(q, g_1, \dots, g_n, \bar{q}) = g^n \sum_{\sigma \in S_n} \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}} \right)_{i_q j_{\bar{q}}} A^{(0)}_{n+2}\left(q, g_{\sigma(1)}, \dots, g_{\sigma(n)}, \bar{q}\right) = g^n \sum_{i=1}^{n!} C_i \, A^{(0)}_{n+2,i}.$$

There are $n!$ partial amplitudes.

Leading colour contribution:

$$\left| \mathcal{A}^{(0)}_{n+2} \right|^2_{\mathrm{lc}} = g^{2n} \sum_{i=1}^{n!} \left(C_i^\dagger C_i \right) \left| A^{(0)}_{n+2,i} \right|^2 = g^{2n} \left(C_1^\dagger C_1 \right) \sum_{i=1}^{n!} \left| A^{(0)}_{n+2,i} \right|^2.$$

Phase space integration is symmetric, can remove sum with $n!$ terms:

$$\int d\phi_n O_n \left| \mathcal{A}^{(0)}_{n+2} \right|^2_{\mathrm{lc}} = n! \, g^{2n} \left(C_1^\dagger C_1 \right) \int d\phi_n O_n \left| A^{(0)}_{n+2,1} \right|^2.$$

Further applications of Monte Carlo techniques

The squared matrix element is summed over all spins and colours.
When using helicity amplitudes we have 2^n **helicity configurations** for $|\mathcal{A}_{n+2}^{(0)}|^2$.

Can replace sum over helicity configurations either by

- Monte Carlo sampling of the configurations or
- Monte Carlo integration over "helicity angles":

$$\sum_{\lambda=\pm} \varepsilon_\mu^{\lambda\,*} \varepsilon_\nu^\lambda \;=\; \frac{1}{2\pi} \int\limits_0^{2\pi} d\phi\, \varepsilon_\mu(\phi)^* \varepsilon_\nu(\phi), \qquad \varepsilon_\mu(\phi) = e^{i\phi}\varepsilon_\mu^+ + e^{-i\phi}\varepsilon_\mu^- .$$

Recall: **Monte Carlo error is independent of the number of dimensions.**
Similar techniques can be used for the colour degrees of freedom.

Take-home messages

- LO calculations can be done **efficiently**.
- The spinor-helicity method, colour decomposition, recurrence relations and Monte Carlo techniques allow the computation of multi-parton observables at leading order.