Standard Model Precision Physics

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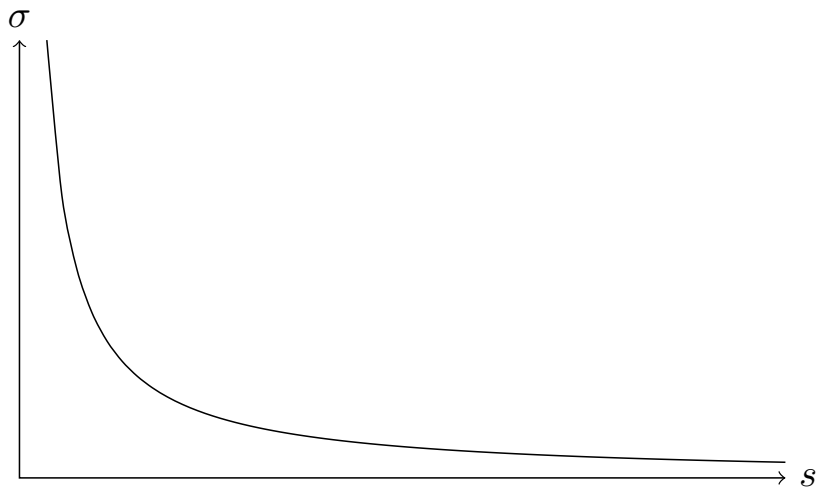
Section 5

Higher-order calculations

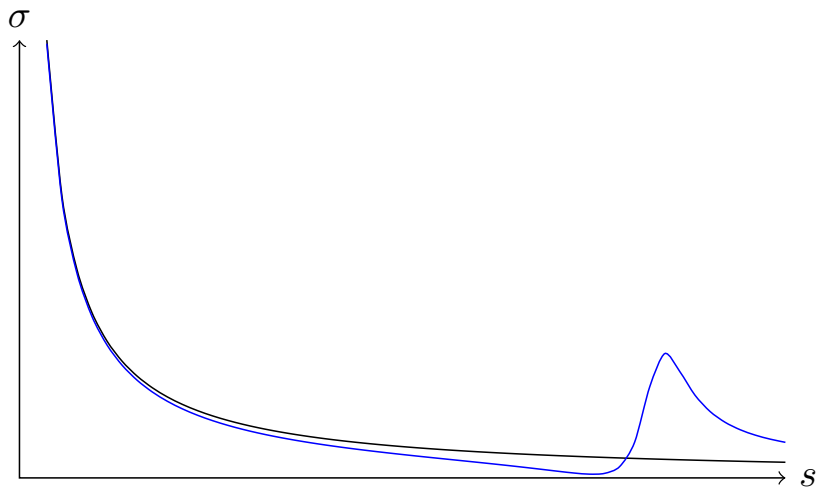
The need for precision

Cross section vs. collider energy within the current paradigma:



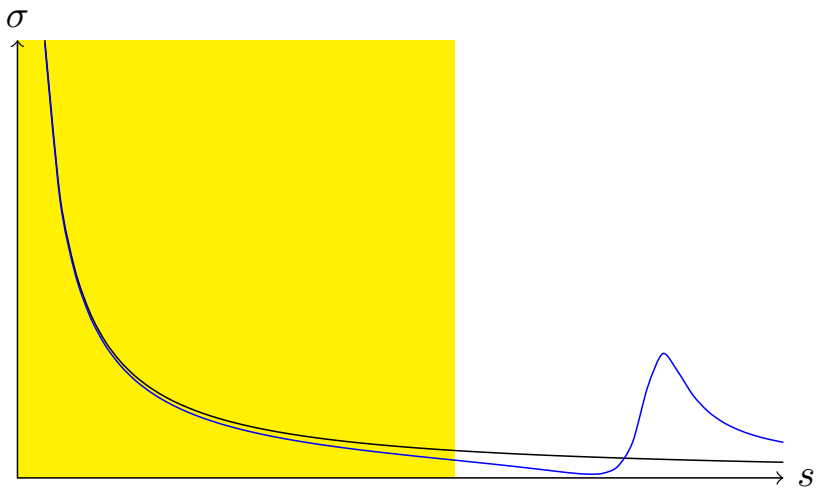
The need for precision

Suppose that there is new physics at higher energies:



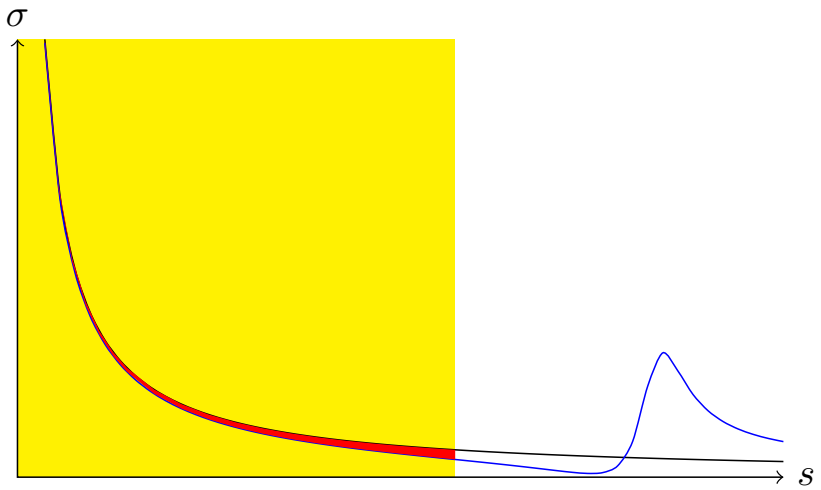
The need for precision

But the center-of-mass energy of the colllider is limited:



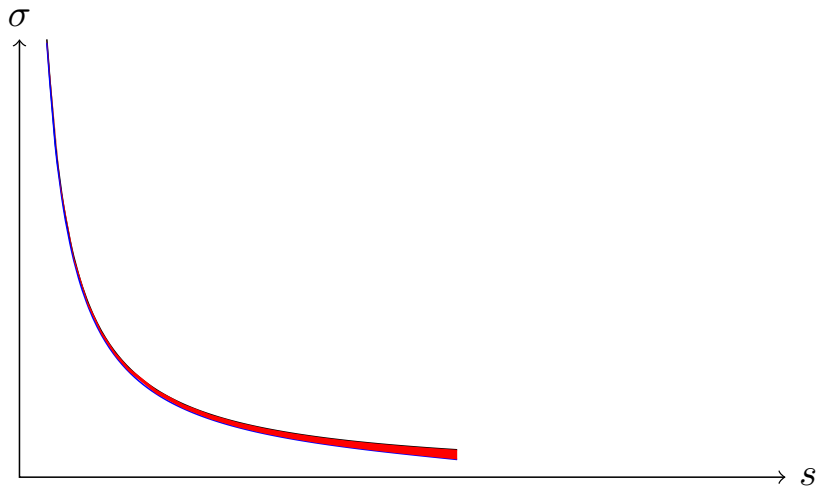
The need for precision

What we can possibly see:



The need for precision

The required precision: Better than the red band!



The master formula for the calculation of observables

$$\langle O_n \rangle^{\mathrm{LO}} = \sum_{a,b} \int dx_1\, f_a(x_1) \int dx_2\, f_b(x_2)\; \frac{1}{2K(\hat{s})}\; \frac{1}{n_a^{\mathrm{spin}}\, n_b^{\mathrm{spin}}\, n_a^{\mathrm{colour}}\, n_b^{\mathrm{colour}}}$$

$$\times \Big\{ \int d\phi_n O(p_1, \ldots, p_n) \Big[\underbrace{\left| \mathcal{A}_{n+2}^{(0)} \right|^2}_{\text{Born}} \Big] \Big\}$$

The master formula for the calculation of observables

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$$\times \Bigg\{ \int d\phi_n O(p_1,\ldots,p_n) \quad \Bigg[\underbrace{\left|\mathcal{A}^{(0)}_{n+2}\right|^2}_{\text{Born}} + \underbrace{2\,\mathrm{Re}\ \mathcal{A}^{(0)\,*}_{n+2}\, \mathcal{A}^{(1)}_{n+2}}_{\text{virtual}} \Bigg]$$

$$+ \int d\phi_{n+1} O(p_1,\ldots,p_{n+1}) \quad \Bigg[\underbrace{\left|\mathcal{A}^{(0)}_{n+3}\right|^2}_{\text{real}} \Bigg] \Bigg\}$$

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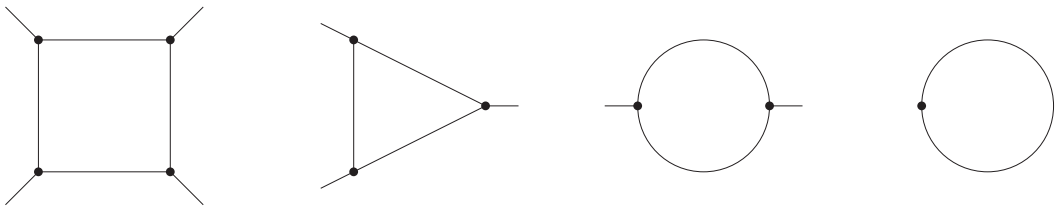
$$+ \int d\phi_{n+2} O(p_1,\ldots,p_{n+2}) \quad \underbrace{\left|\mathcal{A}^{(0)}_{\mathbf{n+4}}\right|^2}_{\text{double real}} \Bigg\}$$

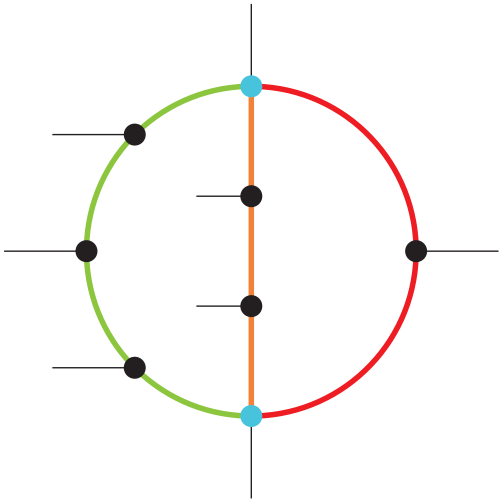
Feynman integrals

- At **one-loop** we only had boxes, triangles, bubbles and tadpoles.



- At **two-loops**: $\mathcal{O}(70)$ topologies

- At most 5 propagators per chain.
- At most 11 propagators in total.



Transcendental functions

Transcendental functions

Up to $O(\varepsilon^0)$ all one-loop amplitudes can be expressed as a sum of algebraic functions of the kinematic variables times two transcendental functions, whose arguments are again algebraic functions of the kinematic variables.

The two transcendental functions are the **logarithm** and the **dilogarithm**:

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Numerical evaluation of the dilogarithm

The **dilogarithm**:

$$\mathrm{Li}_2(x) \quad = \quad \int_0^x \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \quad = \quad \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Map into region $|x| \leq 1$ and $-1 \leq \mathrm{Re}(x) \leq 1/2$, using

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}(\ln(-x))^2, \qquad \mathrm{Li}_2(x) = -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x).$$

Accelaration using Bernoulli numbers B_j:

$$\mathrm{Li}_2(x) \quad = \quad \sum_{j=0}^{\infty} \frac{B_j}{(j+1)!}(-\ln(1-x))^{j+1},$$

Generalisations of the logarithm

- Polylogarithms:

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- Multiple polylogarithms:

$$\mathrm{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) = \sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

This is a nested sum:

$$\ldots \sum_{\mathbf{n_j}=1}^{n_{j-1}-1} \frac{x_j^{\mathbf{n_j}}}{\mathbf{n_j}^{m_j}} \sum_{n_{j+1}=1}^{\mathbf{n_j}-1} \ldots$$

Multiple polylogarithms

Definition based on **nested sums**:

$$\mathrm{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) = \sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on **iterated integrals**:

$$G(z_1,\ldots,z_k;y) = \int_0^y \frac{dt_1}{t_1-z_1} \int_0^{t_1} \frac{dt_2}{t_2-z_2} \ldots \int_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}$$

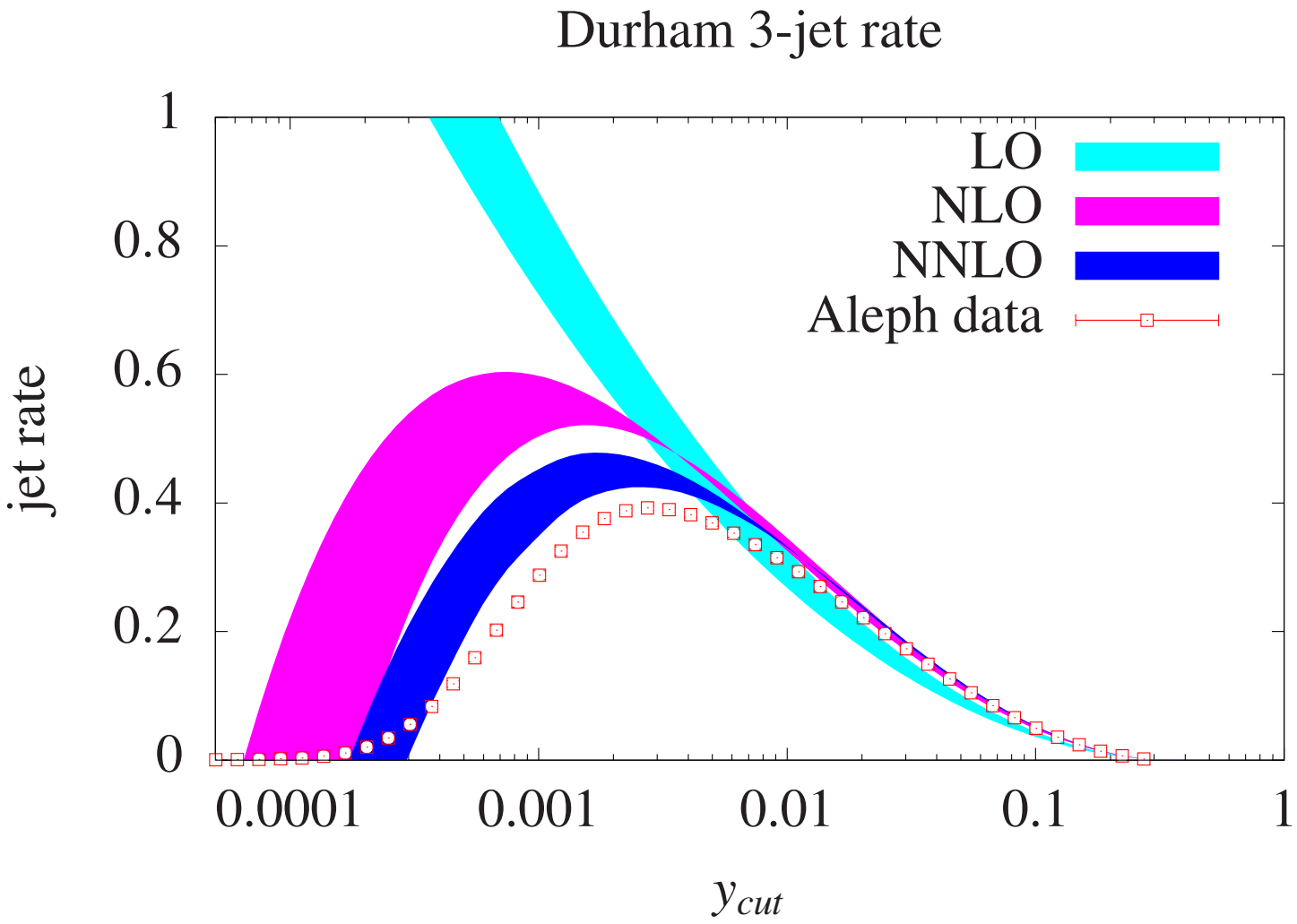
Conversion:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\ldots,\frac{1}{x_1\ldots x_k};1\right)$$

Short hand notation:

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = G(\underbrace{0,\ldots,0}_{m_1-1},z_1,\ldots,z_{k-1},\underbrace{0\ldots,0}_{m_k-1},z_k;y)$$

Results for the three-jet rate in electron-positron annihilation



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  Answer: **Points**.

Geometries

We work with the complex numbers $\mathbb{C}$.

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  Examples are Calabi-Yau manifolds.
- At **two-loops** we will encounter points, curves and surfaces.

How do I find the geometry associated to a particular Feynman integral?

Answer for the experts:

Start with the Baikov representation of the Feynman integral, set $\varepsilon = 0$ and take as many residues as you can. This will result in

$$\int \frac{d^n z}{\sqrt{P(z)}}$$

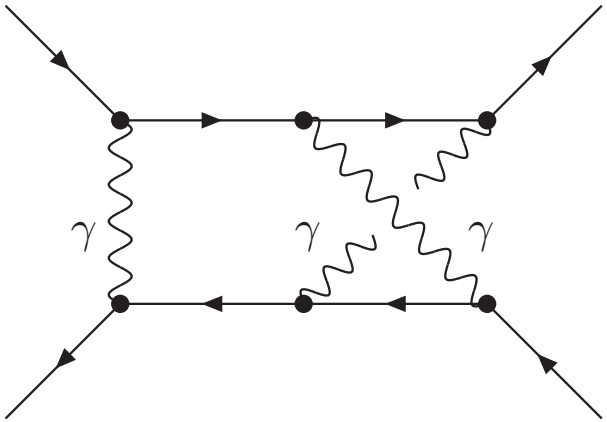
The equation

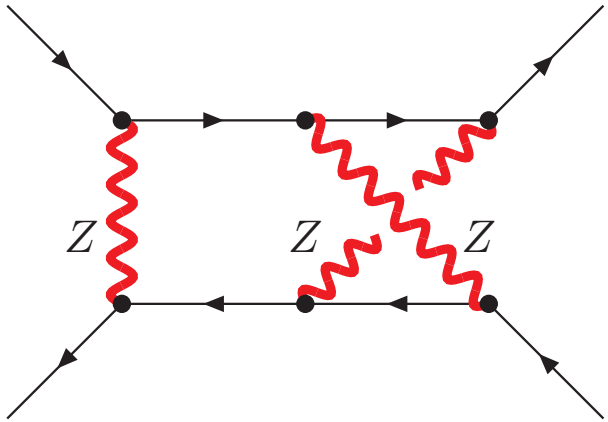
$$\mathbf{y^2 = P(z)}$$

defines the geometry.

Functions related to this geometry will appear in the result for the Feynman integral.

Do I need to worry about this for the two-loop electroweak corrections to Drell-Yan?





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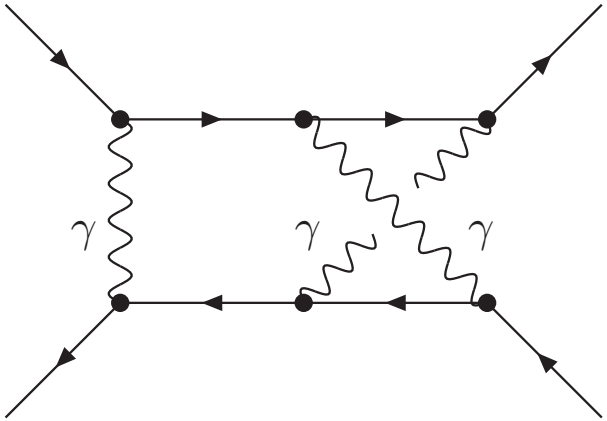
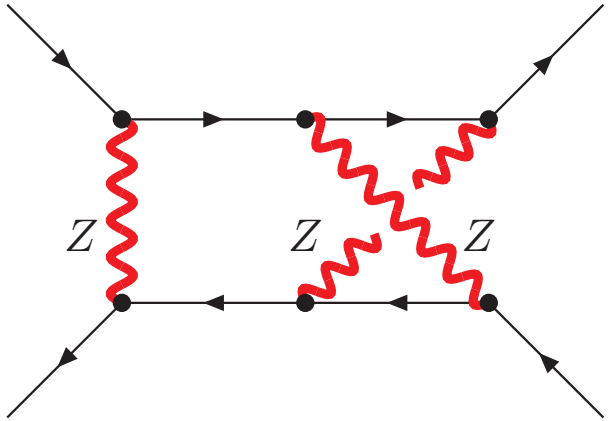


- Geometries:
  - Only points



- Geometries:
  - 1 genus 2 curve
  - 1 K3 surface
  - 5 genus 1 curves
  - Points

Evaluating Feynman integrals

- **Analytic solution**:

$$I = \mathrm{Li}_2(x) - \ln(x)\ln(1-x)$$

- **Series solution** (for example around $x = 0$):

$$I = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \ln(x)\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right] + O\left(x^5\right)$$

- **Numerical solution** (for example for $x = \frac{1}{3}$):

$$I \approx -0.07923572$$

Computing Feynman integrals

Feynman integrals

Associate to a Feynman graph G with n_{ext} external lines, n internal lines and l loops the set of Feynman integrals

$$I_{\nu_1\nu_2\ldots\nu_n}(\varepsilon, x) \quad \sim \quad \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}},$$

with $\nu_j \in \mathbb{Z}$.

This defines the momentum representation of a Feynman integral.

Feynman integrals

The Feynman integral $I_{\nu_1\nu_2\ldots\nu_n}(\varepsilon, x)$ **depends on**

- the dimensional regularisation parameter ε,
- the kinematic variables $x_1, x_2, \ldots$ (dimensionless ratios of scalar products $p_i \cdot p_j$ of the external momenta and internal masses m_j),
- the indices $\nu_1, \ldots, \nu_n$.

Integration-by-parts

Integration-by-parts

Within dimensional regularisation we have for any loop momentum k_i and $a \in \{p_1, \ldots, p_{n_{\text{ext}}}, k_1, \ldots, k_l\}$

$$\int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \; \frac{\partial}{\partial \mathbf{k_i^\mu}} \, a^\mu \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}} \quad = \quad 0.$$

Working out the derivatives leads to **relations among integrals** with different sets of indices $(\nu_1, \ldots, \nu_n)$.

This allows us to express most of the integrals in terms of a few **master integrals** $I_1, \ldots, I_{N_F}$.

Example

Consider the one-loop two-point function with an equal internal mass with $q_1 = k - p$, $q_2 = k$ and $x = -p^2/m^2$.

$$I_{\nu_1\nu_2}(\varepsilon, x) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(-q_1^2 + m^2\right)^{\nu_1} \left(-q_2^2 + m^2\right)^{\nu_2}}.$$

For $q_{\mathrm{IBP}} = p$ we obtain

$$\begin{aligned}
0 &= p^\mu \int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \frac{1}{\left(-q_1^2 + m^2\right)^{\nu_1} \left(-q_2^2 + m^2\right)^{\nu_2}} \\
&= \int \frac{d^D k}{(2\pi)^D} \left[\frac{\nu_1 \left(q_2^2 - q_1^2 - p^2\right)}{\left(-q_1^2 + m^2\right)^{\nu_1+1} \left(-q_2^2 + m^2\right)^{\nu_2}} + \frac{\nu_2 \left(q_2^2 - q_1^2 + p^2\right)}{\left(-q_1^2 + m^2\right)^{\nu_1} \left(-q_2^2 + m^2\right)^{\nu_2+1}} \right] \\
&= \nu_1 \left[\mathbf{I}_{\nu_1\nu_2} - \mathbf{I}_{(\nu_1+1)(\nu_2-1)} + x\mathbf{I}_{(\nu_1+1)\nu_2} \right] + \nu_2 \left[\mathbf{I}_{(\nu_1-1)(\nu_2+1)} - \mathbf{I}_{\nu_1\nu_2} - x\mathbf{I}_{\nu_1(\nu_2+1)} \right].
\end{aligned}$$

Example

We have two possible choices $q_{\mathrm{IBP}} \in \{p, k\}$.

This gives us two equations, which we can arrange as

$$\begin{aligned}
&\nu_1 x\,(4+x)\, I_{(\nu_1+1)\nu_2} = \\
&\qquad [2\,(-\nu_1+\nu_2)+(\nu_1+2\nu_2-D)\,x]\, I_{\nu_1\nu_2} + \nu_1\,(2+x)\, I_{(\nu_1+1)(\nu_2-1)} - 2\nu_2 I_{(\nu_1-1)(\nu_2+1)},
\end{aligned}$$

$$\begin{aligned}
&\nu_2 x\,(4+x)\, I_{\nu_1(\nu_2+1)} = \\
&\qquad [2\,(\nu_1-\nu_2)+(2\nu_1+\nu_2-D)\,x]\, I_{\nu_1\nu_2} - 2\nu_1 I_{(\nu_1+1)(\nu_2-1)} + \nu_2\,(2+x)\, I_{(\nu_1-1)(\nu_2+1)}.
\end{aligned}$$

For $\nu_1 > 0$ and $\nu_2 > 0$ we may use either the first or the second equation to reduce the sum $\nu_1 + \nu_2$:
In both equations, the sum of the indices equals $\nu_1 + \nu_2 + 1$ on the left-hand side, while on the right-hand side the sum of the indices equals for all terms $\nu_1 + \nu_2$.

Example

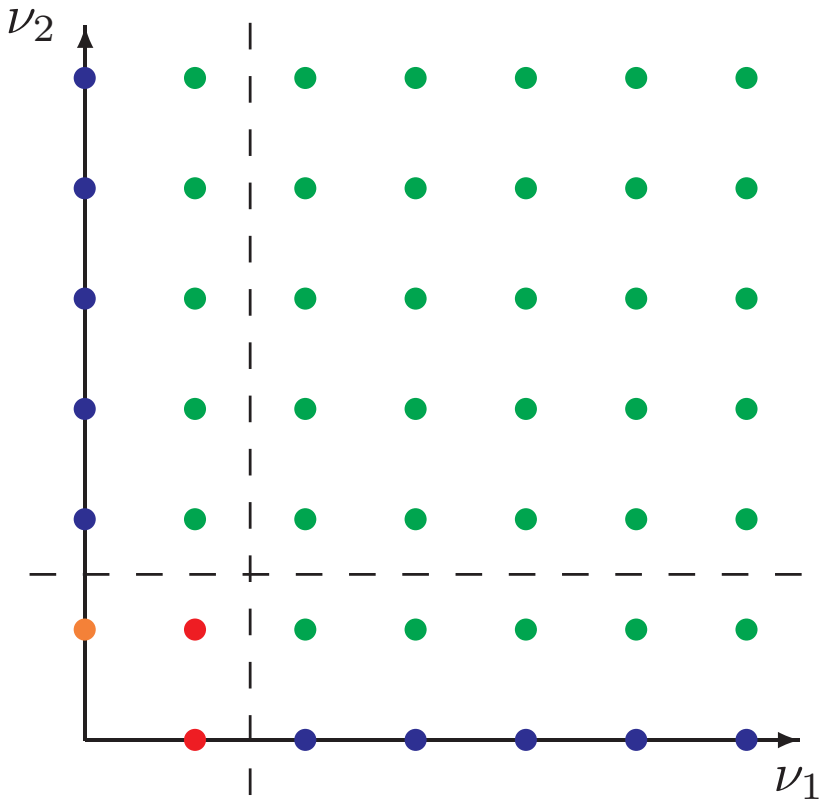
If either $\nu_1 = 0$ (and $\nu_2 > 0$) or $\nu_2 = 0$ (and $\nu_1 > 0$) we have a simpler integral: The integral reduces to a tadpole integral. As the two internal masses are equal, we have

$$I_{\nu 0} \quad = \quad I_{0\nu}\,.$$

For a tadpole integral we repeat the exercise of deriving integration-by-parts identities and we find

$$\nu_2\, I_{0(\nu_2+1)} \quad = \quad \left(\nu_2 - \frac{D}{2}\right) I_{0\nu_2}\,.$$

Example



Integration-by-parts reduction for the one-loop two-point function:
We may reduce any integral $I_{\nu_1\nu_2}$ to a linear combination of I_{11} and I_{10}.

The Laporta algorithm

Integration-by-parts reduction is linear algebra:

1. Gauß elimination
2. Order relation: In each equation we eliminate the most complicated Feynman integral.

In the previous example we used $\nu_1 + \nu_2$ as the main order criterium.

Remark:

- Integration-by-parts reduction is a core algorithm.
- Available computing resources **limit** which calculation can be done.

Linear algebra

Consider the linear system:

$$\begin{aligned} 14732\, I_1 \quad -2514\, I_2 \quad -5\, I_3 \quad -7\, I_4 &= 0 \\ 9872\, I_1 \quad -17294\, I_2 \quad +3\, I_3 \quad -11\, I_4 &= 0 \\ 5068\, I_1 \quad -49336\, I_2 \quad +18\, I_3 \quad -22\, I_4 &= 0 \end{aligned}$$

This system has rank 2.

Linear algebra

Taking I_3 and I_4 as free variables:

$$\begin{aligned} I_1 &= \frac{1237}{3025750} I_3 + \frac{1229}{3025750} I_4 \\ I_2 &= \frac{1231}{3025750} I_3 - \frac{1223}{3025750} I_4 \end{aligned}$$

We obtain a large denominator.

Linear algebra

Taking I_1 and I_2 as free variables:

$$\begin{aligned} I_3 &= 1223 I_1 + 1229 I_2 \\ I_4 &= 1231 I_1 - 1237 I_2 \end{aligned}$$

We would like to avoid large denominators!

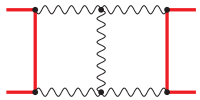
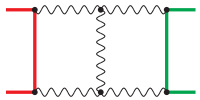
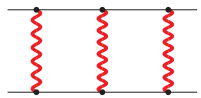
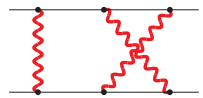
Linear algebra

Select free variables and dependent variables by an order relation:

$$l_1 > l_2 > l_3 > l_4 \qquad \text{(first case)}$$
$$l_4 > l_3 > l_2 > l_1 \qquad \text{(second case)}$$

Lesson: **The order matters for efficiency!**

Order relations

Family				
default order	509 kB	18 433 kB	3 534 kB	116 428 kB
geometric order	18 kB	112 kB	89 kB	3 073 kB
improvement factor	28	165	40	38

The method of differential equations

The method of differential equations

Denote by $x = (x_1, \ldots, x_{N_B})$ the kinematic variables (scalar products of external momenta and internal masses squared).

We want to calculate

$$I_{\nu_1 \nu_2 \ldots \nu_n}(\varepsilon, x)$$

1. *Find a differential equation with respect to the kinematic variables for the Feynman integral.*
2. *Transform the differential equation into a simple form.*
3. *Solve the latter differential equation with appropriate boundary conditions.*

Differential equations

Let x_k be a kinematic variable. Let $I_i \in \{I_1, \ldots, I_{N_F}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the **derivative as a linear combination of the master integrals**.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_F} a_{ij} I_j$$

Differential equations

Let us formalise this:

$$I = (I_1, ..., I_{N_F}), \quad \text{set of master integrals,}$$
$$x = (x_1, ..., x_{N_B}), \quad \text{set of kinematic variables the master integrals depend on.}$$

We obtain a **system of differential equations**

$$[d + A(\varepsilon, x)]\, I = 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

ε-factorised differential equations

Task:

The essential step in computing Feynman integrals is to find a redefinition of the master integrals such that

$$[d + \varepsilon A(x)]\, I \quad = \quad 0,$$

i.e. the **only dependence on** ε *is* **given by** *the* **explicit prefactor**.

Transformations:

Change the basis of the master integrals

$$I' \quad = \quad UI,$$

where $U(\varepsilon, x)$ *is a* $N_F \times N_F$*-matrix. The new connection matrix is*

$$A' \quad = \quad UAU^{-1} + UdU^{-1}.$$

Transformation to an ε-factorised form

1. **Observation**: The basis I obtained from the Laporta algorithm with the geometric order relation has the property that on the maximal cut we have

$$[d + A(\varepsilon, x)]\, I \;=\; 0, \qquad A_{ij}(\varepsilon, x) \;=\; \sum_{k=-(|\mu|_i - |\mu|_j)}^{1} \varepsilon^k A_{ij}^{(k)}(x)\,.$$

2. **Theorem**: From such a form we may always construct algorithmically a matrix U, which leads to a basis $I' = UI$, such that the differential equation for I' is in ε-factorised form.

Solving a differential equation in ε-form

Assume

1. The differential equation for I is in **ε-form**:

$$(d + A)\, I \;=\; 0, \qquad A \;=\; \varepsilon \sum_{j=1}^{N_L} C_j\, \omega_j .$$

2. All master integrals have a **Taylor expansion** in ε:

$$I_{\boldsymbol{\nu}_i}(\varepsilon, x) \;\;=\;\; \sum_{j=0}^{\infty} I_{\boldsymbol{\nu}_i}^{(j)}(x) \cdot \varepsilon^j .$$

3. We know suitable **boundary values** for all master integrals.

Solving a differential equation in ε-form

We plug the Taylor expansion into the differential equation

$$\left(d + \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k \right) \left(\sum_{j=0}^{\infty} I^{(j)}(x) \cdot \varepsilon^j \right) = 0,$$

and compare term-by-term in the ε-expansion.

We obtain

$$\begin{aligned} dI^{(0)}(x) &= 0, \\ dI^{(j)}(x) &= -\sum_{k=1}^{N_L} \omega_k \, C_k \, I^{(j-1)}(x), \quad j \geq 1. \end{aligned}$$

Chen's iterated integrals

Definition

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j .$$

The **iterated integral** is defined by

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k) .$$

Multiple polylogarithms

A simple case:

$$\omega^{\mathrm{mpl}}(z_j) \quad = \quad \frac{d\lambda}{\lambda - z_j}.$$

Definition (Multiple polylogarithms)

$$G(z_1,\ldots,z_k;\lambda) \quad = \quad \int_0^\lambda \frac{d\lambda_1}{\lambda_1 - z_1} \int_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \ldots \int_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}, \qquad z_k \neq 0$$

The method of differential equations

Example

One integral I in one variable x with **boundary condition** $I(0) = 1$. Consider the differential equation

$$(d + A)\, I \;=\; 0, \qquad A \;=\; -\varepsilon\, \frac{dx}{x-1}.$$

Then

$$I(x) \quad = \quad 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + \ldots$$

Take-home messages

- Higher-order calculations are needed for **precision**.
- **NNLO and beyond** is an active field of research.
- Available computing resources limit which calculation can be done: We need more **efficient algorithms**.
- Techniques discussed in this lecture:
  - Multiple polylogarithms
  - Integration-by-parts
  - The method of differential equations
- There are **plenty of opportunities for young people**!