Standard Model Precision Physics

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Section 4

Next-to-leading order calculations

Motivation for higher order corrections

- For precision physics a **leading-order calculation** is **not sufficient**.
  In perturbation theory: Better precision is reached by including higher orders.
- In this lecture we look at the **next-to-leading order** (NLO).

The master formula for the calculation of observables

$$\langle O_n{}^{\mathrm{LO}} \rangle = \sum_{a,b} \int dx_1\, f_a(x_1) \int dx_2\, f_b(x_2)\, \frac{1}{2K(\hat{s})}\, \frac{1}{n_a^{\mathrm{spin}}\, n_b^{\mathrm{spin}}\, n_a^{\mathrm{colour}}\, n_b^{\mathrm{colour}}}$$
$$\times \Big\{ \int d\phi_n O(p_1,\ldots,p_n) \Big[\underbrace{\left|\mathcal{A}_{n+2}^{(0)}\right|^2}_{\text{Born}} \Big] \Big\}$$

The master formula for the calculation of observables

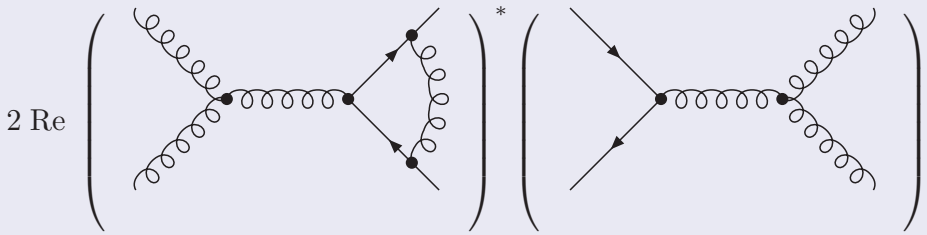
$$\langle O_n^{\mathrm{NLO}} \rangle = \sum_{a,b} \int dx_1\, f_a(x_1) \int dx_2\, f_b(x_2)\, \frac{1}{2K(\hat{s})}\, \frac{1}{n_a^{\mathrm{spin}}\, n_b^{\mathrm{spin}}\, n_a^{\mathrm{colour}}\, n_b^{\mathrm{colour}}}$$

$$\times \Bigg\{ \int d\phi_n O(p_1,\ldots,p_n) \Bigg[\underbrace{\left|\mathcal{A}^{(0)}_{n+2}\right|^2}_{\text{Born}} + \underbrace{\mathbf{2}\ \mathrm{Re}\ \mathcal{A}^{(0)}_{\mathbf{n+2}}{}^{*}\ \mathcal{A}^{(1)}_{\mathbf{n+2}}}_{\text{virtual}} \Bigg]$$

$$+ \int d\phi_{n+1} O(p_1,\ldots,p_n,p_{n+1}) \underbrace{\left|\mathcal{A}^{(0)}_{\mathbf{n+3}}\right|^2}_{\text{real}} \Bigg\}$$

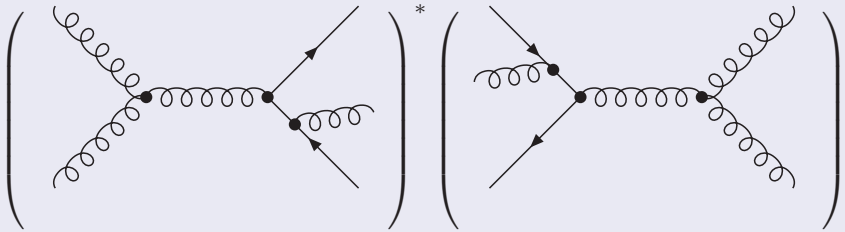
Next-to-leading order calculations

- *Virtual corrections*



We need to know the one-loop amplitude.

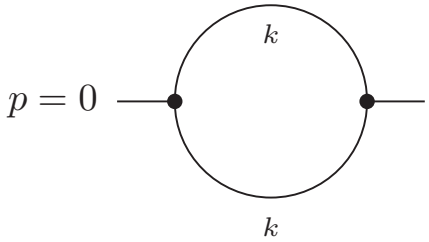
- *Real corrections*



Real part contributes whenever the additional parton is not resolved.

Virtual corrections

Loop integrals



Loop integrals in four space-time dimensions can be divergent!

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = \frac{1}{(4\pi)^2} \int_0^\infty dk^2 \frac{1}{k^2} = \frac{1}{(4\pi)^2} \int_0^\infty \frac{dx}{x}$$

This integral diverges at

- $k^2 \to \infty$ (**UV-divergence**) and at
- $k^2 \to 0$ (**IR-divergence**).

Regularisation

- Cut-off regularisation:

$$\frac{1}{(4\pi)^2}\int_0^\infty \frac{dx}{x} \quad \to \quad \frac{1}{(4\pi)^2}\int_\lambda^\Lambda \frac{dx}{x} = \frac{1}{(4\pi)^2}\left[\ln\Lambda - \ln\lambda\right].$$

- Pauli-Villars regularisation for ultraviolet divergences:

$$\frac{1}{k^2} \quad \to \quad \frac{1}{k^2} - \frac{1}{k^2 - M^2}.$$

- Mass regularisation for infrared divergences:

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2} \quad \to \quad \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)^2} = \frac{1}{(4\pi)^2}\int_0^\infty dk^2 \frac{k^2}{(k^2-m^2)^2}.$$

- Use **dimensional regularisation** to regulate UV- and IR-divergences. We set $D = 4 - 2\varepsilon$ and call ε the dimensional regularisation parameter.

Euler's gamma function

- Concept closely related to: Consider a function $f(z)$, which is defined for any positive integer $n \in \mathbb{N}$ by the factorial function

$$f(n) \quad = \quad n! \quad = \quad 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n.$$

We would like to define $f(z)$ for any value $z \in \mathbb{C}$ (except for a countable set of isolated points, where $f(z)$ is allowed to have poles).

Answer is well known and given by **Euler's gamma function**

$$f(z) \quad = \quad \Gamma(z+1).$$

Dimensional regularisation

- **Divergences are transformed into poles in ε** and for $\varepsilon \neq 0$ the integral is well-defined.
- A Feynman integral I has a **Laurent expansion** in ε:

$$I = \sum_{j=j_{\min}}^{\infty} \varepsilon^j I^{(j)},$$

  where $I^{(j)}$ denotes the coefficient of ε^j.
- We need the first few terms of this Laurent series.
  For NLO calculations we have $j_{\min} = -2$ and we are interested in

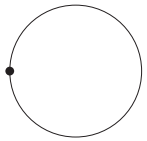
$$I = \frac{1}{\varepsilon^2} I^{(-2)} + \frac{1}{\varepsilon} I^{(-1)} + I^{(0)} + O(\varepsilon).$$

- Note: We would like to get the $I^{(j)}$'s, not necessarily I itself. There are situations where a closed form expression for I is readily obtained, but the Laurent expansion in ε is not immediate.

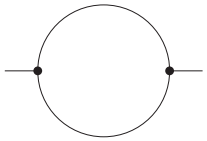
One-loop integrals

At one-loop we only need the following scalar integrals (in various kinematic configurations):

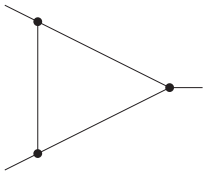
A_0 =



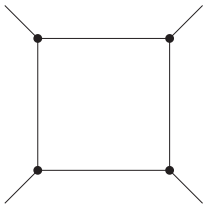
B_0 =



C_0 =



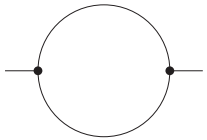
D_0 =



Absence of higher-point functions

- Even for one-loop scattering amplitudes with $n > 4$ external particles **we don't need the one-loop n-point integrals!**
- If all integrals would be finite, this follows from the dimensionality of space-time: There can be only four independent four-vectors in four space-time dimensions.
- Within dimensional regularisation the extra dimensions act effectively as one extra dimension, and there can be pentagon integrals. However, they only appear at order $O(\varepsilon)$.

Example

$$B_0\left(p^2, m_1^2 = 0, m_2^2 = 0\right) \quad = \quad$$ 

$$= \quad \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + 2 - \ln\left(\frac{-p^2}{\mu^2}\right) + O(\varepsilon).$$

$\gamma_E \approx 0.57721566$ Euler's constant,

μ arbitrary scale

All relevant one-loop integrals are known.

Transcendental functions

Up to $O(\varepsilon^0)$ all one-loop amplitudes can be expressed as a sum of algebraic functions of the kinematic variables times two transcendental functions, whose arguments are again algebraic functions of the kinematic variables.

The two transcendental functions are the **logarithm** and the **dilogarithm**:

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Calculation of one-loop amplitudes

The unitarity/cut/OPP method

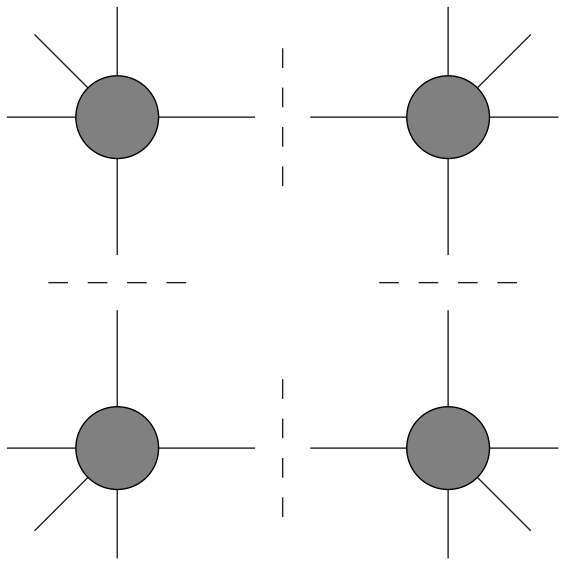
- Scalar integrals are known, need only the coefficients of these integrals
- Coefficients can be obtained by calculating tree-like objects

Cut techniques

Scalar integrals are known, **need only the coefficients** in front **and the rational part** R_n:

$$A_n^{(1)} = \sum_{i,j,k,l} c_{ijkl} I_{ijkl}^{\text{Box}} + \sum_{i,j,k} c_{ijk} I_{ijk}^{\text{Triangle}} + \sum_{i,j} c_{ij} I_{ij}^{\text{Bubble}} + \sum_{i} c_i I_i^{\text{Tadpole}} + R_n$$

- Box coefficients from quadruple cuts.
- Triangle coefficients from triple cuts, after box contribution has been subtracted out.
- Bubble coefficients from double cuts, after box and triangle have been subtracted out.
- Tadpole coefficients from single cuts, after box, triangle and bubble have been subtracted out.
- Rational part from cuts in D dimensions.



The OPP-method (Ossola, Papadopoulos, Pittau)

Consider the integrand of a primitive (cyclic-ordered) massless one-loop amplitude

$$A_n^{(1)} = \int \frac{d^D k}{(2\pi)^D} \frac{P(k)}{\prod_{j=1}^{n} q_j^2}, \qquad P(k) = P\left(k^{(4)}\right) + \tilde{P}\left(k^{(4)}, k^{(-2\epsilon)}\right).$$

The OPP method consists in writing

$$\begin{aligned} P\left(k^{(4)}\right) = & \sum_{i_1<i_2<i_3<i_4} \left[c_{i_1 i_2 i_3 i_4} + \tilde{c}_{i_1 i_2 i_3 i_4}\left(k^{(4)}\right)\right] \prod_{i\notin\{i_1,i_2,i_3,i_4\}} \left(q_i^{(4)}\right)^2 \\ & + \sum_{i_1<i_2<i_3} \left[c_{i_1 i_2 i_3} + \tilde{c}_{i_1 i_2 i_3}\left(k^{(4)}\right)\right] \prod_{i\notin\{i_1,i_2,i_3\}} \left(q_i^{(4)}\right)^2 \\ & + \sum_{i_1<i_2} \left[c_{i_1 i_2} + \tilde{c}_{i_1 i_2}\left(k^{(4)}\right)\right] \prod_{i\notin\{i_1,i_2\}} \left(q_i^{(4)}\right)^2. \end{aligned}$$

- Computational cost of $P(k^{(4)})$ is **tree-like**.
- The c's are the **sought-after coefficients** of the scalar integrals.
- The $\tilde{c}$'s integrate to zero.
- Rational terms:
  - Type 1 from $(q_i^{(4)})^2/q_i^2$
  - Type 2 from $\tilde{P}$.

Divergences

Poles in the dimensional regularisation parameter

- The poles in ε reflect the original divergences in the integrals.
- We have to get rid of these poles, before we can take the limit $\varepsilon \to 0$.
  - **Ultraviolet divergences**: Renormalisation
  - **Infrared divergences**: Cancellation with the real part

Renormalisation

Renormalisation

Parameters appearing in the Lagrangian are not observed quantities, but "bare" quantities.
UV-divergences are **absorbed into** universal **renormalisation constants**:

$$\psi_{\text{bare}} = \sqrt{Z_2}\psi_r, \quad m_{\text{bare}} = Z_m m_r, \quad g_{\text{bare}} = Z_g \cdot \mu^{\varepsilon} \cdot g_{\text{renorm}}, \quad \dots$$

- The renormalisation constants **must** absorb the poles from UV divergences.
- They **may** absorb additional terms.
- A **renormalisation scheme** is defined by the specification which additional terms they absorb.

Renormalisation schemes

- **Modified minimal subtraction scheme** ($\overline{\text{MS}}$-scheme): Absorbs always the combination

$$\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi$$

into the renormalisation constants.

- **On-shell scheme**: Define the renormalisation constants by conditions at a scale where the particles are on-shell $p^2 = m_r^2$.

Example: For a fermion define Z_2 and Z_m by the conditions that the propagator has a pole at m_r with residue 1.

What is the top mass?

- The **$\overline{\text{MS}}$-mass** $m_t^{\overline{\text{MS}}}(\mu)$, defined in the $\overline{\text{MS}}$-scheme. This is a short-distance mass.
- The **pole mass** m_t^{pole}, defined in the on-shell scheme. As the top quark is not a stable particle, it receives sizeable non-perturbative corrections. It is not a short-distance mass.
- The **MSR-mass** $m_t^{\text{MSR}}(\mu, R)$ (read: $m_t^{\overline{\text{MS}}}$ substituted by R) is a two-scale generalisation with a UV-scale μ and an IR-scale R, such that

$$\begin{aligned} m_t^{\text{MSR}}\left(\mu, R = m^{\overline{\text{MS}}}\right) &= m_t^{\overline{\text{MS}}}(\mu), \\ m_t^{\text{MSR}}\left(\mu, R = O(1\,\text{GeV})\right) &= \text{“Monte Carlo mass”}, \\ m_t^{\text{MSR}}\left(\mu, R = 0\right) &= m_t^{\text{pole}}. \end{aligned}$$

What is the Weinberg angle?

- $\overline{\text{MS}}$**-scheme**:

$$\sin^2\theta_W^{\overline{\text{MS}}}(\mu) \quad = \quad \frac{g'^2(\mu)}{g^2(\mu)+g'^2(\mu)}$$

- **On-shell scheme** (with pole masses m_W, m_Z):

$$\sin^2\theta_W^{\text{on-shell}} \quad = \quad 1-\frac{m_W^2}{m_Z^2}$$

- **Effective mixing angle** for fermion f:

$$\sin^2\theta_f^{\text{eff}}(\mu) \quad = \quad \frac{1}{4\,|Q_f|}\left(1-\frac{g_f^V(\mu)}{g_f^A(\mu)}\right)$$

Numbers

Top mass:

$m_t^{\overline{\rm MS}}(m_t)$	$162.69 \pm 0.006 \text{GeV}$
$m_t^{\rm pole}$	$172.52 \pm 0.33 \text{GeV}$

Weinberg angle:

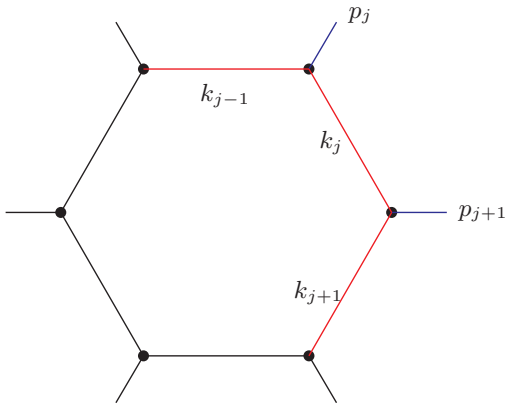
$\sin^2\theta_W^{\overline{\rm MS}}(m_Z)$	0.23122 ± 0.00006
$\sin^2\theta_W^{\rm on-shell}$	0.22342 ± 0.00009
$\sin^2\theta_l^{\rm eff}(m_Z)$	0.23154 ± 0.00006

Infrared divergences in one-loop amplitudes

The origin of infrared divergences

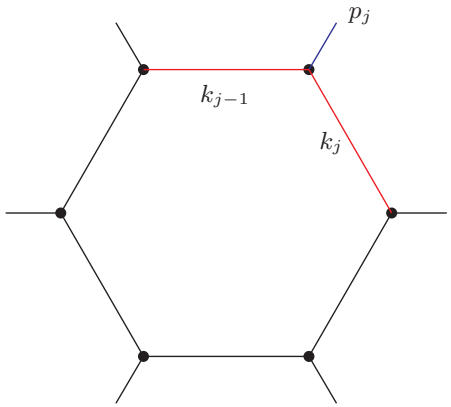
Soft singularities: 3 propagators on-shell

$$k_j \sim 0 \quad \text{and} \quad m_j = 0, \quad p_j^2 = m_{j-1}^2, \quad p_{j+1}^2 = m_{j+1}^2.$$



Collinear singularities: 2 propagators on-shell

$$k_{j-1} \sim x p_j, \quad \text{and} \quad p_j^2 = 0, \quad m_{j-1} = 0, \quad m_j = 0.$$



Infrared structure of one-loop amplitudes

General formula for the infrared poles of a one-loop amplitude:

$$\mathcal{A}_n^{(1)} = \mathbf{I}^{(1)}\,\mathcal{A}_n^{(0)} + \mathcal{F}_n^{(1)},$$

$\mathbf{I}^{(1)}$ contains all infrared poles, $\mathcal{F}_n^{(1)}$ is a finite remainder.

Infrared structure of one-loop amplitudes

General formula for the infrared poles of a one-loop amplitude:

$$\mathcal{A}_n^{(1)} = \mathbf{I}^{(1)}\,\mathcal{A}_n^{(0)} + \mathcal{F}_n^{(1)},$$

$$\text{QCD}: \qquad \mathbf{I}^{(1)} = \frac{\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\sum_i\left(\frac{1}{\varepsilon^2}+\frac{\gamma_i}{\mathbf{T}_i^2}\frac{1}{\varepsilon}\right)\sum_{j\neq i}\mathbf{T}_i\mathbf{T}_j\left(\frac{-2p_ip_j}{\mu^2}\right)^{-\varepsilon},$$

$$\mathbf{T}_q^2 = C_F, \quad \mathbf{T}_g^2 = C_A, \quad \gamma_q = \frac{3}{2}C_F, \quad \gamma_g = \frac{\beta_0}{2}.$$

$\mathbf{I}^{(1)}$ contains all infrared poles, $\mathcal{F}_n^{(1)}$ is a finite remainder.

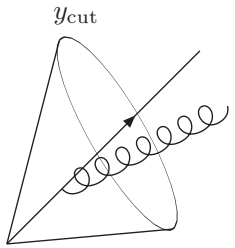
Colour charge operators:

$$\mathbf{T}_q\mathcal{A}(\ldots q_j\ldots) = \left(T_{ij}^a\right)\mathcal{A}(\ldots q_j\ldots),$$
$$\mathbf{T}_g\mathcal{A}\left(\ldots g^b\ldots\right) = \left(if^{cab}\right)\mathcal{A}\left(\ldots g^b\ldots\right).$$

Real corrections

The real correction

- Born matrix element $|\mathcal{A}_{n+1}^{(0)}|^2$ with $(n+1)$ partons.
- Contributes whenever the additional parton is below y_{cut} and is not resolved.
- In particular this is the case in the soft and collinear region.
- **Phase space integration** over soft and collinear region **diverges**.
- We discuss QCD, QED similar/simpler.



Factorisation in the soft region

In the soft limit:

$$\lim_{p_j \to 0} \mathcal{A}_{n+1}^{(0)} \quad = \quad g\mu^{\varepsilon} \varepsilon_{\mu}(p_j)\, \mathbf{J}^{\mu}\, \mathcal{A}_n^{(0)}, \qquad \mathbf{J}^{\mu} = \sum_{i=1}^{n} \mathbf{T}_i \frac{p_i^{\mu}}{p_i \cdot p_j}.$$

$\mathbf{J}^{\mu}$ is called the **eikonal current**.

Squaring the amplitude one finds

$$\lim_{p_j \to 0} \left| \mathcal{A}_{n+1}^{(0)} \right|^2 = -4\pi\alpha_s \mu^{2\varepsilon} \sum_{i=1}^{n} \sum_{k=1, k\neq i}^{n} \mathcal{A}_n^{(0)*} \mathbf{T}_i \cdot \mathbf{T}_k \left[\frac{p_i \cdot p_k}{(p_i \cdot p_j)(p_j \cdot p_k)} \right] \mathcal{A}_n^{(0)}.$$

This is not a complete factorisation, the colour charge operators $\mathbf{T}_i \cdot \mathbf{T}_k$ lead to **colour correlations**.

The soft limit is independent of spins or helicities.

Factorisation in the collinear region

In the collinear limit we parametrise the momenta of the partons i and j as

$$p_i = zp + k_\perp - \frac{k_\perp^2}{z}\frac{n}{2pn}, \qquad p_j = (1-z)p - k_\perp - \frac{k_\perp^2}{1-z}\frac{n}{2pn}, \qquad n^2 = 2pk_\perp = 2nk_\perp = 0$$

Factorisation in the collinear limit:

$$\lim_{p_i||p_j} \mathcal{A}_{n+1}^{(0)} \quad = \quad g\mu^\varepsilon \sum_\lambda \text{Split}^\lambda(p_i,p_j)\, \mathbf{T}_{(ij)\to i+j}\, \mathcal{A}_n^{(0)}(\ldots,p_{(ij)}^\lambda,\ldots).$$

where the **sum is over all polarisations of the intermediate particle** (ij).

Colour charge operators: $\mathbf{T}_{q\to qg} = \mathbf{T}_q$, $\mathbf{T}_{g\to gg} = \mathbf{T}_g$ and

$$\mathbf{T}_{g\to q\bar{q}}\mathcal{A}\left(\ldots g^b \ldots\right) \quad = \quad \left(T_{ij}^b\right)\mathcal{A}\left(\ldots g^b \ldots\right).$$

Factorisation in the collinear region

Squaring the amplitude:

$$\lim_{p_i \| p_j} \left| \mathcal{A}_{n+1}^{(0)} \right|^2 = 4\pi\alpha_s \mu^{2\varepsilon} \sum_{\lambda,\lambda'} \mathcal{A}_n^{\lambda\,(0)^*} \mathbf{P}^{\lambda\lambda'}_{\mathbf{(ij)\to i+j}} \mathcal{A}_n^{\lambda'\,(0)}.$$

$\mathcal{A}_n^{\lambda\,(0)}$ is the amplitude with polarisation factor of particle (ij) removed.

$P^{\lambda\lambda'}_{(ij)\to i+j}$ is called the **Altarelli-Parisi splitting function**.

$$\begin{aligned}
P^{\lambda\lambda'}_{q\to qg} &= C_F \frac{2}{2p_i\cdot p_j} \not{p}' \left[\frac{2z}{1-z} + (1-\varepsilon)(1-z) \right], \\
P^{\lambda\lambda'}_{g\to gg} &= C_A \frac{2}{2p_i\cdot p_j} \left[-g^{\mu\nu} \left(\frac{2z}{1-z} + \frac{2(1-z)}{z} \right) - 4(1-\varepsilon) z(1-z) \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right], \\
P^{\lambda\lambda'}_{g\to q\bar{q}} &= T_R \frac{2}{2p_i\cdot p_j} \left[-g^{\mu\nu} + 4z(1-z) \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right].
\end{aligned}$$

This is not a complete factorisation, the sum over λ and λ' leads to **spin correlations**.
The collinear limit is independent of colour.

Summary on factorisation in soft and collinear limits

- In the soft limit, amplitudes factorise completely in spin space, but **colour correlations** remain.
- In the collinear limit, amplitudes factorise completely in colour space, but **spin correlations** remain.

How to handle correlations: Use colour decomposition and helicity amplitudes.

Cancellation of infrared divergences

The Kinoshita-Lee-Nauenberg theorem

- The **phase space integration** over the unresolved region **diverges**, need a regulator.
- **Unitarity requires** the **same regulator** as in the **virtual part**, therefore use dimensional regularisation.
- Nature ensures that the amplitudes have a nice behaviour in the soft and collinear limits, physicists have to ensure that also the observables have a nice behaviour in these limits:
  Restriction to **infrared-safe observables**.
- For infrared-safe observables infrared divergences cancel in the sum of real and virtual corrections.
  This is the Kinoshita-Lee-Nauenberg theorem: Any infrared-safe observable, summed over all states degenerate according to some resolution criteria, will be finite.

The cancellation of infrared divergences in practise

- The real contribution has $(n+1)$ particles in the final state.
  In four space-time dimensions, the phase space integral is a $3(n+1)-4 = 3n-1$ dimensional integral.

- In $D = 4-2\varepsilon$ **space-time dimensions**, the phase space integral is a

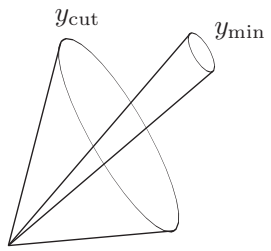
$$(D-1)(n+1)-D = 3n-1-2n\varepsilon$$

  dimensional integral.

- We want to perform the phase space integration by Monte Carlo techniques in four space-time dimensions.

Phase space slicing

Splits the integration of the real emission contribution into a region $y > y_{\min}$ and a region $y < y_{\min}$.
The former is free of singularities and the integration can be performed numerically there.
In the latter the matrix element is **approximated** and the integration over the one-parton phase space is performed analytically.



- Introduces an **error** of **order** $y_{\min}$.
- The first region gives a contribution of the form

$$a \ln^2 y_{\min} + b \ln y_{\min} + c$$

  The logarithms $\ln^2 y_{\min}$ and $\ln y_{\min}$ cancel against the contribution from the second region.
- **But**: Cancelation happens **only numerically**!

n-jettiness

In a modern variant one often uses n-jettiness instead of $y_{\min}$:

$$\tau_n = \sum_{k=1}^{n+r} \min_i \left(\frac{2q_i \cdot p_k}{Q_i} \right)$$

- The momenta $p_1, \ldots, p_{n+r}$ are the final state parton momenta
- The momenta $q_a, q_b, q_1, \ldots, q_n$ are the projections to a Born configuration (including two incoming momenta q_a, q_b).
- $Q_a, Q_b, Q_1, \ldots, Q_n$ are normalisation factors.

This has the advantage that for small τ_n we may use **soft-collinear effective theory**.

The subtraction method

The NLO cross section is rewritten as

$$\begin{aligned}\sigma^{NLO} &= \int_{n+1} d\sigma^R + \int_n d\sigma^V \\ &= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)\end{aligned}$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

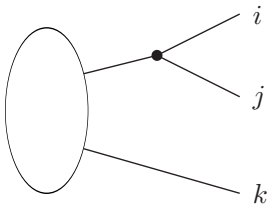
- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the **same pointwise singular behaviour in D dimensions** as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- **Analytic integrability in D dimensions** over the one-parton subspace leading to soft and collinear divergences.

The dipole subtraction terms

The approximation term $d\sigma^A$ is given as a sum over dipoles:

$$d\sigma^A = \sum_{\text{pairs } i,j} \sum_{k \neq i,j} \mathcal{D}_{ij,k}.$$



Each dipole contribution has the following form:

$$\mathcal{D}_{ij,k} = -\frac{1}{2p_i \cdot p_j} \mathcal{A}_n^{\lambda\,(0)\,*}(p_1,\ldots,\tilde{p}_{(ij)},\ldots,\tilde{p}_k,\ldots) \; \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k}^{\lambda\lambda'} \; \mathcal{A}_n^{\lambda'\,(0)}(p_1,\ldots,\tilde{p}_{(ij)},\ldots,\tilde{p}_k,\ldots)$$

- **Colour correlations** through $\mathbf{T}_k \cdot \mathbf{T}_{ij}$.
- **Spin correlations** through $V_{ij,k}^{\lambda\lambda'}$.

The dipoles have the correct soft **and** collinear limit.

Momentum mapping

The amplitudes $\mathcal{A}_n^{\lambda\,(0)}\left(\ldots,\tilde{p}_{(ij)},\ldots,\tilde{p}_k,\ldots\right)$ in the subtraction terms depend on n external momenta.

Need a mapping from a $(n+1)$-parton configuration to a n-parton configuration.

The mapping has to respect:

- **Momentum conservation**
- **On-shell conditions**

$$\begin{aligned}\tilde{p}_{ij} &= p_i + p_j - \frac{y}{1-y}p_k, \qquad y = \frac{s_{ij}}{s_{ijk}},\\ \tilde{p}_k &= \frac{1}{1-y}p_k.\end{aligned}$$

Summary on the subtraction method

In electron-positron annihilation we have:

$$\sigma^{NLO} = \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

With hadrons in the initial state:

$$\sigma^{NLO} = \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + d\sigma^C + \int_1 d\sigma^A\right)$$

$d\sigma^C$ subtracts **initial state collinear singularities**.

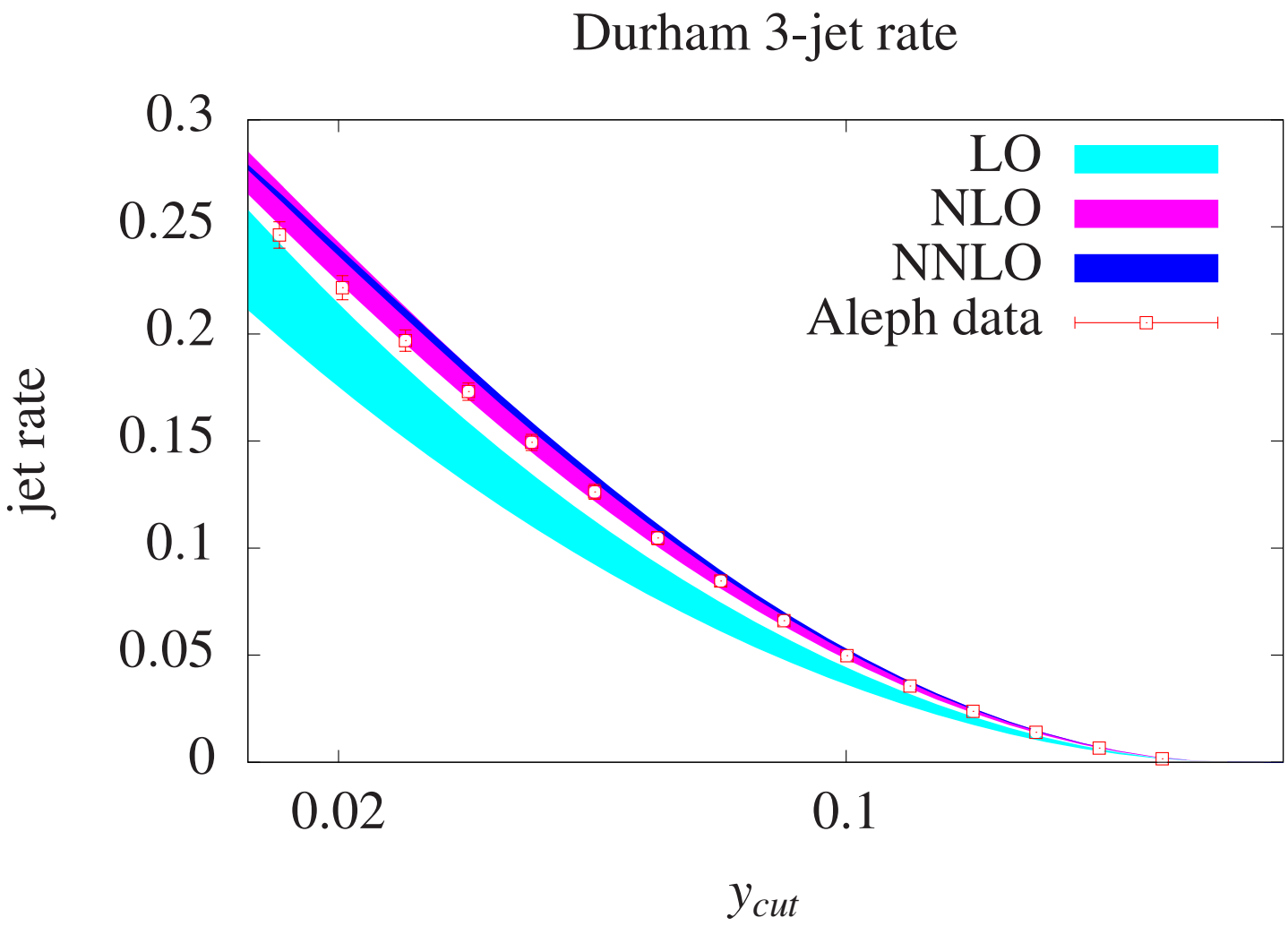
Take-home messages

- NLO calculations involve virtual corrections and real corrections.
- All one-loop integrals known, need only the **coefficients** of the one-loop integrals and the **rational terms**.
- **Cancellation of the infrared divergences** between the virtual corrections and the real corrections.

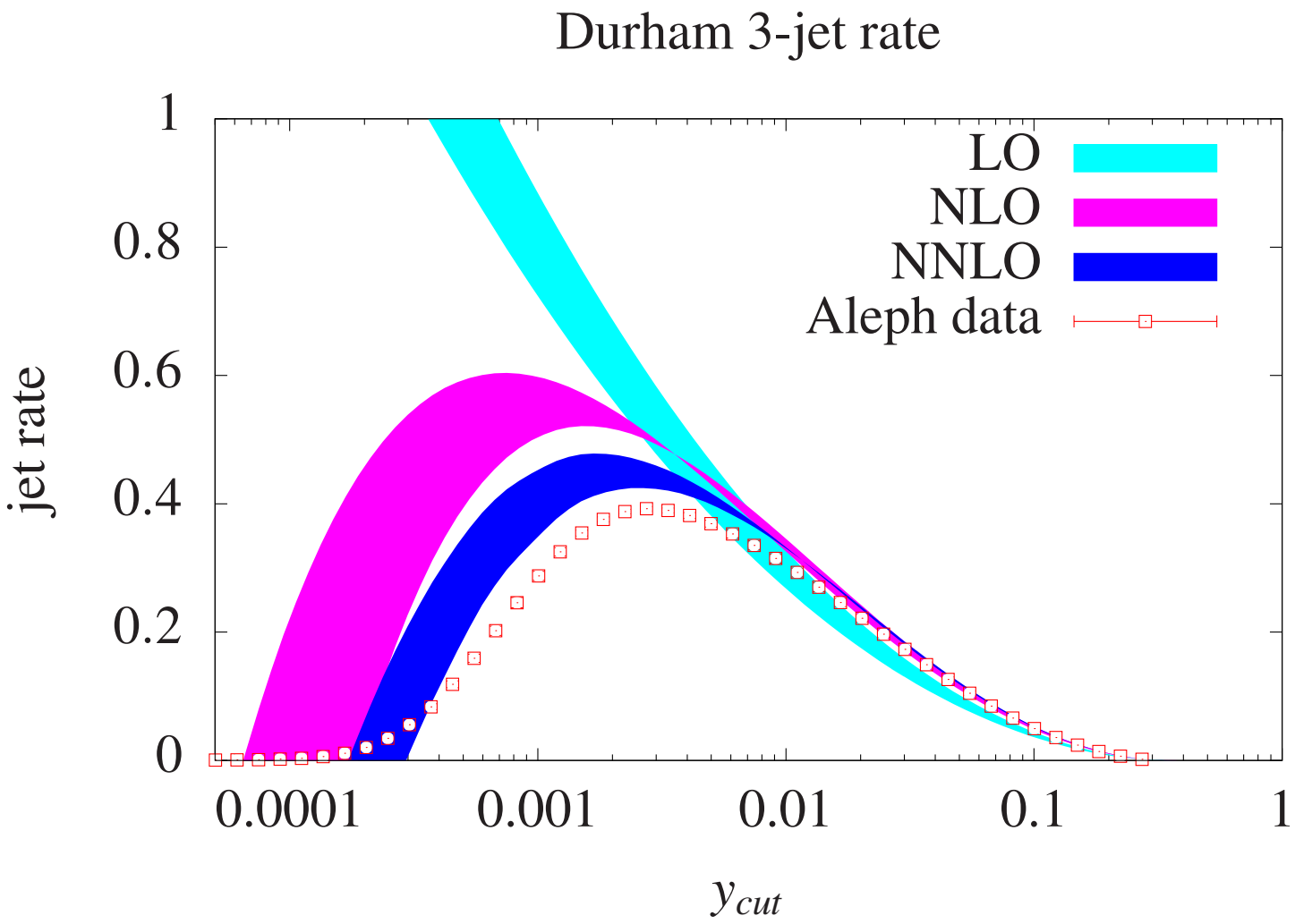
Section 5

Back-up slides

Results for the three-jet rate in electron-positron annihilation



Results for the three-jet rate in electron-positron annihilation



Resummation of large logarithms

Perturbative expansion of a typical observable:

$$\begin{aligned} O(y) &= \frac{\alpha_s}{2\pi} A(y) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(y) + O\left(\alpha_s^3\right), \\ & A(y) = a_2 \ln^2 y + a_1 \ln y + a_0, \\ & B(y) = b_4 \ln^4 y + b_3 \ln^3 y + b_2 \ln^2 y + b_1 \ln y + b_0. \end{aligned}$$

At order α^n we have terms of the form $\alpha_s^n \ln^{2n} y$, $\alpha_s^n \ln^{2n-1} y$, etc.

If y **is small**, we have

$$\alpha_s \ln^2 y \not\ll 1$$

This **spoils the perturbative expansion**.

Resum the large logarithms.