# The Gradient Flow in Perturbation Theory

Lars Georg
*Institute for Theoretical Particle Physics and Cosmology*

The Gradient Flow first arose in the context of perturbation theory in the late 2000s. It provides a bridge between lattice calculations and continuum perturbation theory, since it is both implementable on the lattice and in the continuum. The main idea is to extend the regular fields by an auxiliary parameter, the *flow-time* $t \geq 0$. The behavior of such a new *flowed field* with respect to the flow-time is governed by a linear differential equation called the *flow equation*. The original idea behind this flow equation was a gradient descent towards the stationary point of the classical action,

$$\partial_t \Phi(t,x) = -\frac{\delta S[\Phi]}{\delta \Phi(x)}\bigg|_{\Phi \to \Phi(t,x)}, \tag{1}$$

with the boundary condition that at the flowed field should recover the regular field at vanishing flow-time: $\Phi(t=0,x) = \Phi(x)$.

This choice of flow equation implies that the Feynman rules of the flowed fields acquire an additional exponential damping factor $e^{-tp^2}$, which suppresses UV-modes. In position space, this corresponds to a Gaussian smearing with a smearing radius $R_s(t) \sim \sqrt{t}$. As a consequence, composite operators built from flowed fields, $\tilde{\mathcal{O}}_n(t)$, do not mix under renormalization, unlike their unflowed counterparts $\mathcal{O}_n$. They can instead be renormalized multiplicatively, using only the parameter and flowed field renormalization constants. This makes it very useful to express an observable $R$ in terms of these finite flowed operators

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \left\langle \tilde{\mathcal{O}}_n(t) \right\rangle . \tag{2}$$

The matrix elements are typically computed using lattice simulations, where the introduction of the flow-time ensures the existence of the continuum limit. Our task on the perturbative side is to calculate the new flowed Wilson coefficients $\tilde{C}_n(t)$. We can calculate these coefficients by making the following observation: the flowed fields reduce to the regular fields at $t=0$, and therefore the flowed operators must also admit an expansion in terms of the regular operators as $t \to 0$

$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{=} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m . \tag{3}$$

This relation is known as the *short-flow-time expansion* (SFTX) and contains the central object of interest: the matching matrix $\zeta(t)$. From this expansion it follows immediately that $\tilde{C}_n(t) = C_k \left[\zeta^{-1}(t)\right]_{kn}$. Thus, the matching matrix $\zeta(t)$ provides the essential bridge between the "flowed world" and the regular theory.

The extraction of $\zeta$ is conceptually very similar to matching procedures in effective field theories (EFTs). We employ the *method of projectors*: one computes suitable Greens functions of the SFTX, applies appropriate derivatives with respect to the external scales, and finally sets these scales to zero. In this limit, the right-hand side of Eq. (3) reduces to its leading-order contributions, while on the left-hand side the $t$-dependent terms survive. By evaluating these matrix elements explicitly, one can determine the entries of the matching matrix.

Our actual calculations are carried out using a tool chain: first, we generate the Feynman diagrams with `qgraf`; next, we assign abstract Feynman rules and distribute the momenta using `tapir` and `exp`; finally, we perform the explicit computations, such as applying derivatives and taking traces, in `FORM`. At 2-loop, the resulting integrals are known analytically, at 3-loop, we calculate them numerically using `ftint`.

The underlying goal is to work towards a full flowed standard model and apply the Gradient Flow to EFTs like SMEFT or LEFT.