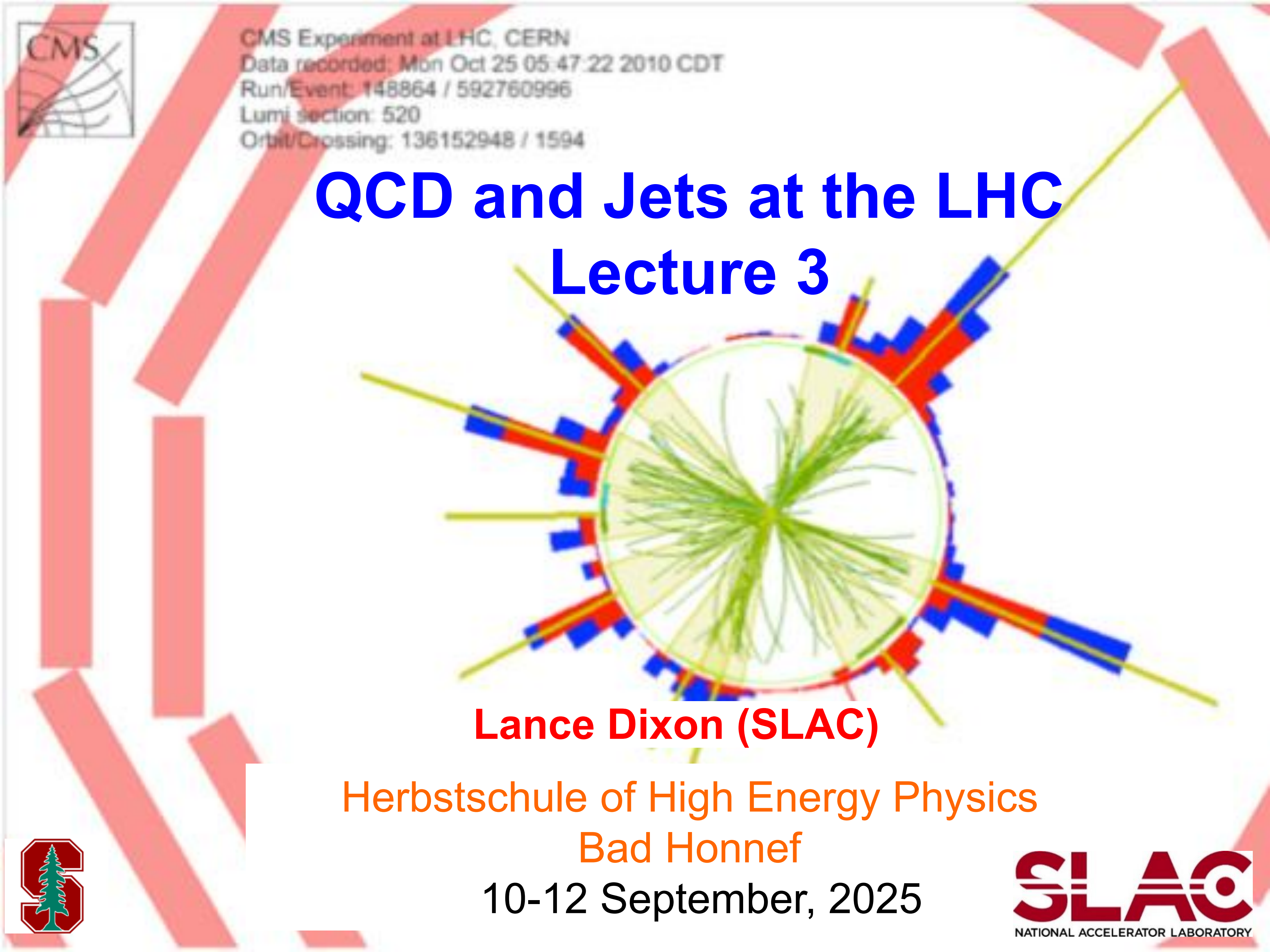
QCD and Jets at the LHC
Lecture 3



Lance Dixon (SLAC)

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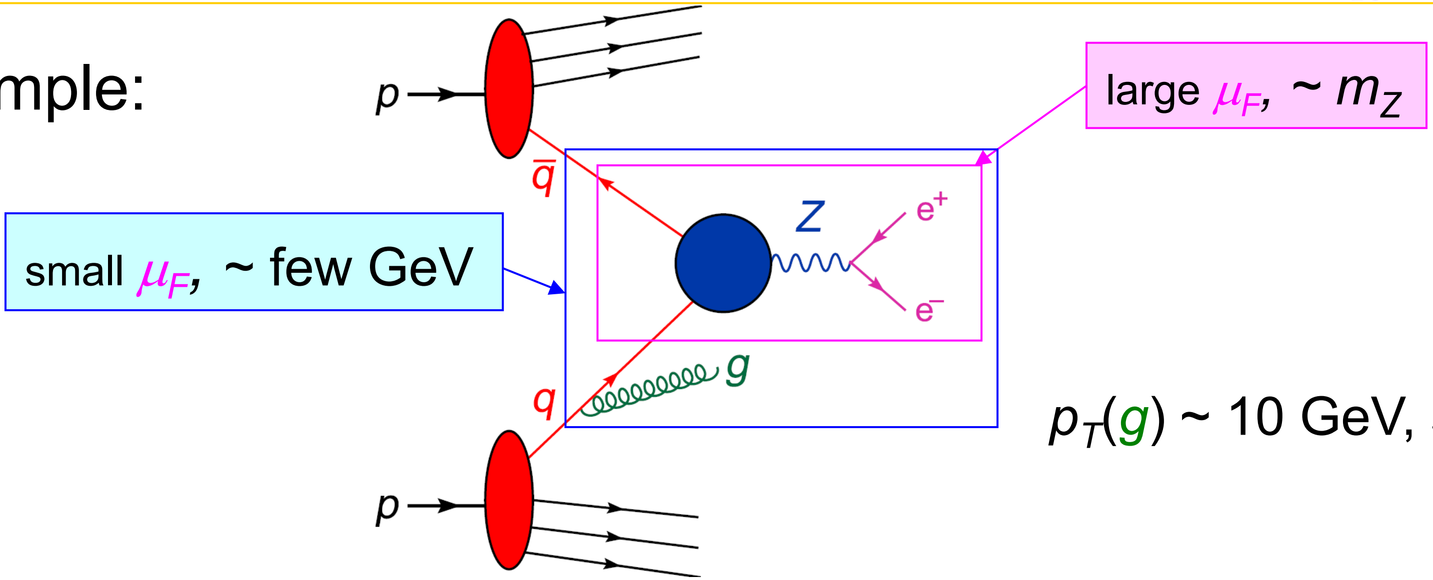
Bad Honnef
10-12 September, 2025

Parton evolution

Partons in the proton are not quite free: the parton distributions $f_a(x, \alpha_S, \mu_F)$ evolve as scale μ_F at which they are resolved varies

$$\sigma^{pp\to X}(s;\alpha_s,\mu_R,\mu_F) = \sum_{a,b}\int_0^1 dx_1 \int_0^1 dx_2\ f_a(x_1,\alpha_s,\mu_F)\, f_b(x_2,\alpha_s,\mu_F) \times \hat{\sigma}^{ab\to X}(sx_1x_2;\alpha_s,\mu_R,\mu_F)$$

Example:



large μ_F, ~ m_Z

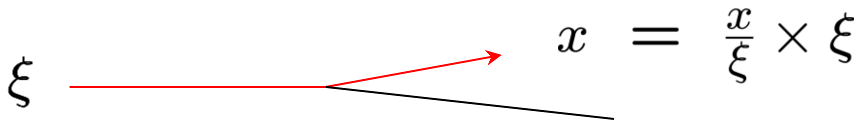
small μ_F, ~ few GeV

$p_T(g)$ ~ 10 GeV, say

Parton evolution (cont.)

- parton distributions are nonperturbative
- must be measured experimentally, e.g. in ep collisions at HERA (DESY)
- experimental data typically at much lower μ_F than 100-1000 GeV
- fortunately, evolution at $\mu_F \geq 2\ \mathrm{GeV}$ is perturbative
- DGLAP equation:

$$\mu^2 \frac{\partial}{\partial \mu^2} f_a(x,\mu) = \frac{\alpha_s(\mu)}{2\pi} \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}(x/\xi, \alpha_s(\mu))\, f_b(\xi,\mu)$$



$$x = \frac{x}{\xi} \times \xi$$

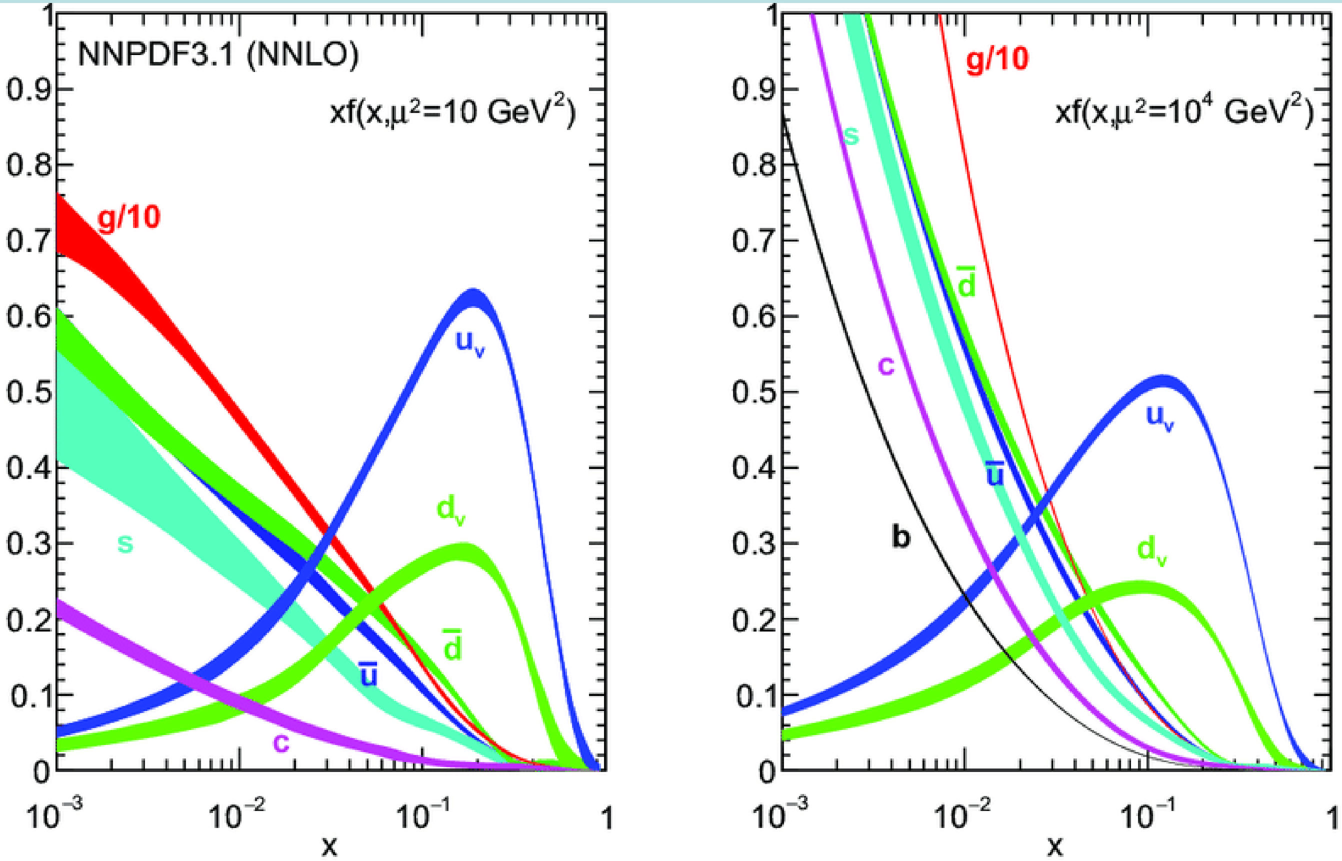
$$P_{ab}(x,\alpha_s) = P_{ab}^{(0)}(x) + \frac{\alpha_s}{2\pi} P_{ab}^{(1)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(2)}(x) + \cdots$$

LO (1974) NLO (1980) NNLO (2004) NNNLO (partial)

e.g. $P_{qq}^{(0)}(x) = C_F[\frac{(1+x^2)}{(1-x)_+} + \frac{3}{2}\delta(1-x)]$

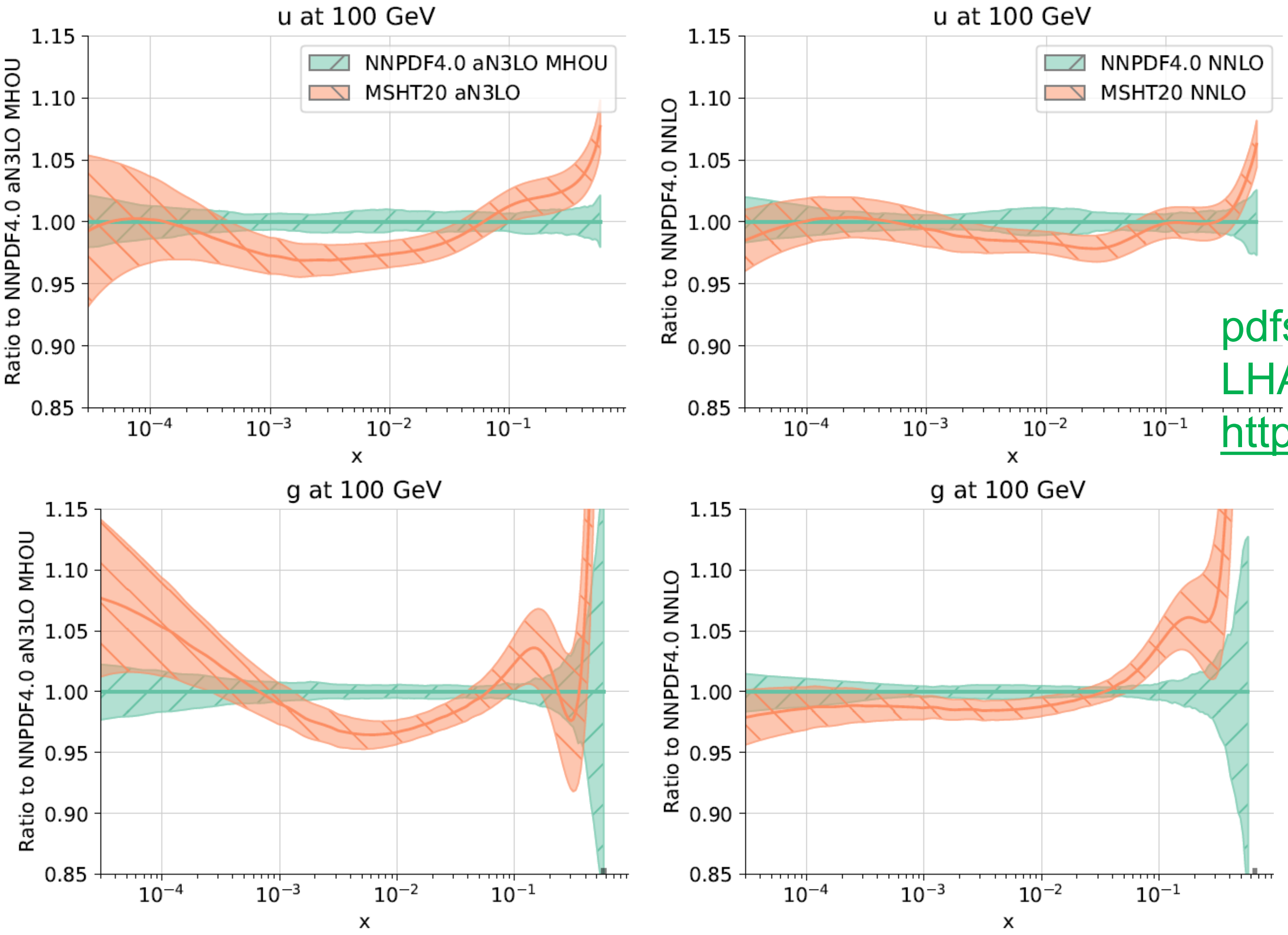
Precision PDFs

- From deep inelastic scattering experiments, especially HERA, other experiments, and now LHC data, know PDFs to few percent
- Essential input for all LHC predictions



Still few % differences between groups

NNPDF + MSHT, 2411.05373



pdfs available via
LHAPDF interface:
https://www.lhapdf.org/

Infrared safety

Infrared-safe observables *O*:

- Behave smoothly in soft limit as any parton momentum → 0

- Behave smoothly in collinear limit as any pair of partons → parallel (||)

$$O_n(\ldots, k_s, \ldots) \to O_{n-1}(\ldots, \cancel{k_s}, \ldots) \qquad k_s \to 0$$

$$O_n(\ldots, k_a, k_b, \ldots) \to O_{n-1}(\ldots, k_P, \ldots) \qquad k_a \,||\, k_b$$

- Cannot predict perturbatively infrared-unsafe quantities, such as:
  - number of partons (hadrons) in event
  - observables requiring no radiation in some region (rapidity gaps or overly strong isolation cuts)
  - p_T(*W*, *Z* or Higgs) precisely at $p_T = 0$

Infrared safety (cont.)

Examples of IR safe quantities at LHC:

- most kinematic distributions of "electroweak" objects, *W, Z, Higgs* (photons tricky because of collinear issues)
- jets, defined by cluster or (suitable) cone algorithm

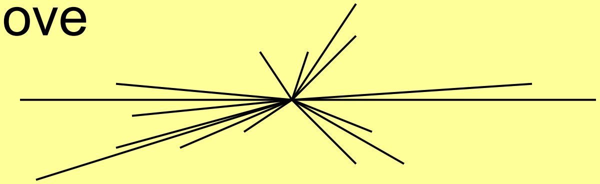
jet cluster algorithm $n = -2, 0, 2$ ←→ **anti-k$_T$, CA, k$_T$**

- Construct list of objects, starting with particles i, plus "the beam" b
- Define "distance" between objects, vanishing in soft/collinear limits:

$$d_{ij} = \min\{k_T^{(i)}, k_T^{(j)}\}^n \, [(\eta^{(i)} - \eta^{(j)})^2 + (\phi^{(i)} - \phi^{(j)})^2]/R^2 \qquad d_{ib} = [k_T^{(i)}]^n \qquad \eta \equiv -\ln\tan\frac{\theta}{2}$$

- If a d_{ij} is smallest, cluster together i and j.

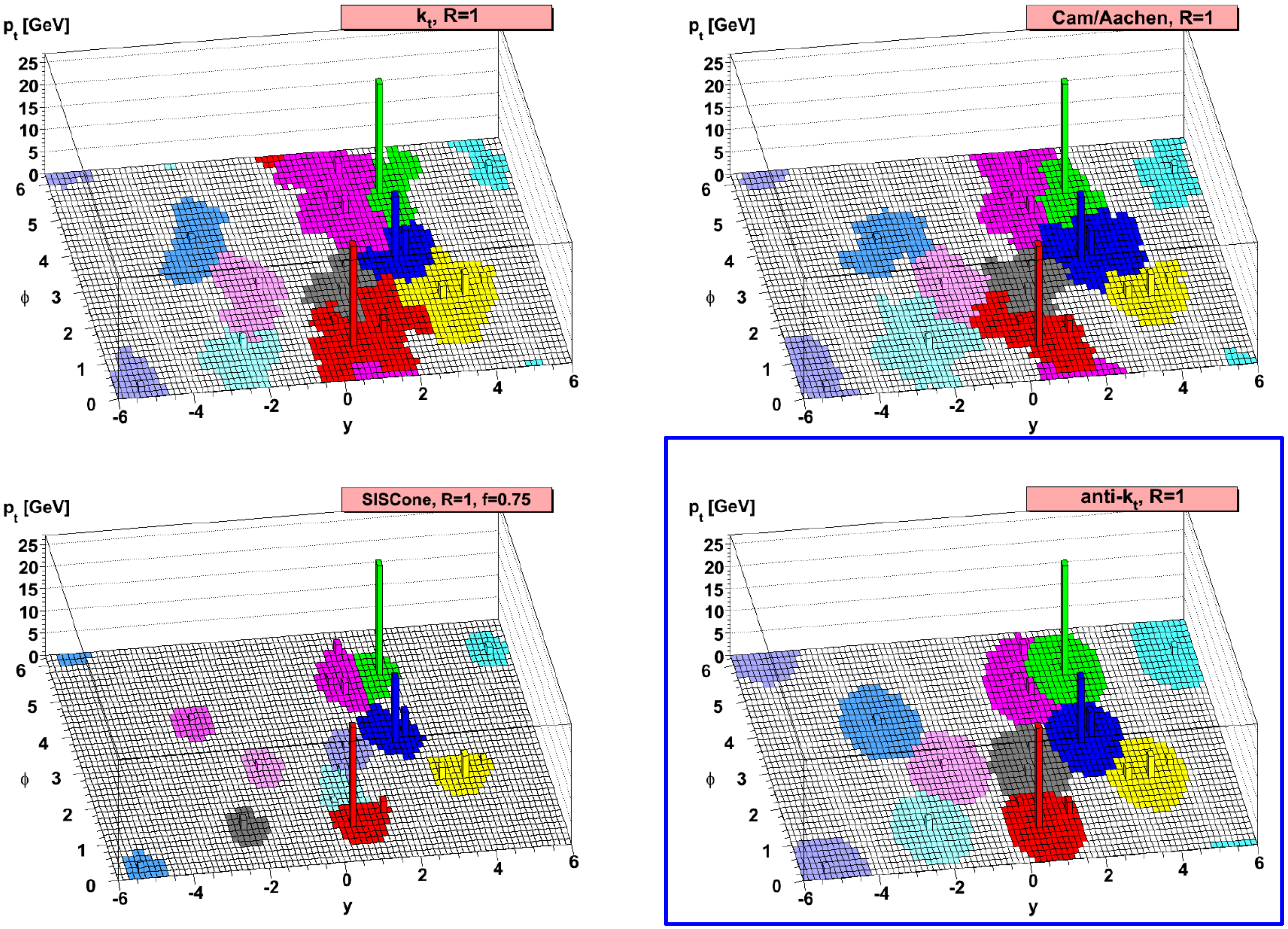
If a d_{ib} is smallest, declare i to be a jet and remove it from the list of particles



- Repeat until all objects are jets

Jets are an incredibly versatile, powerful concept at hadron colliders

“Catchment area” for soft particle depends heavily on jet algorithm



Cacciari, Salam, Soyez 0802.1189, anti-k_t

anti-k_t picks up soft radiation from underlying event more predictably

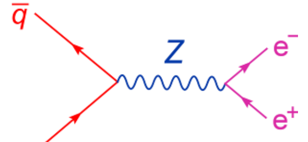
→ easier to correct for underlying event

→ anti-k_t widely adopted at LHC

Overall structure of higher-order QCD corrections

Example of *Z* production at hadron colliders

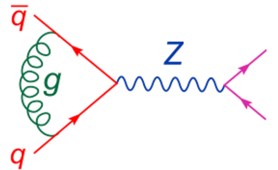
LO $\hat{\sigma}^{(0)}$



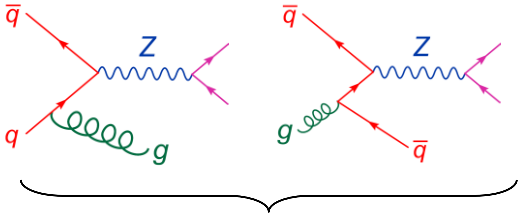
tree

convolute with pdfs
apply cuts

NLO $\hat{\sigma}^{(1)}$



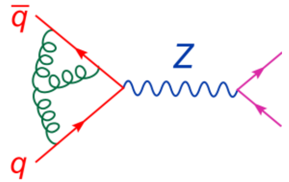
1 loop



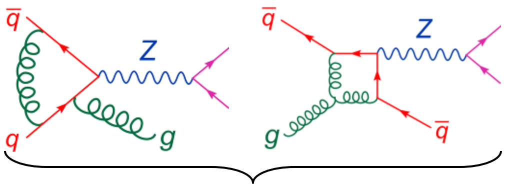
tree + 1 parton

dim. reg. $D = 4 - 2\epsilon$
first, cancel infrared divergences ($1/\epsilon^2$)
between virtual & real

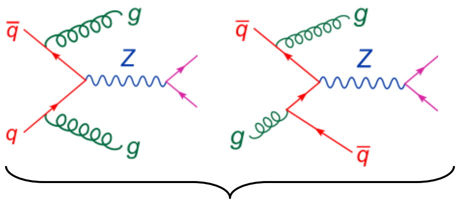
NNLO $\hat{\sigma}^{(2)}$



2 loops



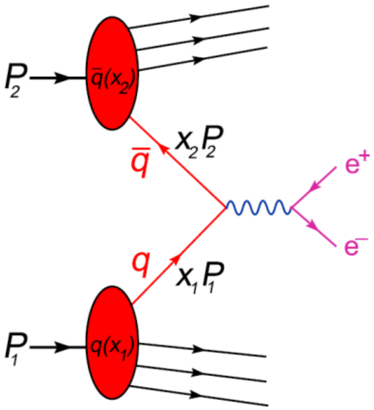
1 loop + 1 parton



tree + 2 partons

intricate ($1/\epsilon^4$) IR cancellations

The Drell-Yan process
(simplest hadron collider process)



LO partonic cross section:

$$\hat{s} = x_1 x_2 s = M^2_{e^+e^-}$$

$$\hat{\sigma}(q\bar{q} \to e^+e^-) = \frac{1}{2\hat{s}}\frac{1}{4N_c^2}\sum_{h,c}|\mathcal{A}_4|^2$$

$$= \frac{4\pi\alpha^2}{3}\frac{1}{N_c}Q_q^2$$

$$\frac{d\hat{\sigma}}{dM^2} = \frac{\sigma_0}{N_c}Q_q^2\delta(\hat{s}-M^2), \qquad \sigma_0 \equiv \frac{4\pi\alpha^2}{3M^2}$$

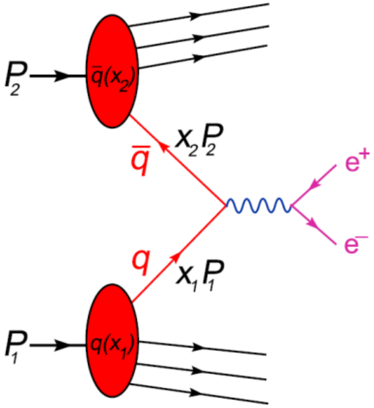
LO hadronic cross section:

$$\frac{d\sigma}{dM^2} = \int_0^1 dx_1 dx_2 \sum_q [q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)]\frac{d\hat{\sigma}}{dM^2}$$

$$= \frac{\sigma_0}{N_c}\int_0^1 dx_1 dx_2 \delta(x_1 x_2 s - M^2)\sum_q Q_q^2[q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)]$$

$$= \frac{\sigma_0\, s}{N_c}\int_0^1 dx_1 dx_2 \delta(x_1 x_2 - \tau)\sum_q Q_q^2[q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)], \qquad \tau \equiv \frac{M^2}{s}$$

Drell-Yan rapidity distribution



rapidity $$Y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$$

$$\exp(2Y) = \frac{E+p_z}{E-p_z} = \frac{P_2\cdot P_Z}{P_1\cdot P_Z} = \frac{\frac{1}{x_2}p_{\bar{q}}\cdot P_Z}{\frac{1}{x_1}p_q\cdot P_Z} = \frac{x_1}{x_2}$$

combined with mass measurement,

$$x_1 x_2 = \tau = \frac{M^2}{s}$$

double distribution $$\frac{d^2\sigma}{dM^2dY} = \frac{\sigma_0}{N_c\, s}\sum_q Q_q^2[q(x_1)\bar{q}(x_2)+\bar{q}(x_1)q(x_2)]$$

measures product of quark and antiquark distributions at

$$x_1 = \sqrt{\tau}e^{Y} \qquad x_2 = \sqrt{\tau}e^{-Y}$$

NLO QCD corrections to Drell-Yan production

$$\frac{d\sigma^{\rm NLO}}{dM^2} = \frac{\sigma_0}{N_c\, s}\int_0^1 dx_1 dx_2 dz\, \delta(x_1 x_2 z - \tau)\sum_q Q_q^2\Big[$$

$$q(x_1,\mu_F)\bar{q}(x_2,\mu_F)\Big(\delta(1-z) + \frac{\alpha_s(\mu_R)}{2\pi} C_F D_q(z,\mu_F)\Big)$$

$$+g(x_1,\mu_F)(q(x_2,\mu_F)+\bar{q}(x_2,\mu_F))\frac{\alpha_s(\mu_R)}{2\pi} T_R D_g(z,\mu_F)$$

$$+(x_1 \leftrightarrow x_2)\Big]$$

where

$$D_q(z,\mu_F) = 4(1+z^2)\left(\frac{\ln(1-z)+\ln(M/\mu_F)}{1-z}\right)_+ -2\frac{1+z^2}{1-z}\ln z + \delta(1-z)\left(\frac{2}{3}\pi^2 - 8\right)$$

singular distribution as $z \longrightarrow 1$

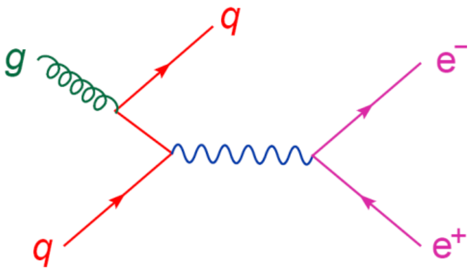
[See backup slides for more details of the calculation.]

QCD corrections to DY (cont.)

and

$$D_g(z,\mu_F) \;=\; (z^2+(1-z)^2)\left[\ln\frac{(1-z)^2}{z}+2\ln\frac{M}{\mu_F}\right]+\frac{1}{2}+3z-\frac{7}{2}z^2$$

comes from the $qg \to q\gamma^*$ subprocess:



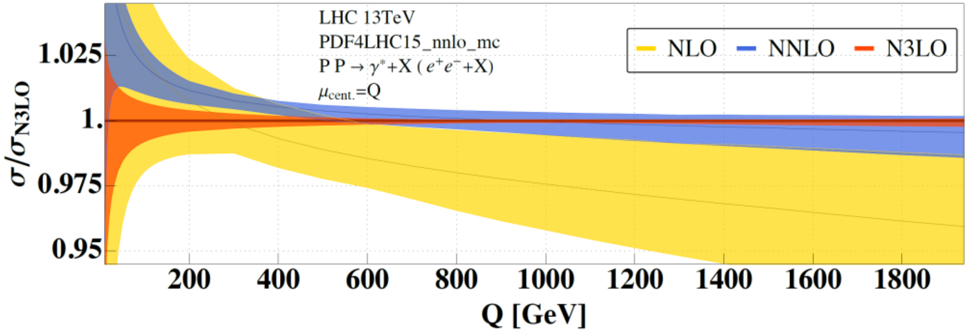
- Cross section related by crossing to $q\bar{q} \to g\gamma^*$

- Remove $g \to q\bar{q}$ collinear singularity in same way

- Note that there is no $\frac{1}{1-z}$ (soft gluon) singularity in this term, and no $\delta(1-z)$ virtual term.

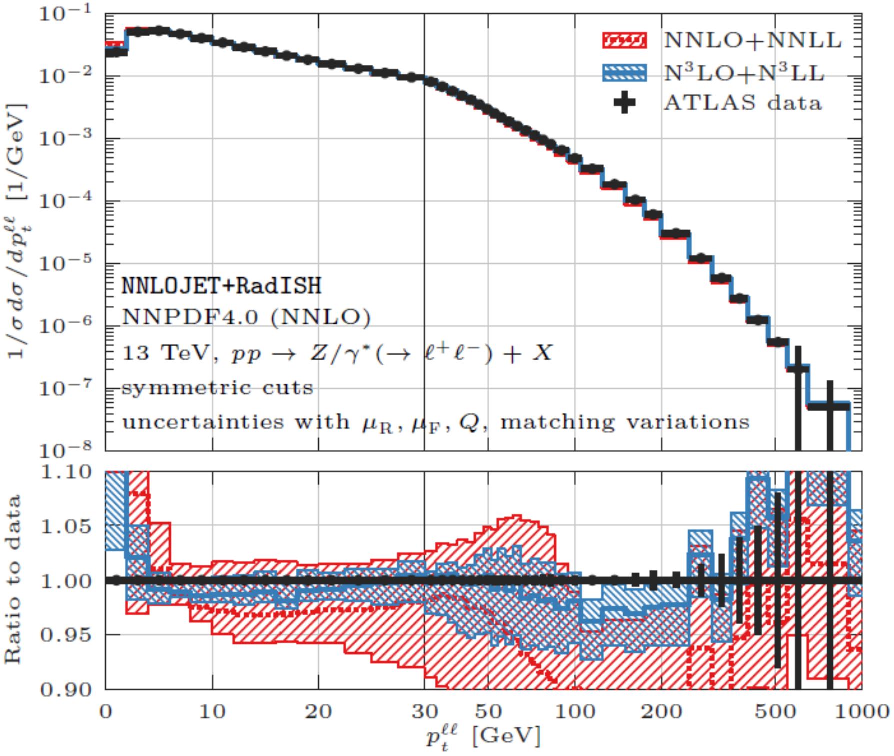
DY/Z now known at N3LO

Total cross section



Duhr, Dulat, Mistlberger, 2001.07717;
Baglio, Duhr, Mistlberger, Szafron,
n3loxs, 2209.06138

Differential distributions

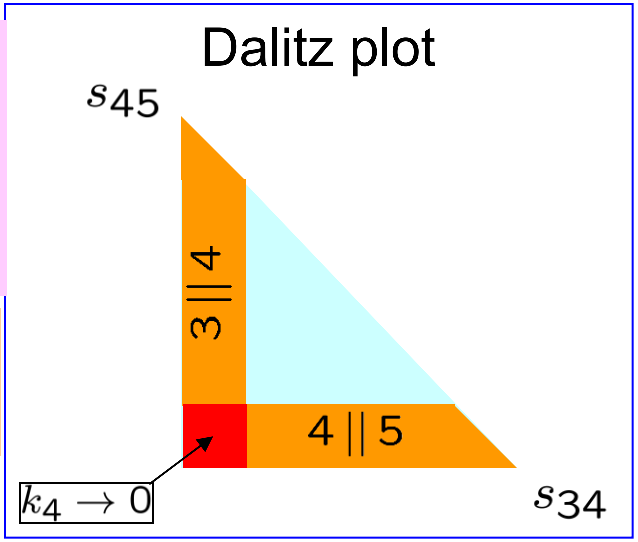


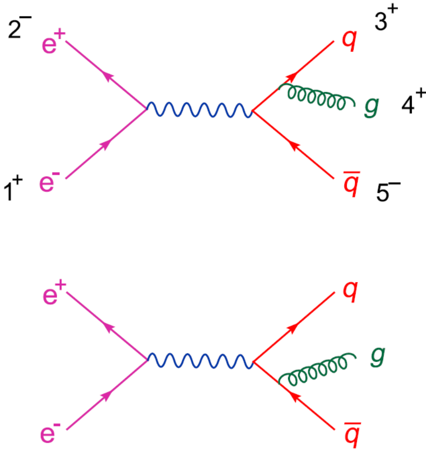
Chen, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli, 2203.01565

Real radiation in general case

Cannot perform the phase-space integral analytically in D=4-2ε, especially not for generic experimental cuts

Also can't do it numerically, because of $1/\varepsilon^2$ poles





2 solutions:

1. Slice out singular regions of phase-space, with (thin) width s_{min} Perform integral there approximately. Rest of integral done numerically. Check cancellation of s_{min} dependence.
2. Subtract a function that mimics the soft/collinear behavior of the radiative cross section, and which you can integrate (analytically). Integral of the difference can be done numerically.

Subtraction methods

for more complex, differential processes

$$\begin{aligned}
\sigma_n^{\mathrm{NLO}} &= \int d\sigma_n^{\mathrm{NLO}} = \int_n d\sigma^V + \int_{n+1;\epsilon} d\sigma^R \\
&= \int_n d\sigma^V + \int_{n+1;\epsilon} d\sigma^A + \int_{n+1;\epsilon=0} [d\sigma^R - d\sigma^A] \\
&= \int_n [d\sigma^V + \int_1 d\sigma^A]_{\epsilon=0} + \int_{n+1;\epsilon=0} [d\sigma^R - d\sigma^A]
\end{aligned}$$

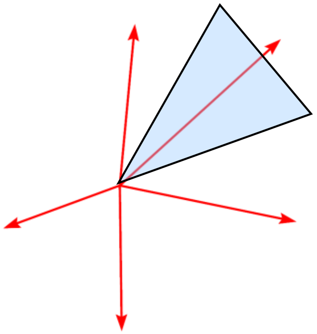
- Subtraction term $d\sigma^A$ should match $d\sigma^R$ pointwise on $(n+1)$ phase space
- Factorization of $d\sigma^A$ needed to allow integral to be split, combined with $d\sigma^V$

Two types of subtraction methods

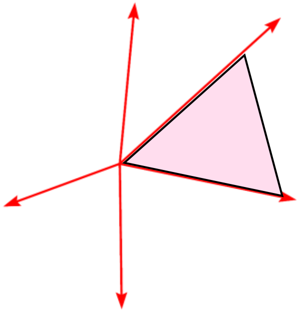
Long history of developments, including:

Ellis, Ross, Terrano (1980)
Frixione, Kunszt, Signer (1995)

- More recently, Lorentz-invariant subtraction terms built up for general processes in 2 different ways:



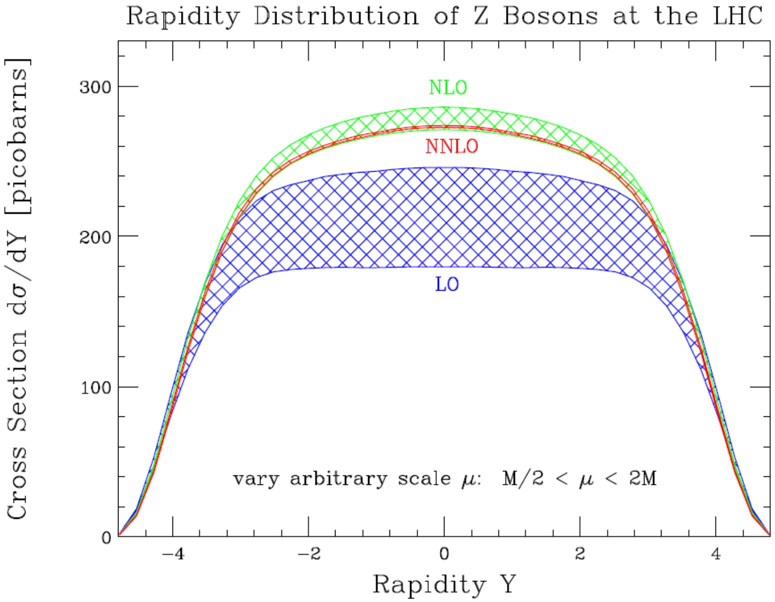
1) Start with collinear approximation,
add "half of soft behavior" on each side.
"Dipole" subtraction Catani, Seymour (1996)
[N.B.: not the dipole shower used in MC community]



2) Start with soft radiation pattern,
add "half of collinear behavior" on each side.
"Antenna" subtraction NLO Kosower (1997,2003),
NNLO Gehrmann,
Gehrmann-de-Ridder,
Glover, Heinrich, … 2005→current

Why are (N)NLO corrections so large?

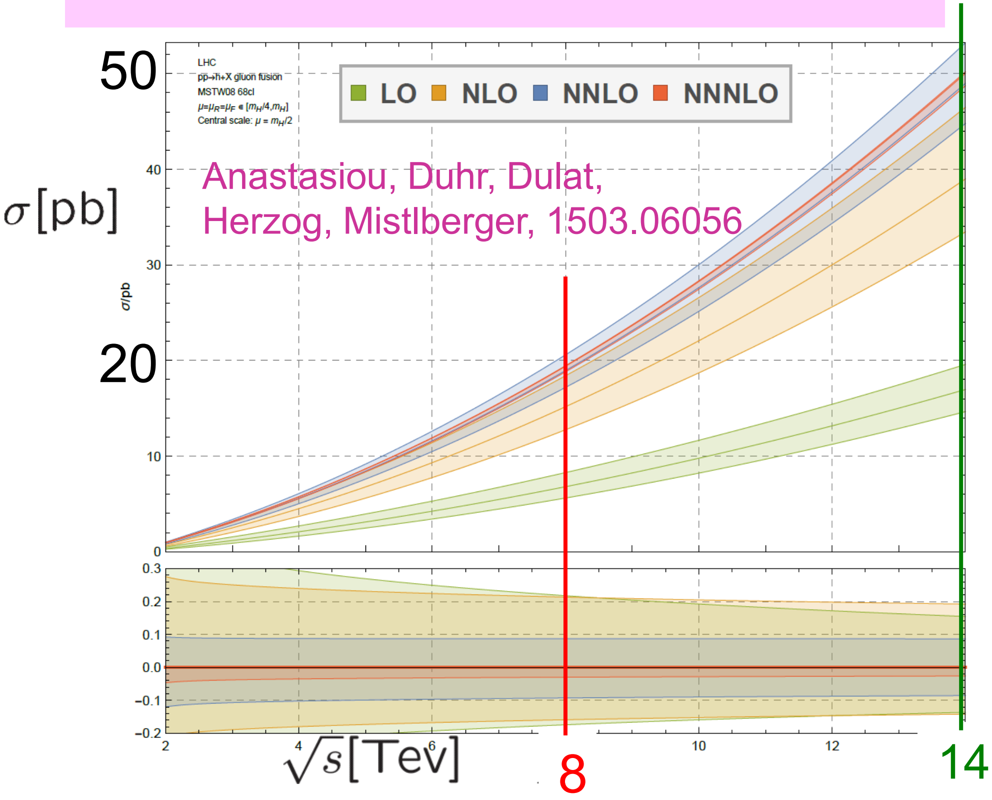
+ 30% typical for quark-initiated ($W, Z, \dots$)


Rapidity Distribution of Z Bosons at the LHC

This is much bigger than, e.g.

$$R_{e^+e^-} = 1 + \frac{\alpha_s}{\pi} \approx 1 + \frac{0.1}{\pi} \approx 1 + 0.03$$

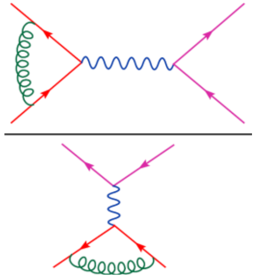
+ 100% for some gluon-initiated ($gg \to \text{Higgs} + X$)


Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056

Some answers (not all for all processes)

1. LO parton distribution fits not very reliable due to large theory uncertainties
2. New processes can open up at NLO. In W or Z production at LHC, $qg \to \gamma^* q$ opens up, and $g(x)$ is very large – but the qg correction is negative!
3. Large π^2 from analytic continuation from space-like region where pdfs are measured (DIS) to time-like region (Drell-Yan/W/Z):

2 Re



$$= 1 + \frac{\alpha_s}{\pi} C_F \left(-\frac{1}{\epsilon^2}\right) \text{Re}\left[\left(\frac{\mu^2}{-Q^2}\right)^\epsilon - \left(\frac{\mu^2}{+Q^2}\right)^\epsilon\right]$$

$$= 1 + \frac{\alpha_s}{\pi} C_F \left(-\frac{1}{\epsilon^2}\right) \text{Re}\left[\exp(i\pi\epsilon) - 1\right] = 1 + \frac{\alpha_s}{\pi} C_F \frac{\pi^2}{2}$$

4. Soft-gluon/Sudakov resummation

- A prevalent theme in QCD whenever one is at an edge of phase space
- Infrared-safe but sensitive to a second, smaller scale
- Same physics as in (high-energy) QED: $e^+e^- \to e^+e^-(\gamma)$
- What is prob. of no photon with $E > \Delta E$ **and** $\theta > \Delta\theta$?

$$P = 1 - \frac{\alpha}{\pi}\int_{\Delta E}\frac{dE}{E}\int_{\Delta\theta}\frac{d\theta}{\theta} + \cdots \quad = \quad 1 - \frac{\alpha}{\pi}\ln\Delta E\ln\Delta\theta + \cdots$$

$$= \exp\left(-\frac{\alpha}{\pi}\ln\Delta E\ln\Delta\theta\right) + \cdots$$

soft

collinear

leading double logarithms -- in contrast to single logs of renormalization group, DGLAP equations.

exponentiation because soft emissions are independent in QED

Hadron collider examples

$p_T(Z)$, $p_T(W)$ [latter needed for m_W measurement at hadron colliders]

Production of heavy states, like
- top quark at Tevatron or LHC
- even a light Higgs boson at the LHC, via $gg \rightarrow H$

Called threshold resummation or $\tau \rightarrow 1$ limit, where $\tau = M^2/s$.

Can be important for $\tau << 1$ though.
For $M = m_H = 125$ GeV at 14 TeV LHC, $\tau = 10^{-4}$!
Radiation is being suppressed because you are
running out of phase space – parton distributions are falling fast.

Threshold Resummation

Can see the first threshold log in the NLO corrections to Drell-Yan/*W/Z* production:

$$
\begin{aligned}
C_F D_q(z,\mu_F) \;&=\; 4C_F(1+z^2)\boxed{\left(\frac{\ln(1-z)+\ln(M/\mu_F)}{1-z}\right)_+} \\
&\quad -2\frac{1+z^2}{1-z}\ln z \;+\; \delta(1-z)\left(\frac{2}{3}\pi^2-8\right)
\end{aligned}
$$

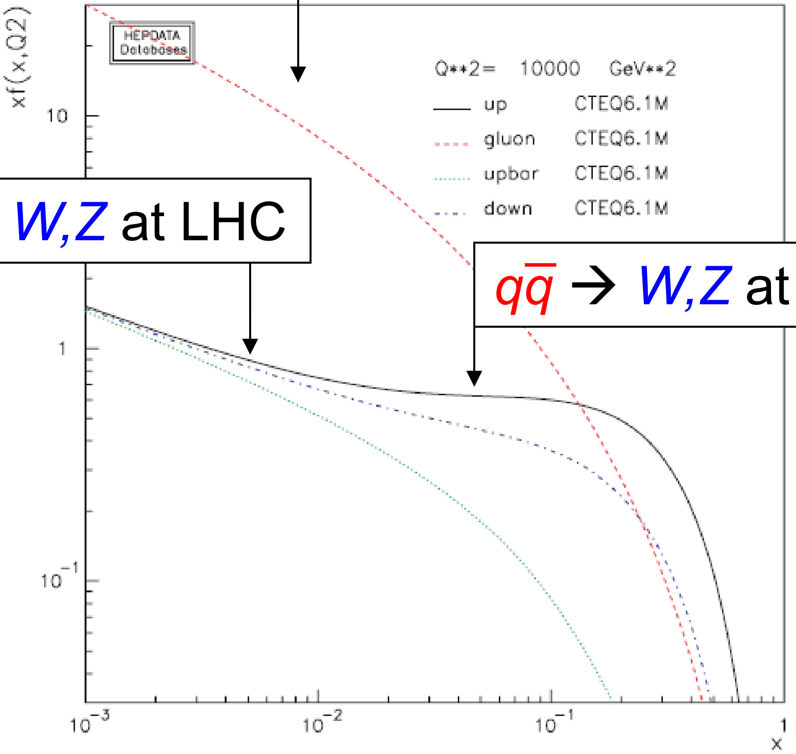
It is a double-log expansion:

$$
D_q^{(n)}(z,\mu_F) \;\propto\; (C_F\alpha_s)^n\left[\left(\frac{\ln^{2n+1}(1-z)}{1-z}\right)_+ + \cdots\right]
$$

For *gg* -> *H*, same leading behavior at large *z*.
Except color factor is much bigger: *C_A = 3*, not *C_F = 4/3*

$$
D_{gg\to H}^{(n)}(z,\mu_F) \;\propto\; (C_A\alpha_s)^n\left[\left(\frac{\ln^{2n+1}(1-z)}{1-z}\right)_+ + \cdots\right]
$$

Fast falling pdfs -- worse for gluons



gg → *light Higgs* at LHC

$q\bar{q}$ → *W,Z* at LHC

$q\bar{q}$ → *W,Z* at Tevatron

Q: If it is called Sudakov suppression, why does it increase the cross section?

A. Because the same suppression happens in the DIS process used to measure the pdfs. Both parton distributions "bigger than you thought": 2 – 1 > 0.

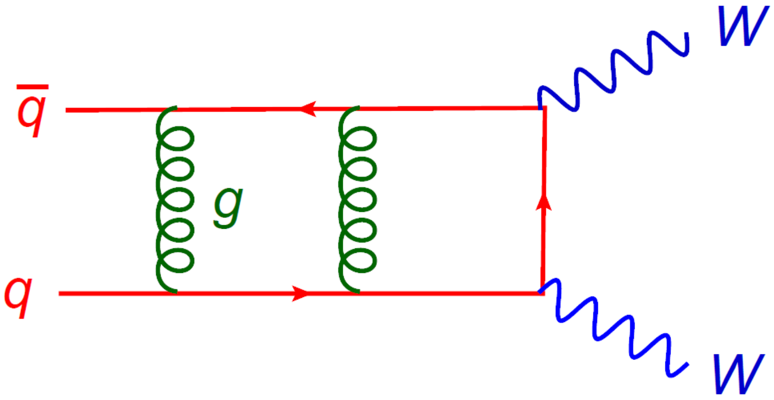
pdfs

partonic cross section

NNLO

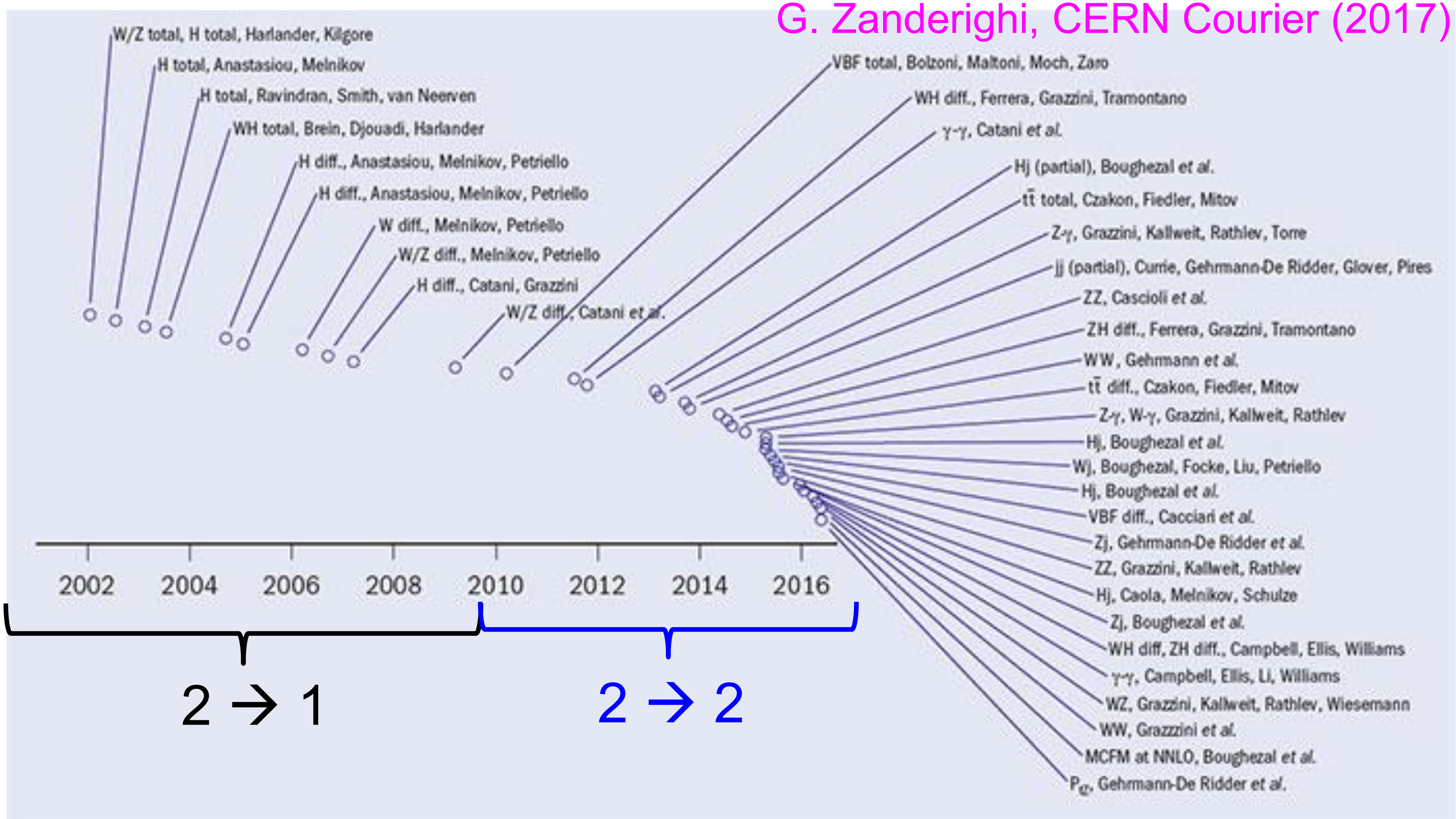
- Required for high precision at LHC, because NLO results often have 10% or more residual uncertainties
- High precision needed for:
- parton distributions
  - evolution (NNLO DGLAP kernels)
  - fits to DIS, Drell-Yan, and jet data
- LHC production of single *W*s and *Z*s
  - "partonic" luminosity monitor
  - precision m_W
- Higgs production via gluon fusion and extraction of Higgs couplings
- LHC production of $t\bar{t}$
- pairs of *W*s and *Z*s
- More recently, the beginnings of 2 → 3 processes …

Massless internal 2 → 2



- Here, 2 loop integrals typically are multiple polylogarithms (MPLs), e.g. Goncharov, 1105.2076
- Together with advances in handling real radiation, and stable one-loop 2 → 3 amplitudes, made possible a large class of 2 → 2 processes at NNLO

NNLO QCD @ LHC



G. Zanderighi, CERN Courier (2017)

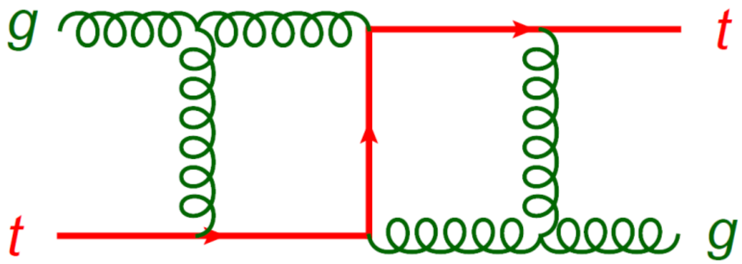
2 → 1

2 → 2

NNLO 2 → 2 enabled by understanding multiple polylogarithms (MPLs)

Top pair production

- At subleading color at 2 loops (NNLO) in the partonic process $gg \to t\bar{t}$, one finds

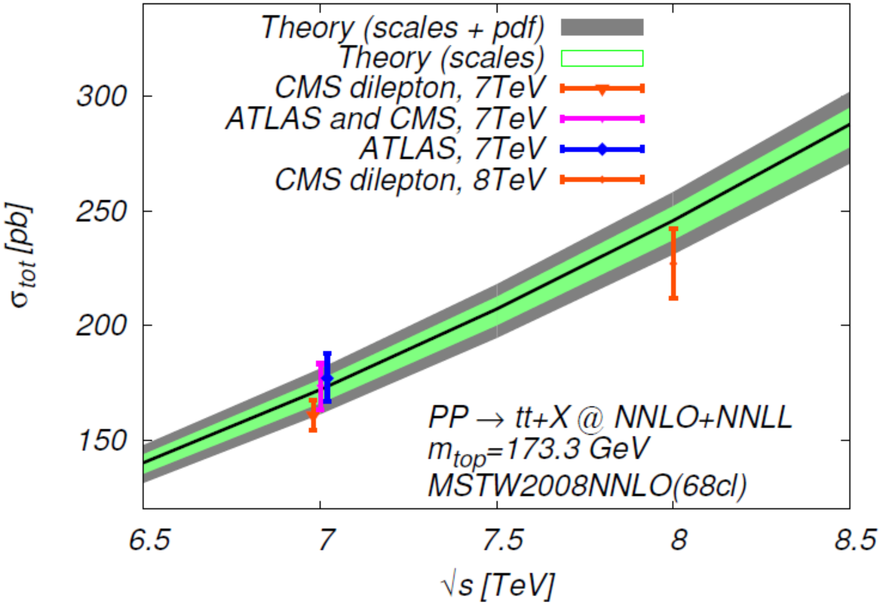


- More complicated function:

elliptic polylogarithm

- Done numerically first

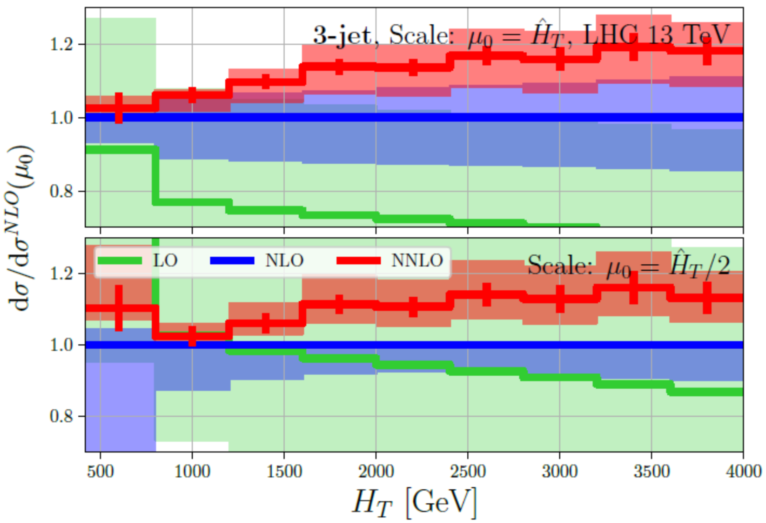
Czakon, Fiedler, Mitov, 1303.6254

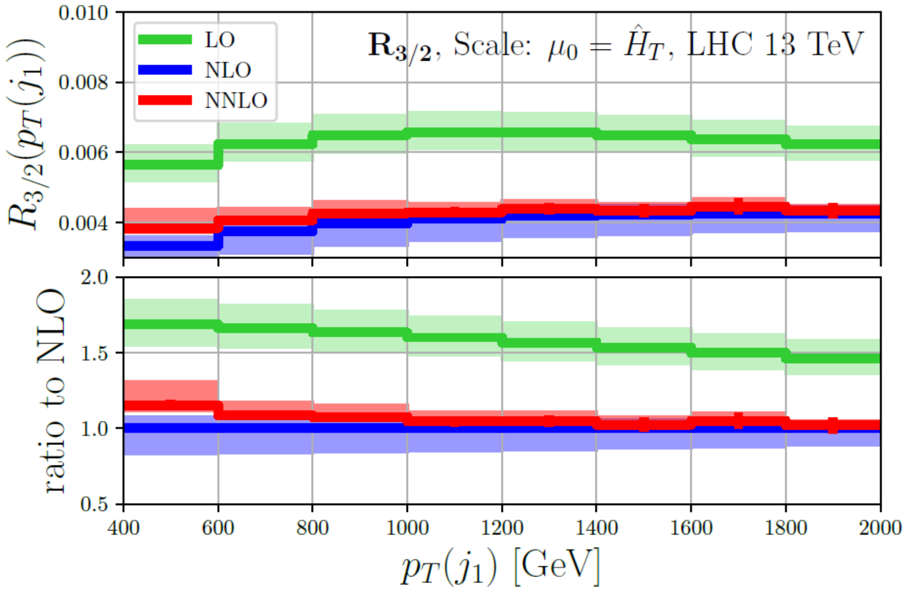


NNLO 3 jet production

Czakon, Mitov, Poncelet, 2106.05331

- State of art: much computing power required!



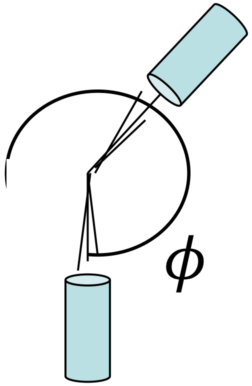


$$H_T = \sum_i E_{T,i}$$

Application: NNLO energy correlators for α_S

ATLAS, 2301.09351

Asymmetry in transverse energy-energy correlator (ATEEC)

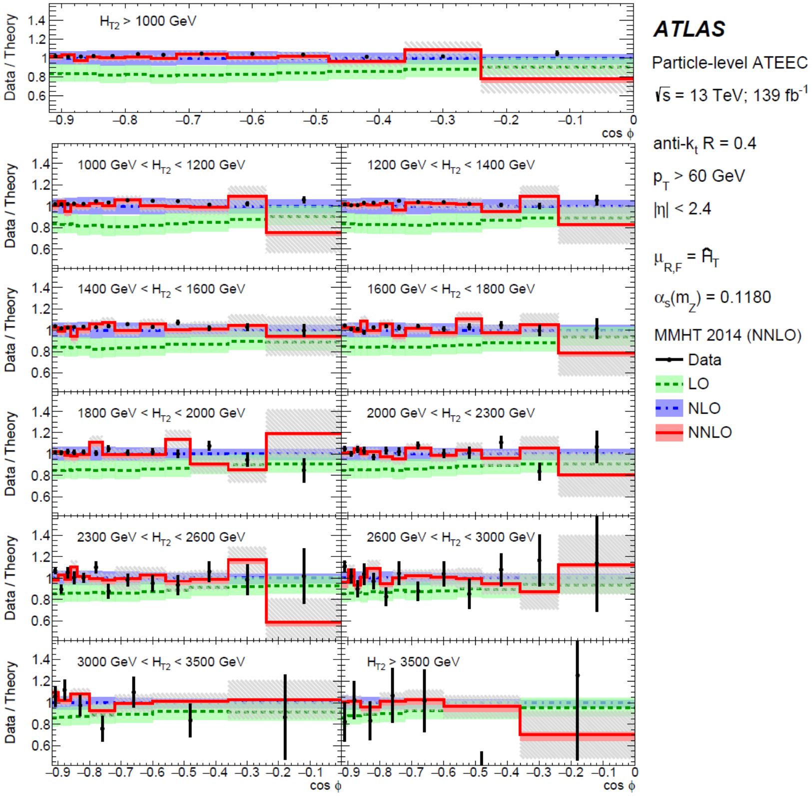


$$\frac{1}{\sigma}\frac{d\Sigma}{d\cos\phi} \equiv \frac{1}{\sigma}\sum_{ij}\int \frac{d\sigma}{dx_{Ti}dx_{Tj}d\cos\phi}x_{Ti}x_{Tj}dx_{Ti}dx_{Tj} = \frac{1}{N}\sum_{A=1}^{N}\sum_{ij}\frac{E^A_{Ti}E^A_{Tj}}{\left(\sum_k E^A_{Tk}\right)^2}\delta(\cos\phi - \cos\varphi_{ij})$$

$$\frac{1}{\sigma}\frac{d\Sigma^{\rm asym}}{d\cos\phi} = \frac{1}{\sigma}\frac{d\Sigma}{d\cos\phi}\bigg|_{\phi} - \frac{1}{\sigma}\frac{d\Sigma}{d\cos\phi}\bigg|_{\pi-\phi}$$

- Leads to

$$\alpha_S(M_Z) = 0.1185 \pm 0.0009(exp)^{+0.0025}_{-0.0012}(th)$$



EECs represent another class of IR safe QCD observables. Although no jets in measurement, can still use calculation by Czakon, Mitov, Poncelet, 2106.05331

Levels of Approximation

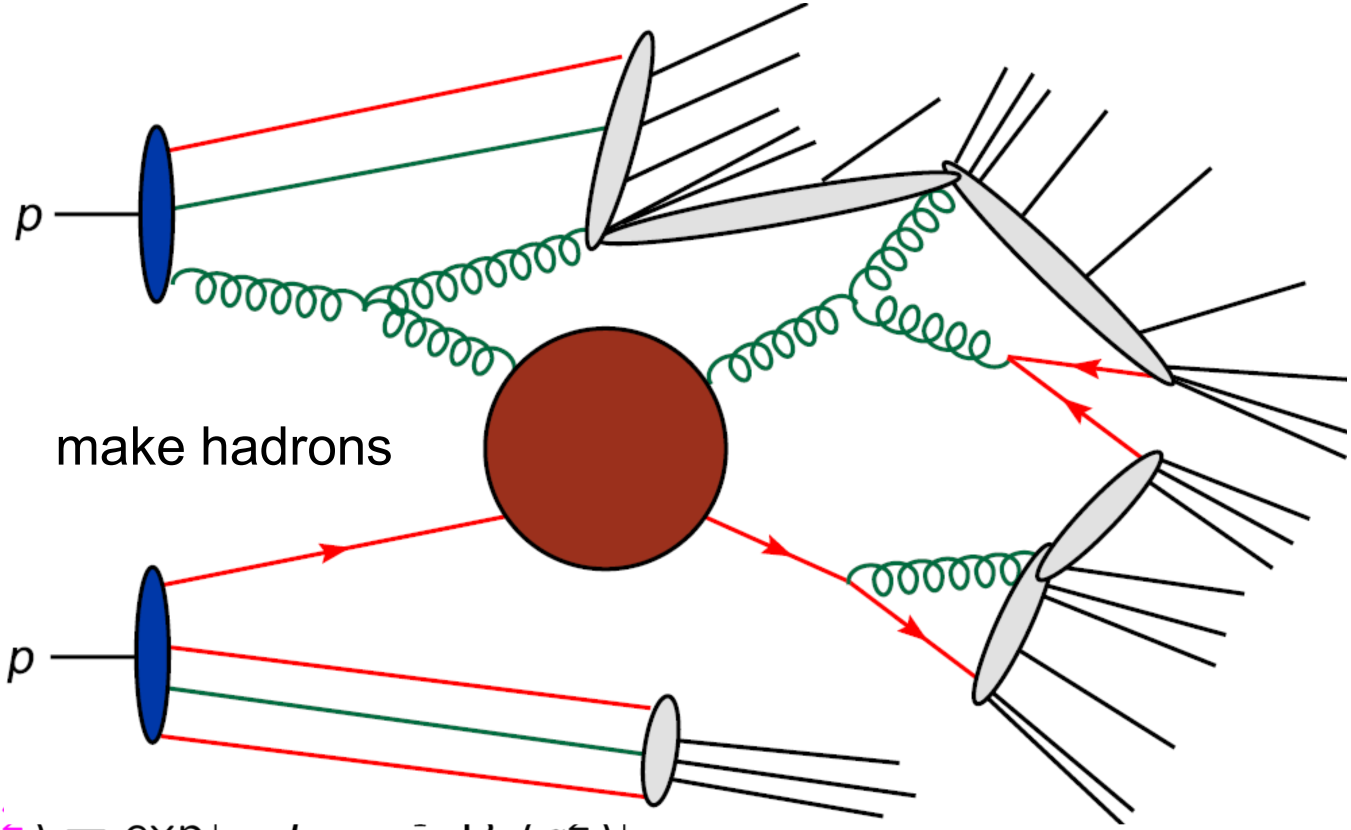
- Monte Carlos (PYTHIA, HERWIG,SHERPA)
- LO, fixed-order matrix elements (MEs)
- LO MEs matched to parton showers (ALPGEN, SHERPA, MADGRAPH/EVENT, …)
- NLO MEs (parton level) (MCFM, BLACKHAT, MADGRAPH, OPENLOOPS,…)
- NLO MEs matched to showers (MC@NLO, POWHEG, SHERPA)
- NNLO MEs (FeWZ, HNNLO, DYNNLO, …)
- MC@NNLO (MiNNLO, …)
- N3LO MEs

Monte Carlos

- Based on properties of soft and collinear radiation in QCD
- Partons surrounded by “cloud” of soft and collinear partons
- Leading double logs of Q_{hard}/Q_{soft} exponentiate, can be generated probabilistically
- Shower starts with basic 2 → 2 parton scattering
  -- or basic production process for *W, Z, tt*, etc.
- Further radiation approximate, requires infrared cutoff
- Shower can be evolved down to very low Q_{soft} , where models for hadronization and spectator interactions can be applied
- Complete hadron-level event description attained
- Normalization of event rates unreliable
- Event “shapes” sometimes unreliable

Monte Carlos in pictures

Splitting probability: $\mathrm{P}_g(q^2) = \int_0^1 dz \frac{\alpha_s(q^2)}{2\pi} \widehat{P}_{gg}(z) \Theta(q^2 - q_0^2)$



make hadrons

Sudakov

$\Delta_g(Q^2, q^2) = \exp\left[- \int_{q^2} \frac{}{\tilde{q}^2} \mathrm{P}_g(q^2)\right]$

based on color flow

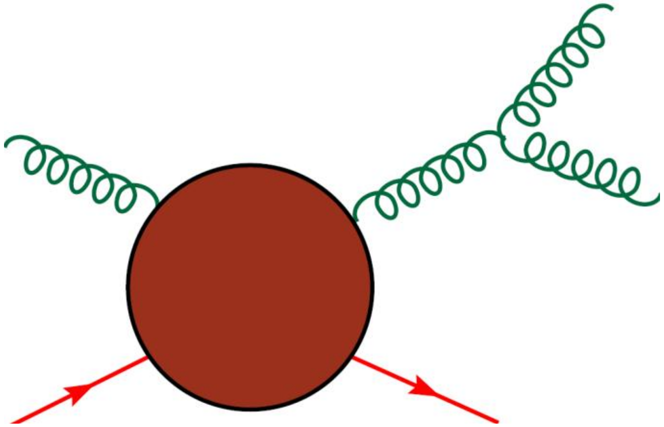
Matching MEs to showers

- Would like to have both:
  - accurate hard radiation pattern of MEs
  - hadron-level event description of parton-shower MCs
- Why not just use 2 → 3,4,… parton processes as starting point for the shower?
- Problem of double-counting:

  When does radiation "belong"

  to the shower, and

  when to the hard matrix element?

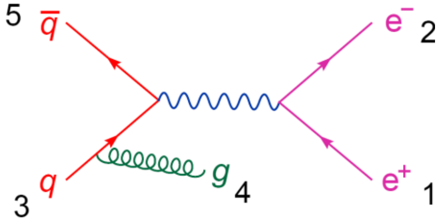


Conclusions

- LHC physics is very rich: wide range of processes and rates
- Since we don't know what form new physics will take, need theoretical control over many types of processes → higher order QCD (also EW)!
- Much recent progress in high precision, both experimentally and theoretically (up to N3LO in some cases!)
- Higher experimental precision coming with HL-LHC, so theory must try to keep up!
- High multiplicity final states remain very difficult beyond NLO
- Not only because amplitudes are tough, but so are real radiative corrections
- Large effort needed for future progress!

Extra Slides

NLO QCD corrections to Drell-Yan production



$$|A_5|^2 = \frac{s_{13}^2 + s_{15}^2 + s_{23}^2 + s_{25}^2}{s_{12}s_{34}s_{45}}$$

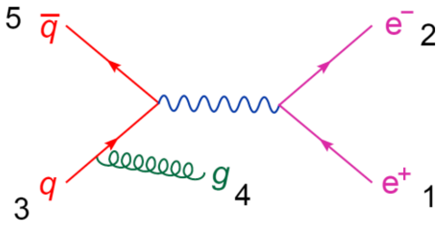
As at LO, average over decay direction of e^+ and e^-:

$$\langle k_1^\mu k_1^\nu \rangle_\Omega \equiv \int \frac{d\Omega_{e^+e^-}}{4\pi} k_1^\mu k_1^\nu = -\frac{s_{12}}{12}\eta^{\mu\nu} + \frac{1}{3}(k_1+k_2)^\mu (k_1+k_2)^\nu = \langle k_2^\mu k_2^\nu \rangle_\Omega$$

$$\langle s_{13}^2 \rangle_\Omega = \langle s_{23}^2 \rangle_\Omega = \frac{1}{3}(s_{13}+s_{23})^2 = \frac{1}{3}(s_{34}+s_{35})^2$$

$$\Rightarrow \quad \langle |A_5|^2 \rangle_\Omega = \frac{2}{3}\frac{(s_{34}+s_{35})^2 + (s_{35}+s_{45})^2}{s_{12}s_{34}s_{45}}$$

Phase space for DY @ NLO



Could use gluon energy, angle in CM frame, E_4 , θ

Trade for $z, y \in [0,1]$ defined by:

$$z = \frac{s_{12}}{s_{35}}$$

$$y = \frac{1-\cos\theta}{2}$$

$$E_4 = -\frac{s_{34}+s_{45}}{2\sqrt{s_{35}}} = -\frac{s_{12}-s_{35}}{2\sqrt{s_{35}}} = \frac{1-z}{2}\sqrt{s_{35}}$$

$$\begin{aligned} s_{34} &= -\sqrt{s_{35}}E_4(1-\cos\theta) = -y(1-z)s_{35} \\ \Rightarrow \quad s_{45} &= -\sqrt{s_{35}}E_4(1-\cos\theta) = -(1-y)(1-z)s_{35} \\ s_{12} &= M^2 = z s_{35} \end{aligned}$$

cross section: $$\langle |A_5|^2 \rangle_\Omega = \frac{2}{3\,M^2}\frac{(1-y(1-z))^2+(1-(1-y)(1-z))^2}{y(1-y)\,(1-z)^2}$$

P.S. measure in $D = 4-2\varepsilon$

$$\propto \left(\frac{\mu^2}{s_{35}}\right)^\epsilon \frac{d^{3-2\epsilon}p_4}{2E_4} \propto \left(\frac{\mu^2\,z}{M^2}\right)^\epsilon dE_4\,E_4^{1-2\epsilon}\,d\cos\theta(\sin^2\theta)^{-\epsilon}d\Omega^{1-2\epsilon}$$

$$\propto \left(\frac{\mu^2}{M^2}\right)^\epsilon dy\,dz\,[y(1-y)]^{-\epsilon}\,z^\epsilon\,(1-z)^{1-2\epsilon}$$

QCD corrections to DY (cont.)

Integral to do:

$$I = \left(\frac{\mu^2}{M^2}\right)^{\epsilon} z^{\epsilon}\,(1-z)^{-1-2\epsilon} \times \int_0^1 dy\,[y(1-y)]^{-\epsilon} \frac{(1-y(1-z))^2 + (1-(1-y)(1-z))^2}{y(1-y)}$$

Hard collinear divergences are at $y = 0,1$

related by symmetry

Separate using

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$$

Expand $1/y$ term in cross section about $y = 0$

$$\begin{aligned} I &= 2\left(\frac{\mu^2}{M^2}\right)^{\epsilon} z^{\epsilon}\,(1-z)^{-1-2\epsilon} \int_0^1 dy\, y^{-1-\epsilon} \left[1 + z^2 - 2y(1-y)(1-z)^2\right] \\ &\qquad \times (1 - \epsilon \ln(1-y)) \\ &= 2\left(\frac{\mu^2}{M^2}\right)^{\epsilon} z^{\epsilon}\,(1-z)^{-1-2\epsilon} \left[-\frac{1+z^2}{\epsilon} - (1-z)^2 + \mathcal{O}(\epsilon)\right] \end{aligned}$$

QCD corrections to DY (cont.)

Including a few other omitted prefactors:

$$\frac{d\hat{\sigma}^{\text{NLO, real}}}{dM^2} = \frac{\sigma_0}{N_c\, s} Q_q^2 \frac{\alpha_s}{2\pi} C_F \Bigg[2\left(-\frac{1}{\epsilon} - \ln(4\pi) + \gamma\right)\frac{1+z^2}{1-z} -2\frac{1+z^2}{1-z}\left(-2\ln(1-z) + \ln z - \ln\frac{M^2}{\mu^2}\right) -2(1-z)^2\Bigg]$$

divergence absorbed into *q(x)* in $\overline{\text{MS}}$ factorization scheme

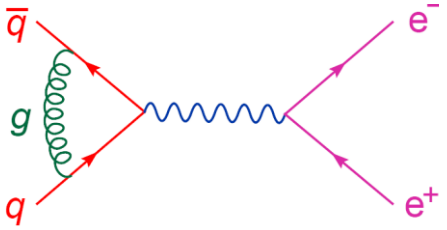
correction to cross section

artifact of my using unconventional FDH scheme with 2 gluon helicities, vs. standard 2-2ε of CDR – drop!

$$\begin{aligned} q(x,\mu) &= q_0(x) + \frac{\alpha_s(\mu)}{2\pi}\left(-\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E\right)\int_x^1 \frac{d\xi}{\xi}\left[P_{qq}^{(0)}(x/\xi)q_0(\xi) + P_{qg}^{(0)}(x/\xi)g_0(\xi)\right] \\ &= q_0(x) + \frac{\alpha_s(\mu)}{2\pi}\left(-\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E\right)\int_x^1 \frac{dz}{z}\left[C_F\frac{1+z^2}{1-z}q_0(x/z) + P_{qg}^{(0)}(z)g_0(x/z)\right] \end{aligned}$$

QCD corrections to DY (cont.)

Finally, virtual graph has support only at $z = 1$ → kinematics same as at LO. Regulates $\frac{1}{1-z}$ into plus distribution. Final result:



$$\frac{d\sigma^{\mathrm{NLO}}}{dM^2} = \frac{\sigma_0}{N_c\, s}\int_0^1 dx_1 dx_2 dz\, \delta(x_1 x_2 z - \tau) \sum_q Q_q^2 \Big[q(x_1,\mu_F)\bar{q}(x_2,\mu_F)\Big(\delta(1-z) + \frac{\alpha_s(\mu_R)}{2\pi} C_F D_q(z,\mu_F)\Big) + g(x_1,\mu_F)(q(x_2,\mu_F) + \bar{q}(x_2,\mu_F))\frac{\alpha_s(\mu_R)}{2\pi} T_R D_g(z,\mu_F) + (x_1 \leftrightarrow x_2)\Big]$$

where

$$D_q(z,\mu_F) = 4(1+z^2)\left(\frac{\ln(1-z) + \ln(M/\mu_F)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln z + \delta(1-z)\left(\frac{2}{3}\pi^2 - 8\right)$$

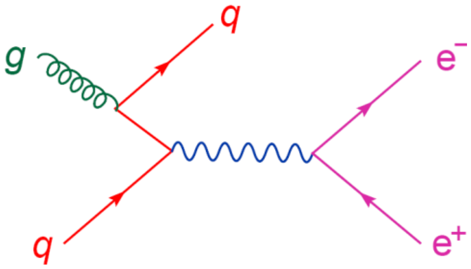
singular distribution as $z \longrightarrow 1$

QCD corrections to DY (cont.)

and

$$D_g(z,\mu_F) \;=\; (z^2+(1-z)^2)\left[\ln\frac{(1-z)^2}{z}+2\ln\frac{M}{\mu_F}\right]+\frac{1}{2}+3z-\frac{7}{2}z^2$$

comes from the $qg \to q\gamma^*$ subprocess:



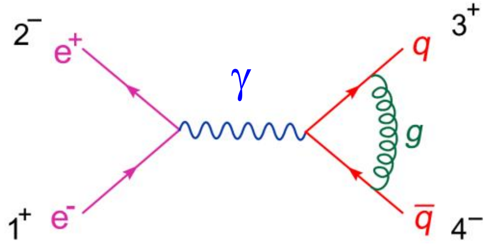
- Cross section related by crossing to $q\bar{q} \to g\gamma^*$

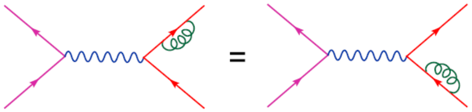
- Remove $g \to q\bar{q}$ collinear singularity in same way

- Note that there is no $\frac{1}{1-z}$ (soft gluon) singularity in this term.

Virtual Corrections

The simplest process:





= 0 in dim. reg.

cancellation of UV & IR divergences!

overlap of soft & collinear IR divergences

$$\mathcal{A}_4^{1-\text{loop}} = \mathcal{A}_4^{\text{tree}} \frac{\alpha_s}{4\pi} \exp[\epsilon(\ln(4\pi)-\gamma_E)] \times 2C_F \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - \frac{7}{2} - \frac{\delta_R}{2} + \frac{\pi^2}{12}\right]$$

for $2-2\epsilon\delta_R$ virtual-gluon helicity states

$\delta_R = 1$ for CDR & HV schemes; $\delta_R = 0$ for FDH $\approx$ $\overline{\text{DR}}$ scheme

More complicated 1-loop amplitudes

$ggggg$

$$+++++ = \frac{i}{96\pi^2}\frac{s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}+s_{45}s_{51}+s_{51}s_{12}+\varepsilon(1,2,3,4)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle},$$

$$-++++) = \frac{i}{48\pi^2}\frac{1}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]}\Bigg[(s_{23}+s_{34}+s_{45})[2\,5]^2-[2\,4]\,\langle 4\,3\rangle\,[3\,5]\,[2\,5]$$
$$-\frac{[1\,2]\,[1\,5]}{\langle 1\,2\rangle\,\langle 1\,5\rangle}\left(\langle 1\,2\rangle^2\langle 1\,3\rangle^2\frac{[2\,3]}{\langle 2\,3\rangle}+\langle 1\,3\rangle^2\langle 1\,4\rangle^2\frac{[3\,4]}{\langle 3\,4\rangle}+\langle 1\,4\rangle^2\langle 1\,5\rangle^2\frac{[4\,5]}{\langle 4\,5\rangle}\right)\Bigg]$$

$$V^g = -\frac{1}{\epsilon^2}\sum_{j=1}^{5}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^{\epsilon}+\sum_{j=1}^{5}\ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}}\right)\ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}}\right)+\frac{5}{6}\pi^2-\frac{\delta_R}{3}$$

the following functions for the $(1^-,2^-,3^+,4^+,5^+)$ helicity configuration,

~ 1 page

$--+++$

$$V^f = -\frac{5}{2\epsilon}-\frac{1}{2}\left[\ln\left(\frac{\mu^2}{-s_{23}}\right)+\ln\left(\frac{\mu^2}{-s_{51}}\right)\right]-2, \qquad V^s = -\frac{1}{3}V^f+\frac{2}{9}$$

$$F^f = -\frac{1}{2}\frac{\langle 1\,2\rangle^2\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}}$$

$$F^s = -\frac{1}{3}\frac{[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\,\langle 4\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3}-\frac{1}{3}F^f$$
$$-\frac{1}{3}\frac{\langle 3\,5\rangle\,[3\,5]^3}{[1\,2]\,[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]}+\frac{1}{3}\frac{\langle 1\,2\rangle\,[3\,5]^2}{[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]}+\frac{1}{6}\frac{\langle 1\,2\rangle\,[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]}{s_{23}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,s_{51}}$$

and the corresponding ones for the $(1^-,2^+,3^-,4^+,5^+)$ helicity configuration,

$$V^f = -\frac{5}{2\epsilon}-\frac{1}{2}\left[\ln\left(\frac{\mu^2}{-s_{34}}\right)+\ln\left(\frac{\mu^2}{-s_{51}}\right)\right]-2, \qquad V^s = -\frac{1}{3}V^f+\frac{2}{9}$$

$$F^f = -\frac{\langle 1\,3\rangle^2\langle 4\,1\rangle[2\,4]^2}{\langle 4\,5\rangle\langle 5\,1\rangle}\frac{\mathrm{Ls}_1\left(\frac{-s_{23}}{-s_{51}},\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^2}+\frac{\langle 1\,3\rangle^2\langle 5\,3\rangle[2\,5]^2}{\langle 3\,4\rangle\langle 4\,5\rangle}\frac{\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{34}},\frac{-s_{51}}{-s_{34}}\right)}{s_{34}^2}$$
$$-\frac{1}{2}\frac{\langle 1\,3\rangle^3\left(\langle 1\,5\rangle\,[5\,2]\,\langle 2\,3\rangle-\langle 3\,4\rangle\,[4\,2]\,\langle 2\,1\rangle\right)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}}$$

$-+-++$

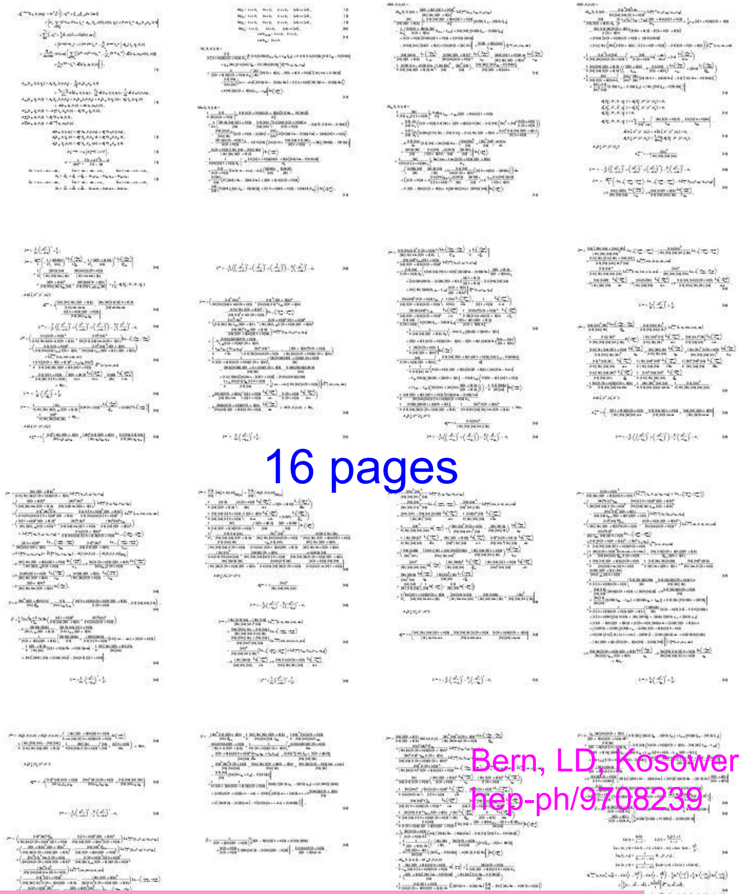
$$F^s = -\frac{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle^2[2\,4]^2}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle^2}\frac{2\,\mathrm{Ls}_1\left(\frac{-s_{23}}{-s_{51}},\frac{-s_{34}}{-s_{51}}\right)+\mathrm{L}_1\left(\frac{-s_{23}}{-s_{51}}\right)+\mathrm{L}_1\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^2}$$
$$+\frac{\langle 3\,2\rangle\langle 2\,1\rangle\langle 1\,5\rangle\langle 5\,3\rangle^2[2\,5]^2}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle^2}\frac{2\,\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{34}},\frac{-s_{51}}{-s_{34}}\right)+\mathrm{L}_1\left(\frac{-s_{12}}{-s_{34}}\right)+\mathrm{L}_1\left(\frac{-s_{51}}{-s_{34}}\right)}{s_{34}^2}$$
$$+\frac{2}{3}\frac{\langle 2\,3\rangle^2\langle 4\,1\rangle^3[2\,4]^3}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3}-\frac{2}{3}\frac{\langle 2\,1\rangle^2\langle 5\,3\rangle^3[2\,5]^3}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{12}}{-s_{34}}\right)}{s_{34}^3}$$
$$+\frac{\mathrm{L}_2\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^3}\Bigg(\frac{1}{3}\frac{\langle 1\,3\rangle\,[2\,4]\,[2\,5]\left(\langle 1\,5\rangle\,[5\,2]\,\langle 2\,3\rangle-\langle 3\,4\rangle\,[4\,2]\,\langle 2\,1\rangle\right)}{\langle 4\,5\rangle}$$
$$+\frac{2}{3}\frac{\langle 1\,2\rangle^2\langle 3\,4\rangle^2\,\langle 4\,1\rangle\,[2\,4]^3}{\langle 4\,5\rangle\,\langle 5\,1\rangle\,\langle 2\,4\rangle}-\frac{2}{3}\frac{\langle 3\,2\rangle^2\langle 1\,5\rangle^2\,\langle 5\,3\rangle\,[2\,5]^3}{\langle 5\,4\rangle\,\langle 4\,3\rangle\,\langle 2\,5\rangle}\Bigg)$$
$$+\frac{1}{6}\frac{\langle 1\,3\rangle^3\left(\langle 1\,5\rangle\,[5\,2]\,\langle 2\,3\rangle-\langle 3\,4\rangle\,[4\,2]\,\langle 2\,1\rangle\right)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}}+\frac{1}{3}\frac{[2\,4]^2[2\,5]^2}{[1\,2][2\,3][3\,4]\langle 4\,5\rangle[5\,1]}$$
$$-\frac{1}{3}\frac{\langle 1\,2\rangle\langle 4\,1\rangle^2[2\,4]^3}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle[2\,3][3\,4]s_{51}}+\frac{1}{3}\frac{\langle 3\,2\rangle\langle 5\,3\rangle^2[2\,5]^3}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle[2\,1][1\,5]s_{34}}+\frac{1}{6}\frac{\langle 1\,3\rangle^2\,[2\,4]\,[2\,5]}{s_{34}\,\langle 4\,5\rangle\,s_{51}}.$$

Bern, LD, Kosower,
hep-ph/9302280

More legs,
or massive
legs, rapidly
increases
complexity!

Some
helicity
config's
more
complex
than others

$Vq\bar{q}gg \qquad V = W, Z, \gamma^*$



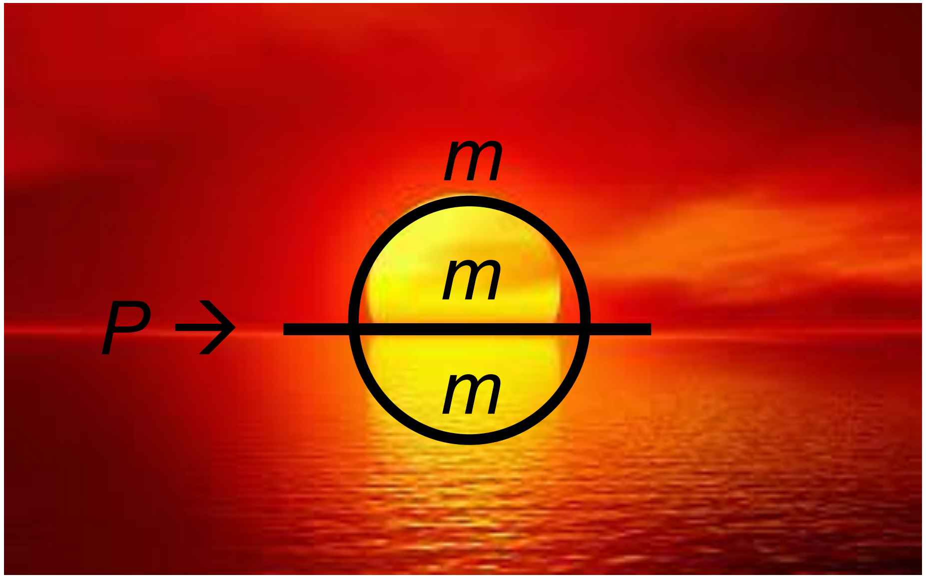
16 pages

Bern, LD, Kosower,
hep-ph/9708239

→ "numerical" approaches eventually

Two loop integrals

- Become non-polylogarithmic – “elliptic polylogarithms” – very quickly if there are internal particle masses, e.g. the massive sunset integral

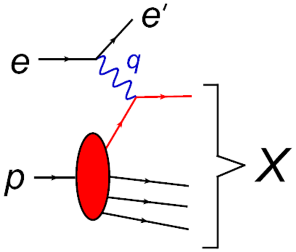


Broadhurst-Fleischer-Tarasov, 9304303, Berends-Böhm-Buza-Scharf (1994), Laporta-Remiddi, 0406160, Adams-Bogner-Weinzierl, 1302.7004, Bloch-Vanhove, 1309.5865,…

IR safe/unsafe
examples from ep scattering

(Similar discussion for pp.)

Deep Inelastic Scattering (DIS)
ep → eX (inclusive in hadronic state X: OK)

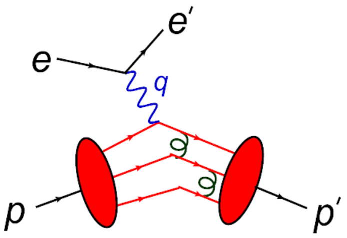


$q^2 = -Q^2$ large

$x = \frac{Q^2}{2P\cdot q}$ fixed

$M_{\rm had}^2 = (q+P)^2 = \frac{1-x}{x} Q^2$ large

ep → ep exclusive scattering (very small rate)



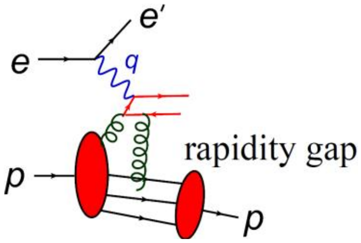
$A \sim 1/Q^4$

$q^2 = -Q^2$ large

$x \to 1$

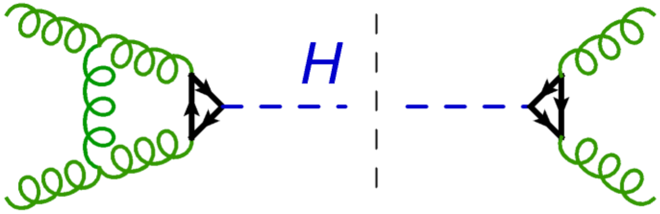
$M_{\rm had}^2 = m_{\rm p}^2$ small

Diffraction (forbids soft gluon radiation)

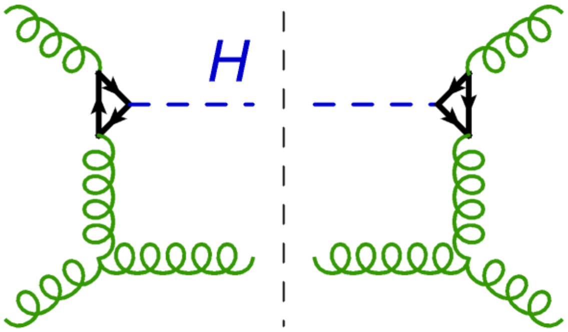


Some NLO QCD Feynman diagrams

Amplitude | Amplitude* = cross section

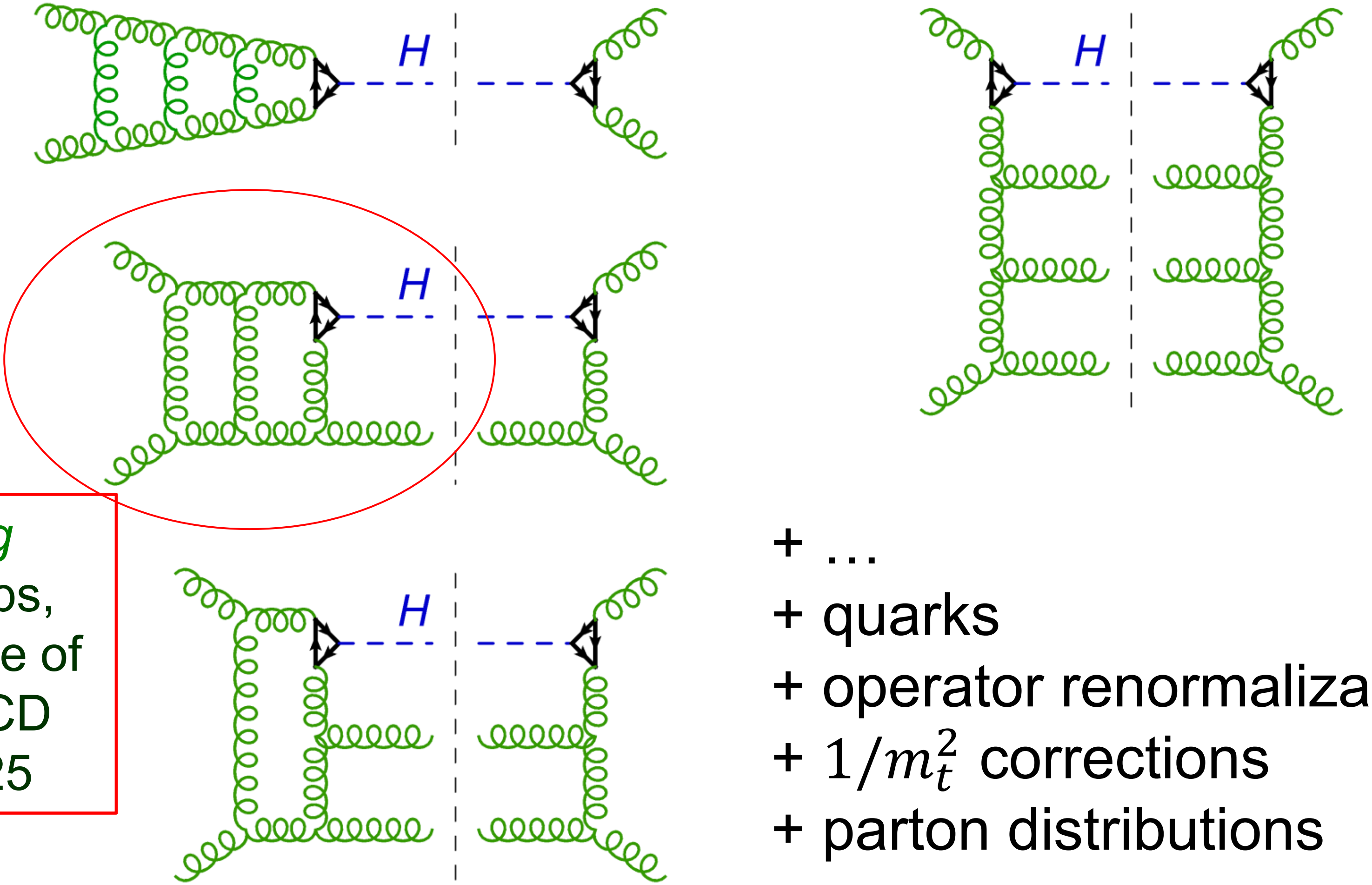


virtual *gg* → *H*



real, *gg* → *Hg*

Very few of the NNNLO QCD diagrams

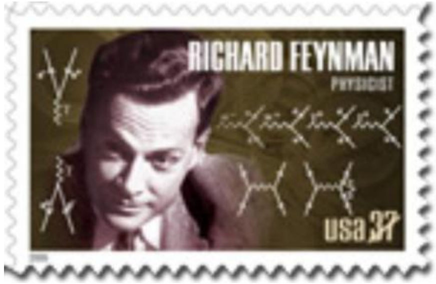


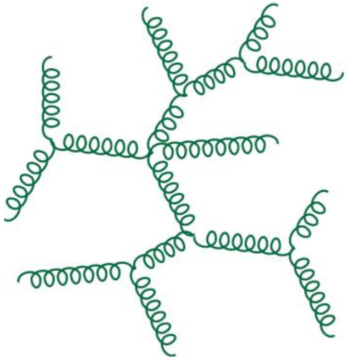
gg → *Hg*
@ 2 loops,
was state of
art in QCD
until 2025

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

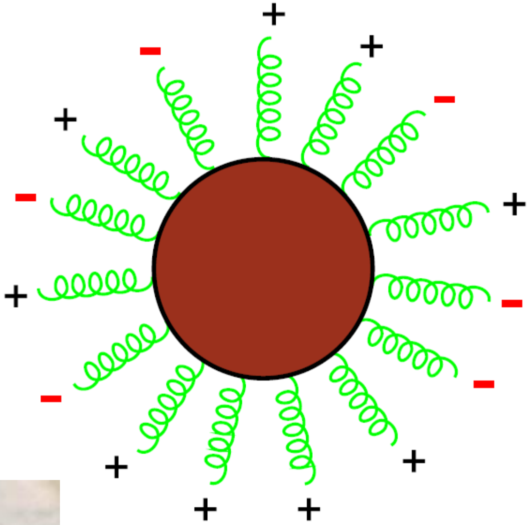
Scattering amplitudes are underlying building blocks

Granularity vs. Fluidity





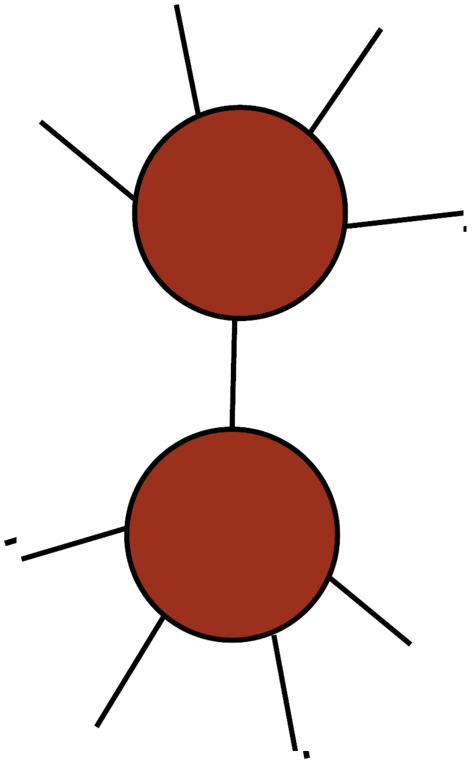
+ …



Fluid Tree Amplitudes

Tree amplitude is a rational function of kinematic variables.
Falls apart into simpler tree amplitudes in special limits

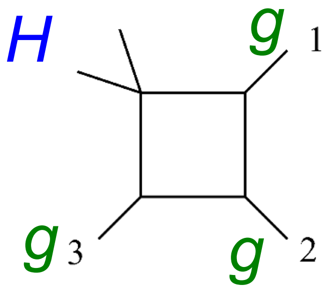
20 years ago:
picture led directly to BCFW (on-shell) recursion relations:
Reconstruct amplitude from poles in complex plane, where it factorizes into simpler amplitudes
Britto, Cachazo, Feng, Witten, hep-th/0501052



Beyond tree level

- Loop level Feynman diagrams come with an instruction to integrate over all loop momenta

- For example, at one loop the amplitude for *gg* → *Hg* involves the "scalar box" integral



$$= \int \frac{d^4p}{p^2(p-p_1)^2(p-p_1-p_2)^2(p-p_1-p_2-p_3)^2}$$

$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

where the dilogarithm is $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t}\ln(1-t)$

One loop not too bad

- For any number of external particles, all one-loop integrals (even in dimensional regularization, $D = 4 - 2\epsilon$) can be reduced to scalar box integrals + simpler

Brown-Feynman (1952), Melrose (1965), ‘t Hooft-Veltman (1974), Passarino-Veltman (1979), van Neerven-Vermaseren (1984), Bern, LD, Kosower (1992)

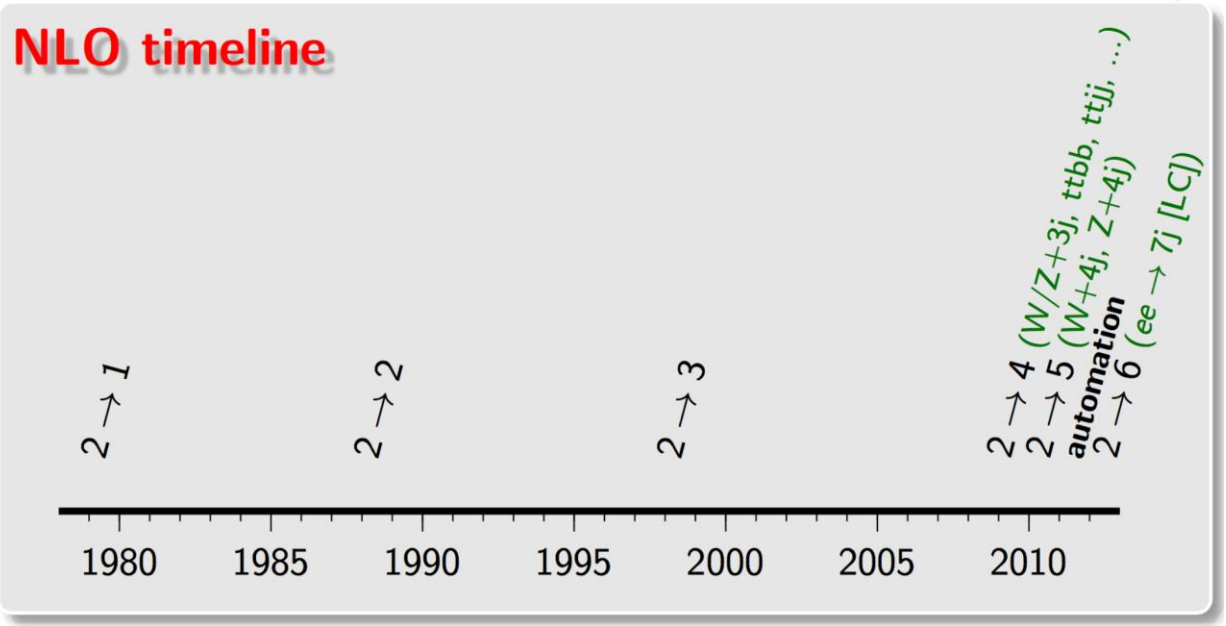
→ combinations of $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t)$

where x is (many different) functions of the kinematic variables (Mandelstam invariants),

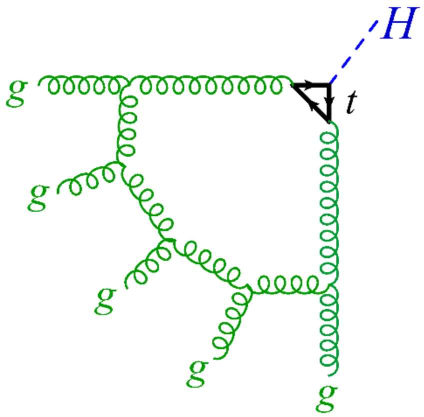
plus logarithms

1-loop progress → NLO QCD @ LHC

G. Salam (2012)



2010: NLO $W+4j$ [BlackHat+Sherpa: Berger et al]	[unitarity]
2011: NLO $WWjj$ [Rocket: Melia et al]	[unitarity]
2011: NLO $Z+4j$ [BlackHat+Sherpa: Ita et al]	[unitarity]
2011: NLO $4j$ [BlackHat+Sherpa: Bern et al]	[unitarity]
2011: first automation [MadNLO: Hirschi et al]	[unitarity + feyn.diags]
2011: first automation [Helac NLO: Bevilacqua et al]	[unitarity]
2011: first automation [GoSam: Cullen et al]	[feyn.diags(+unitarity)]
2011: $e^+e^- \to 7j$ [Becker et al, leading colour]	[numerical loops]



2013: NLO $H+3j$ in gluon fusion [GoSam, Sherpa, MadEvent: Cullen et al.]

Dipole subtraction

Catani, Seymour, hep-ph/9602227, hep-ph/9605323

$$d\sigma^A = \sum_{\text{dipoles}} d\sigma^B \otimes dV_{\text{dipole}}$$

includes sum over colors, convolution over momentum fractions

$$\int_{n+1} d\sigma^A = \sum_{\text{dipoles}} \int_n d\sigma^B \otimes \int_1 dV^{\text{dipole}}$$

$$= \int_n d\sigma^B \otimes I$$

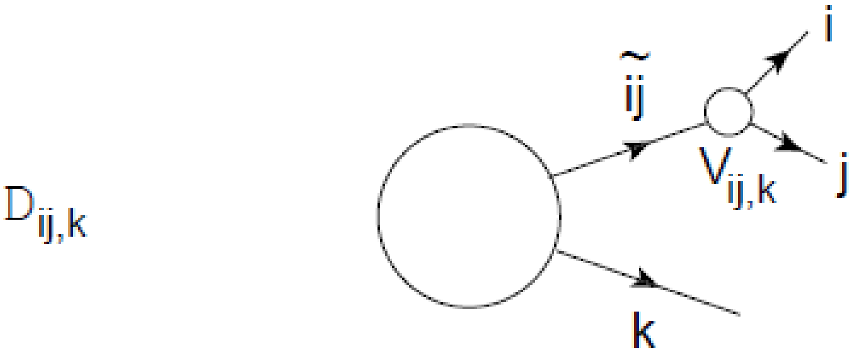
Poles in ε cancel universal IR poles in

$$d\sigma^V = d\sigma^B \otimes I^{(1)}$$

In the case of hadrons in the initial state, some terms have a more complicated structure, involving convolution over an initial-state splitting

Momentum map

If no initial state hadrons, have only final-state dipoles ij and spectators k



map (n+1) -body to n-body:

$$p_i^\mu + p_j^\mu + p_k^\mu = \tilde{p}_{ij}^\mu + \tilde{p}_k^\mu$$

Spectator k recoils so that $\widetilde{ij}$ can be massless:

$$\tilde{p}_k^\mu = \frac{1}{1-y_{ij,k}} p_k^\mu \ , \qquad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{y_{ij,k}}{1-y_{ij,k}} p_k^\mu$$

$$y_{ij,k} = \frac{p_i p_j}{p_i p_j + p_j p_k + p_k p_i}$$

$y_{ij,k} \to 0$, for $p_i \to 0$, $p_j \to 0$ and $p_i p_j \to 0$

Spectator k also used to define collinear fractions:

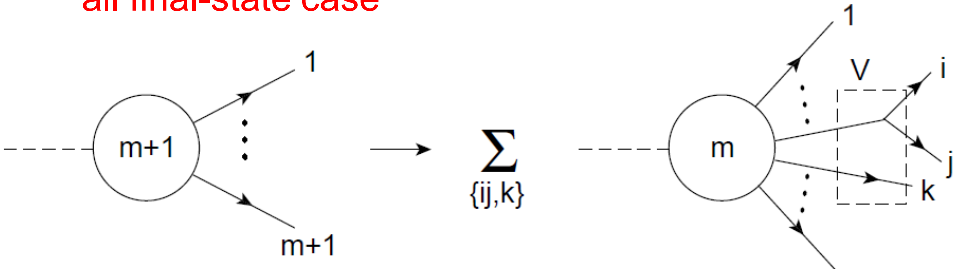
$$\tilde{z}_i = \frac{p_i p_k}{p_j p_k + p_i p_k} = \frac{p_i \tilde{p}_k}{\tilde{p}_{ij} \tilde{p}_k}$$

Sample dipole

Using Altarelli-Parisi kernels, build dipole subtraction function $D_{ij,k}$ for each pair of partons i,j that can get singular, and for each “spectator” parton k

The $D_{ij,k}$ multiply the LO cross section, at a boosted phase-space point:

all final-state case



$$\mathcal{D}_{ij,k}\,(p_1, ..., p_{m+1}) = -\frac{1}{2p_i \cdot p_j}$$
$$\cdot\, {}_m\!< 1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 | \, \frac{\boldsymbol{T}_k \cdot \boldsymbol{T}_{ij}}{\boldsymbol{T}_{ij}^2} \, \boldsymbol{V}_{ij,k} \, | 1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 >_m$$

$\boldsymbol{T}_k$ = color operators

$$< s | \boldsymbol{V}_{q_i g_j, k}(\tilde{z}_i; y_{ij,k}) | s' > \; = \; 8\pi\mu^{2\epsilon}\alpha_S\, C_F \left[\frac{2}{1 - \tilde{z}_i(1 - y_{ij,k})} - (1 + \tilde{z}_i) - \epsilon(1 - \tilde{z}_i) \right] \delta_{ss'}$$

All dipole integrals can be done analytically

Multi-loop much more complex

- At L loops, instead of just Li_2's, get special functions with up to $2L$ integrations
- Weight $2L$ "iterated integrals"
- **Best case**: generalized polylogarithms, defined iteratively by

$$\boxed{G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)}$$

and $G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

- Still very intricate multi-variate functions

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