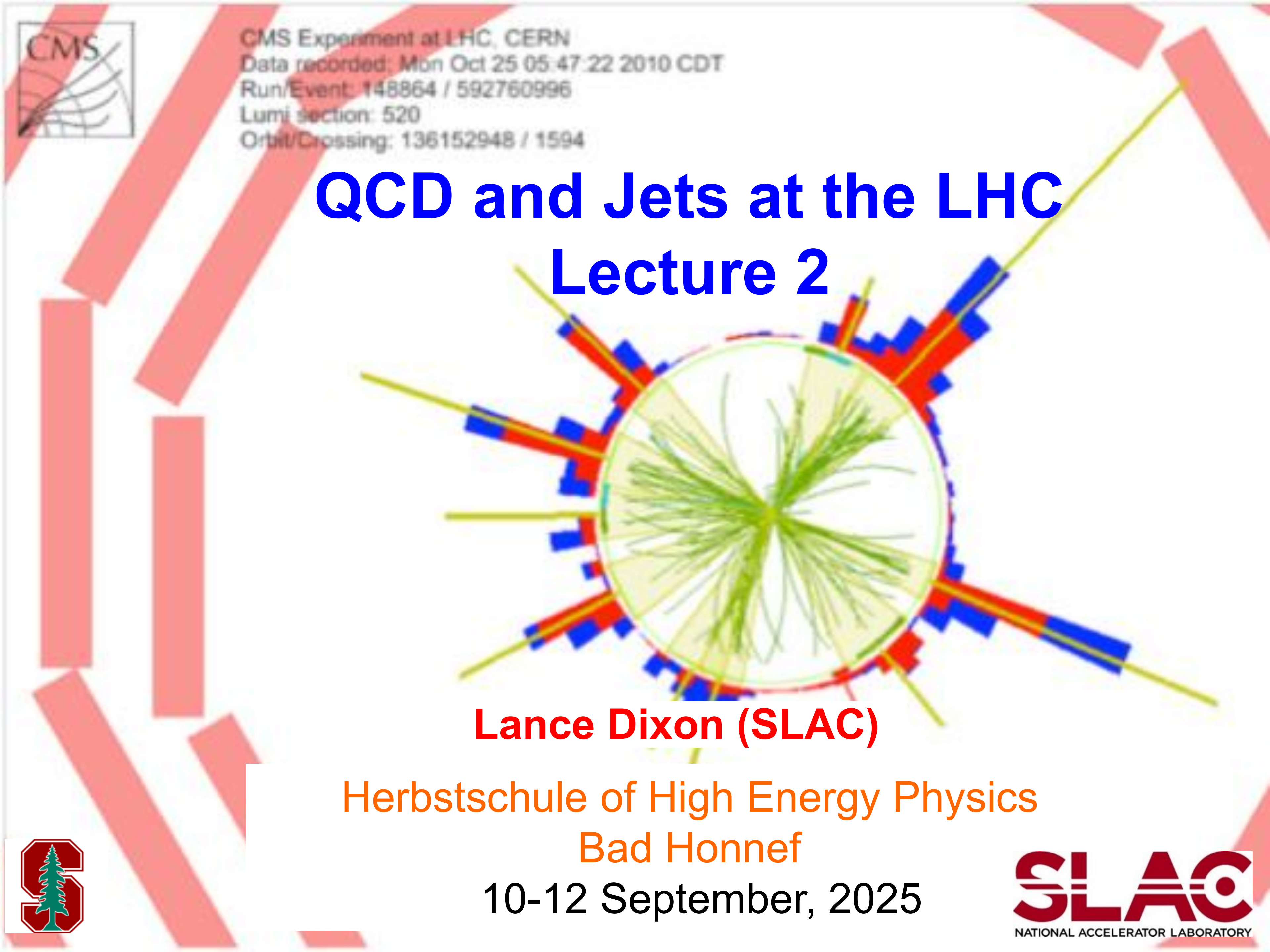


QCD and Jets at the LHC
Lecture 2

Lance Dixon (SLAC)

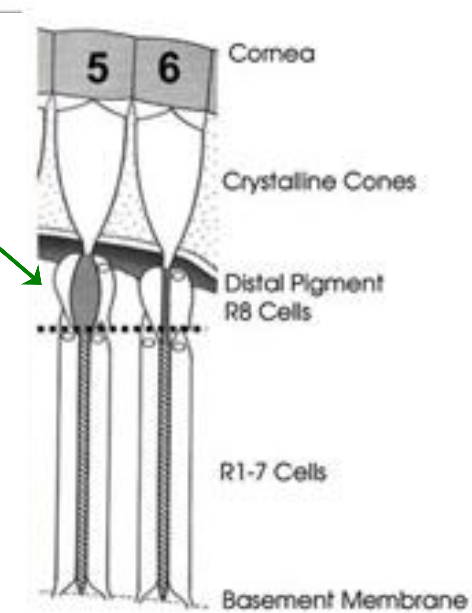
Herbstschule of High Energy Physics
Bad Honnef
10-12 September, 2025

The Tail of the Mantis Shrimp

- Reflects left and right circularly polarized light differently

- Led biologists to discover that its eyes have differential sensitivity
- It communicates via the helicity formalism

λ/4 plate



"It's the most private communication system imaginable. No other animal can see it."

- Roy Caldwell (U.C. Berkeley)

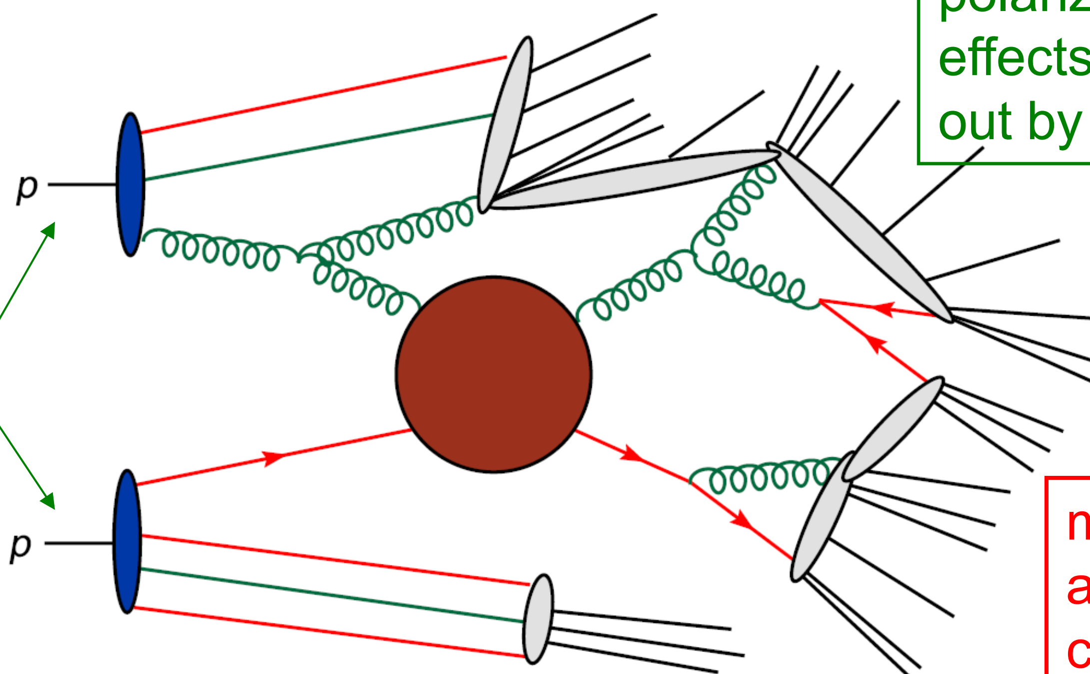
What the Biologists Didn't Know

Particle theorists have also evolved capability to communicate results via helicity formalism

LHC experimentalists are blind to it

~ any final-state polarization effects washed out by fragmentation

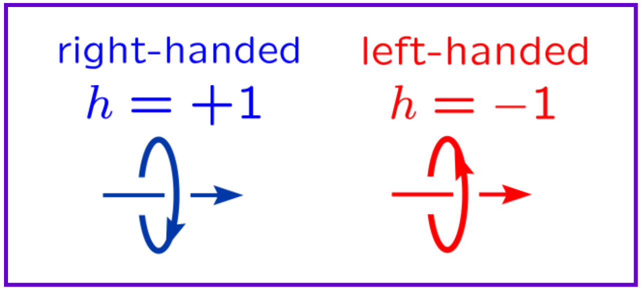
unpolarized

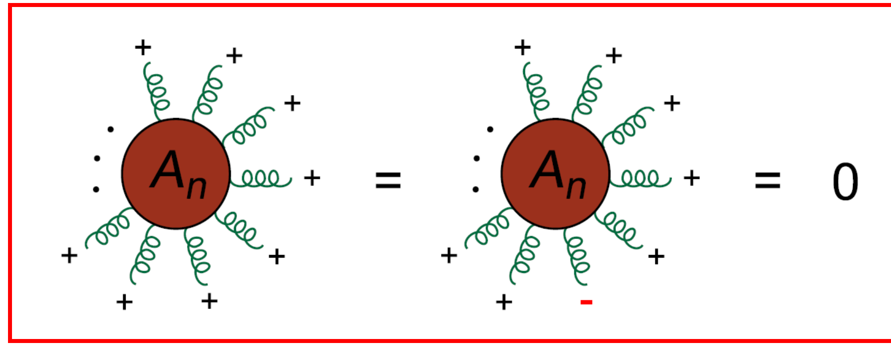


must sum over all helicity configurations

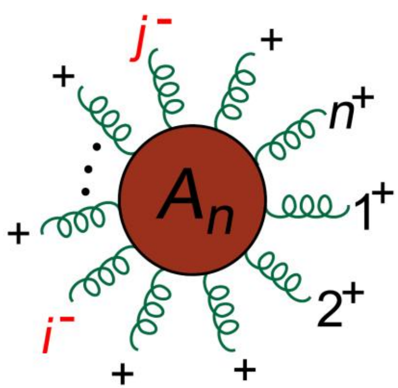
Helicity Formalism Exposes Tree-Level Simplicity in QCD

Many helicity amplitudes either vanish or are very short





Analyticity makes it possible to recycle this simplicity into loop amplitudes



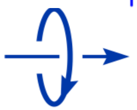
$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

Parke-Taylor formula (1986)

Basic variables: two-component spinors & spinor products

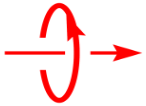
right-handed: $(\lambda_i)_\alpha = u_+(k_i)$

$h = +1/2$



left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

$h = -1/2$



Lorentz products: ~~$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$~~

Better to use spinor products:

$$\bar{u}_-(k_i)u_+(k_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\, j\rangle$$

$$\bar{u}_+(k_i)u_-(k_j) = \varepsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_j)_{\dot{\beta}} = [i\, j]$$

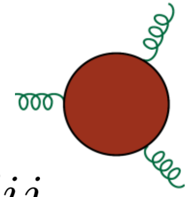
Which always obey: $\langle i\, j\rangle [j\, i] = s_{ij}$

If momenta k_i are **real**, they are **complex square roots** of s_{ij}:

$$\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

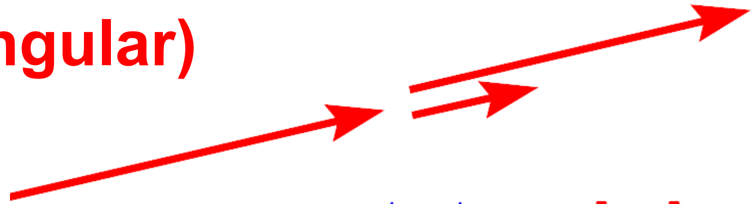
Special Complex Momenta

- Make sense of most basic process with all 3 particles massless



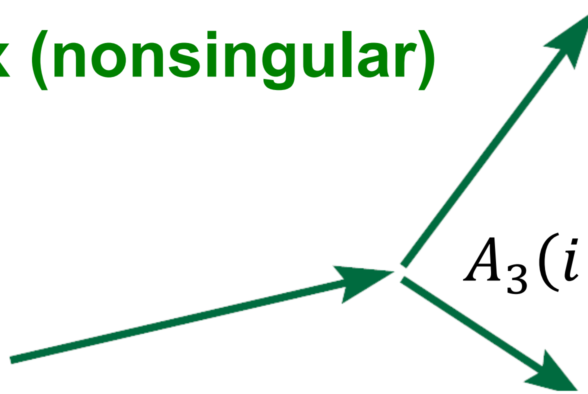
$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 = 0 \quad \forall\ i,j \qquad \langle i\, j\rangle [j\, i] = s_{ij}$$

real (singular)



$$\langle i\, j\rangle = [i\, j] = s_{ij} = 0 \qquad \forall\ i,j$$

complex (nonsingular)

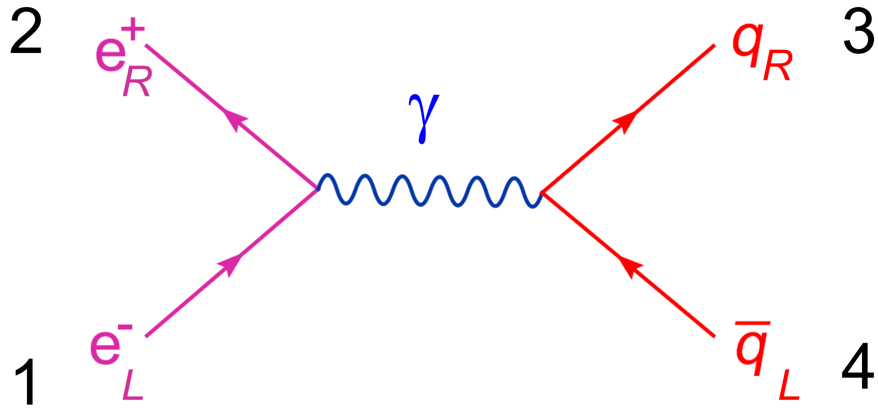


$[i\, j] = 0$ but $\langle i\, j\rangle \neq 0$

$$A_3(i^-, j^-, k^+) = \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 1\rangle}$$

makes sense

Most famous (simplest) Feynman diagram



add helicity information, numeric labels

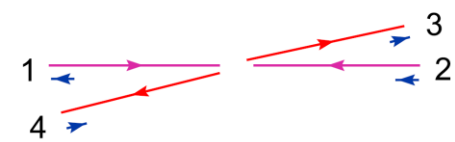
$$\mathcal{A}_4 = 2ie^2 Q_e Q_q \delta^{\bar{\imath}_4}_{i_3} A_4$$

Fierz identity

$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)^{\dot\beta\beta} = 2\ \delta_\alpha^\beta\,\delta_{\dot\alpha}^{\dot\beta}$$

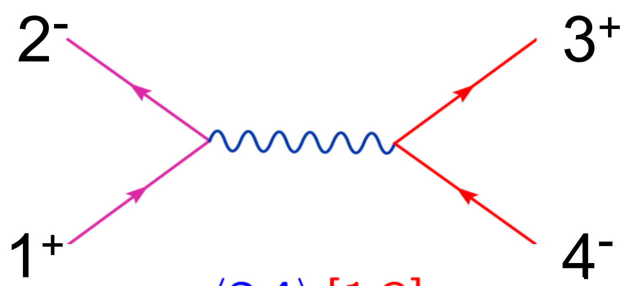
$$\begin{aligned} A_4 &= \frac{1}{2\,s_{12}}\overline{v_-}(k_2)\gamma^\mu u_-(k_1)\ \overline{u_+}(k_3)\gamma_\mu v_+(k_4) \\ &= \frac{1}{2\,s_{12}}(\sigma^\mu)_{\alpha\dot\alpha}(\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}\ (\sigma_\mu)^{\dot\beta\beta}(\tilde\lambda_3)_{\dot\beta}(\lambda_4)_\beta \\ &= \frac{1}{s_{12}}(\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}(\lambda_4)_\alpha(\tilde\lambda_3)_{\dot\alpha} \end{aligned}$$

$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}} = e^{i\phi}\frac{s_{13}}{s_{12}} = \frac{-e^{i\phi}}{2}(1-\cos\theta)$$



helicity suppressed as 1 || 3 or 2 || 4

Sometimes useful to rewrite answer



Crossing symmetry more manifest
if we switch to all-outgoing helicity labels
(flip signs of incoming helicities)

$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}}$$
$$= \frac{\langle 2\,4\rangle\,[1\,3]}{\langle 1\,2\rangle[2\,1]}\frac{\langle 1\,3\rangle}{\langle 1\,3\rangle}$$
$$= -\frac{\langle 2\,4\rangle\,[2\,4]\langle 2\,4\rangle}{\langle 1\,2\rangle[2\,4]\langle 4\,3\rangle}$$

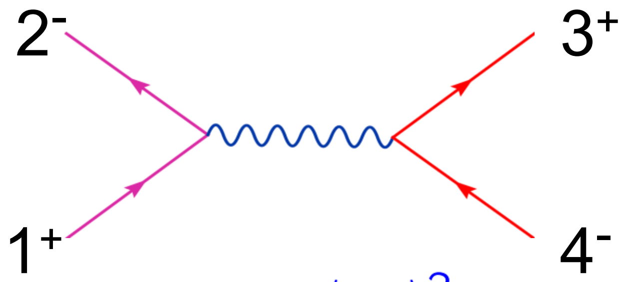
$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$ "holomorphic"

or $$A_4 = \frac{[1\,3]^2}{[1\,2][3\,4]}$$ "antiholomorphic"

useful identities

$$\langle i\,j\rangle = -\langle j\,i\rangle$$
$$[i\,j] = -[j\,i]$$
$$\langle i\,i\rangle = [i\,i] = 0$$
$$\langle i\,j\rangle\,[j\,i] = s_{ij}$$
$$\sum_{j=1}^{4}\langle i\,j\rangle\,[j\,k] = 0$$
$$s_{12} = s_{34}$$
$$s_{13} = s_{24}$$

Symmetries for all other helicity config's

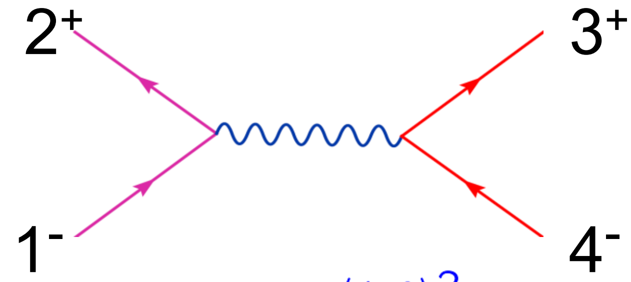


$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

C

$1 \leftrightarrow 2$

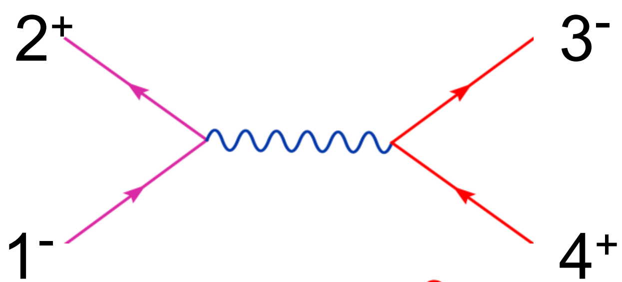
$\Rightarrow$



$$A_4 = -\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

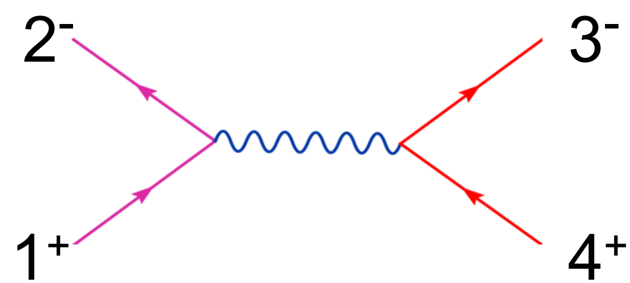
P $\Downarrow$ $\langle\,\rangle \leftrightarrow [\,]$

$\Downarrow$



$$A_4 = \frac{[2\,4]^2}{[1\,2][3\,4]}$$

$\Rightarrow$



$$A_4 = -\frac{[1\,4]^2}{[1\,2][3\,4]}$$

Unpolarized, helicity-summed cross sections

(the norm in QCD)

$$\frac{d\sigma(e^+e^- \to q\bar{q})}{d\cos\theta} \propto \sum_{\text{hel.}} |A_4|^2 = 2\left\{\left|\frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2 + \left|\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2\right\}$$

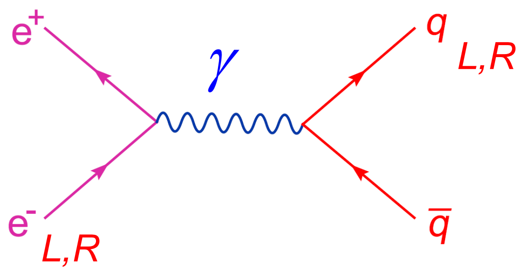
$$= 2\,\frac{s_{24}^2 + s_{14}^2}{s_{12}^2}$$

$$= \frac{1}{2}\left[(1-\cos\theta)^2 + (1+\cos\theta)^2\right]$$

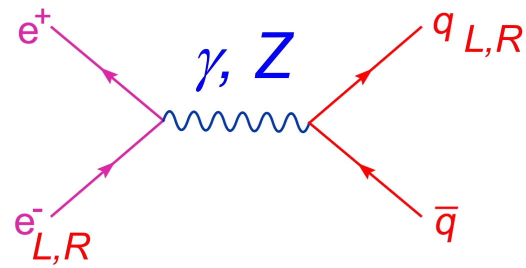
$$= 1 + \cos^2\theta$$

Reweight helicity amplitudes → electroweak/QCD processes

For example, *Z* exchange



$\Rightarrow$

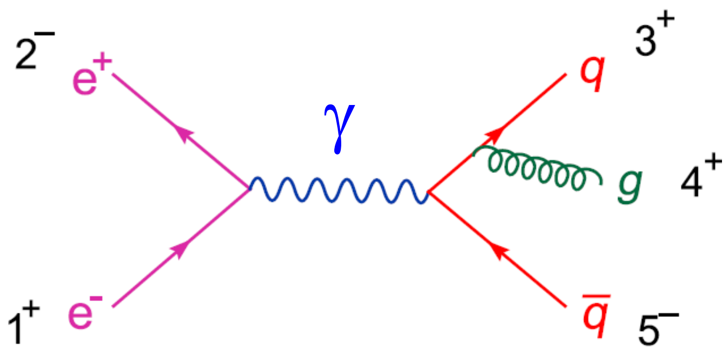


$$Q_e Q_q \quad \Rightarrow \quad Q_e Q_q + \frac{v^e_{L,R} v^q_{L,R} \; s}{s - M_Z^2 + i\Gamma_Z M_Z}$$

$$v^f_L = \frac{2I^f_3 - 2Q_f \sin^2\theta_W}{\sin 2\theta_W} \qquad v^f_R = -\frac{2Q_f \sin^2\theta_W}{\sin 2\theta_W}$$

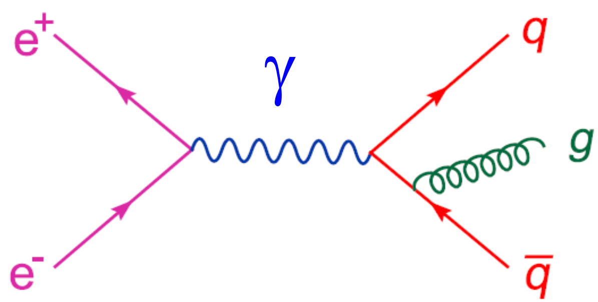
Next most famous pair of Feynman diagrams

(to a higher-order QCD person)



$$\mathcal{A}_5 = 2ie^2 g Q_e Q_q (T^{a_4})_{i_3}^{\bar{\imath}_5} A_5$$

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$



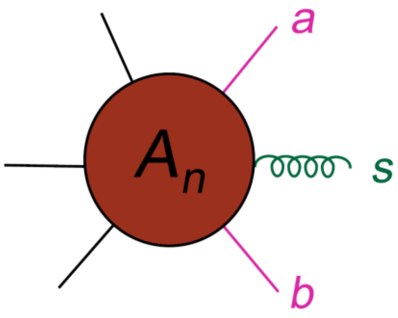
…→ $$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle}$$

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$

1. Soft gluon behavior

$k_4 \to 0$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} = \frac{\langle 3\,5\rangle}{\langle 3\,4\rangle\langle 4\,5\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,5\rangle}$$
$$\to \mathcal{S}(3,4^+,5) \times A_4(1^+,2^-,3^+,5^-)$$



Universal "eikonal" factors for emission of soft gluon s between two hard partons a and b

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle} \quad \left[= \frac{\varepsilon\cdot p_a}{k_s\cdot p_a} - \frac{\varepsilon\cdot p_b}{k_s\cdot p_b}\right]$$
$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Soft emission is from the classical chromoelectric current: independent of parton type (q *vs.* g) and helicity – only depends on momenta of a,b, and color charge

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$ (cont.)

2. Collinear behavior

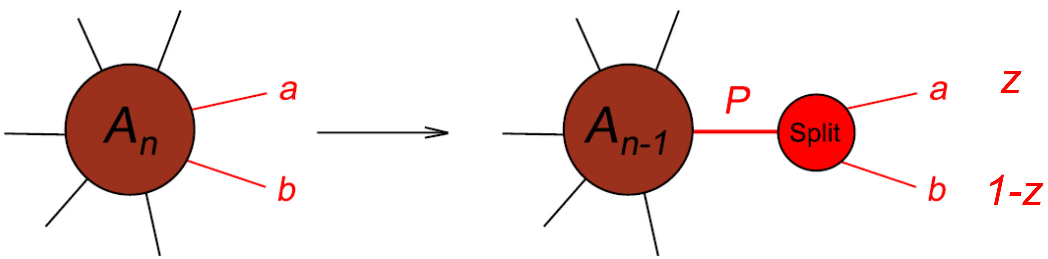
$k_3 \,||\, k_4$: $\quad k_3 = z\, k_P, \quad k_4 = (1-z)\, k_P$

$k_P \equiv k_3 + k_4, \quad k_P^2 \to 0$

$\lambda_3 \approx \sqrt{z}\lambda_P, \quad \lambda_4 \approx \sqrt{1-z}\lambda_P,$ etc.

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{1}{\sqrt{1-z}\,\langle 3\,4\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle P\,5\rangle}$$

$$\to \text{Split}_-(3_q^+, 4_g^+) \times A_4(1^+, 2^-, P^+, 5^-)$$



Time-like kinematics (fragmentation). Space-like (parton evolution) related by crossing

Universal collinear factors, or splitting amplitudes $\text{Split}_{-h_P}(a^{h_a}, b^{h_b})$ depend on parton type and helicity h

Collinear limits (cont.)

We found, from $k_3 \,||\, k_4$:

$$\text{Split}_-(a_q^+, b_g^+) = \frac{1}{\sqrt{1-z}\,\langle a\, b\rangle}$$

Similarly, from $k_4 \,||\, k_5$:

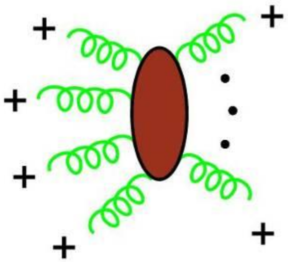
$$\text{Split}_+(a_g^+, b_{\bar{q}}^-) = \frac{1-z}{\sqrt{z}\,\langle a\, b\rangle}$$

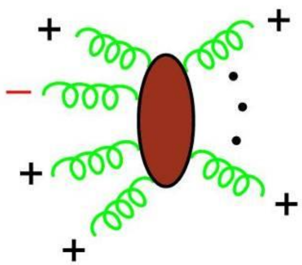
$$\Downarrow$$

Applying C and P:

$$\text{Split}_-(a_q^+, b_g^-) = -\frac{z}{\sqrt{1-z}\,[a\, b]}$$

Simplest pure-gluonic amplitudes

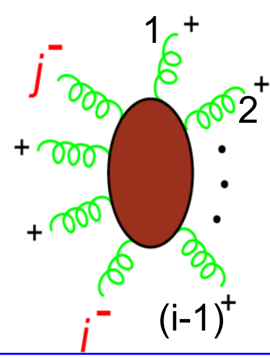
$$A_n^{\text{tree}}(1^\pm, 2^+, \ldots, n^+) \;=$$ 

= 

= 0

Maximally helicity-violating (MHV) amplitudes:

$$A_n^{ij,\ \text{MHV}} \;=\; A_n^{\text{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

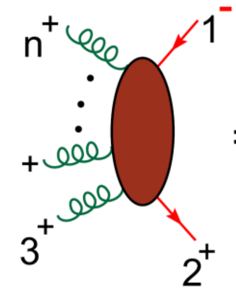
= 

$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

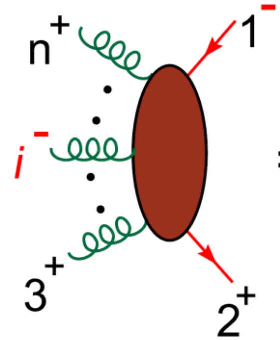
Parke-Taylor formula (1986)

MHV amplitudes with massless quarks

Helicity conservation on fermion line → $A_n^{\rm tree}(1_{\bar q}^{\pm}, 2_q^{\pm}, 3^{h_3}, \ldots, n^{h_n}) \equiv 0$

more vanishing ones: $A_n^{\rm tree}(1_{\bar q}^{-}, 2_q^{+}, 3^{+}, \ldots, n^{+}) =$  $= 0$

the MHV amplitudes:

$A_n^{\rm tree}(1_{\bar q}^{-}, 2_q^{+}, \ldots, i^{-}, \ldots, n^{+}) =$  $= \dfrac{\langle 1\, i\rangle^3 \langle 2\, i\rangle}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$

Related to pure-gluon MHV amplitudes by N=4 supersymmetry: after stripping off color factors, massless quarks ~ gluinos

Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor (1985); Kunszt (1986)

Properties of MHV amplitudes

1. Verify soft limit

$k_s \to 0$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a\, s\rangle\langle s\, b\rangle \cdots \langle n\, 1\rangle} = \frac{\langle a\, b\rangle}{\langle a\, s\rangle\langle s\, b\rangle} \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a\, b\rangle \cdots \langle n\, 1\rangle}$$

$$\to \text{Soft}(a, s^+, b) \times A_{n-1}^{ij,\ \text{MHV}}$$

2. Extract gluonic collinear limits:

$k_a \,||\, k_b \quad (b = a+1)$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a-1, a\rangle\langle a\, b\rangle\langle b, b+1\rangle \cdots \langle n\, 1\rangle} = \frac{1}{\sqrt{z(1-z)}\langle a\, b\rangle} \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a-1, P\rangle\langle P, b+1\rangle \cdots \langle n\, 1\rangle}$$

$$\to \text{Split}_-(a^+, b^+) \times A_{n-1}^{ij,\ \text{MHV}}$$

So

$$\text{Split}_-(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

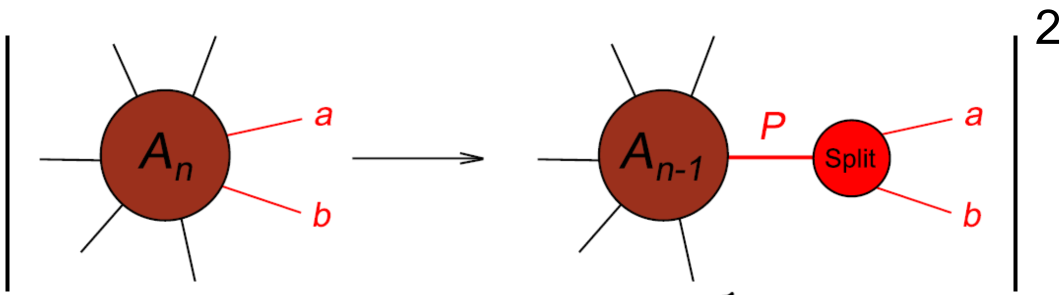
and

$$\text{Split}_+(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

$$\text{Split}_+(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

plus parity conjugates

From splitting amplitudes to probabilities



$$d\sigma_n \sim d\sigma_{n-1} \times \frac{1}{s_{ab}} \times P(z)$$

$$P(z) \propto \sum_{h_P, h_a, h_b} |\text{Split}_{-h_P}(a^{h_a}, b^{h_b})|^2 \, s_{ab}$$

$q \to qg$:

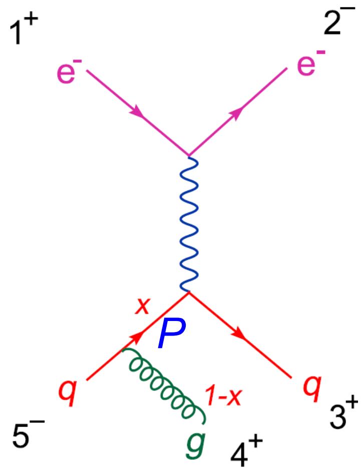
$$P_{qq}(z) \propto C_F \left\{ \left| \frac{1}{\sqrt{1-z}} \right|^2 + \left| \frac{z}{\sqrt{1-z}} \right|^2 \right\} \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$

$$= C_F \frac{1+z^2}{1-z} \qquad z < 1$$

Note soft-gluon singularity as $z_g = 1 - z \to 0$

Space-like splitting

- The case relevant for parton evolution
- Related by crossing to time-like case
- Have to watch out for flux factor, however



$$q \to qg: \quad k_P = x\, k_5, \quad k_4 = (1-x)\, k_5$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{\frac{1}{x}}{\sqrt{\frac{1-x}{x}}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

$$= \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1-x}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

absorb into flux factor: $d\sigma_5 \propto \frac{1}{s_{15}}$ $d\sigma_4 \propto \frac{1}{s_{1P}} = \frac{1}{x\, s_{15}}$

When dust settles, get exactly the same splitting kernels (at LO)

Similarly for gluons

$g \to gg$:

$$P_{gg}(z) \propto C_A \left\{ \left| \frac{1}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{z^2}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{(1-z)^2}{\sqrt{z(1-z)}} \right|^2 \right\}$$

$$= C_A \frac{1+z^4+(1-z)^4}{z(1-z)} \qquad C_A = N_c$$

$$= 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \qquad z < 1$$

Again a soft-gluon singularity. Gluon number not conserved.
But momentum is. Momentum conservation mixes $g \to gg$ with

$g \to q\bar{q}$:

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right] \qquad T_R = \frac{1}{2}$$

(can deduce, up to color factors, by taking $e^+ \,||\, e^-$ in $\mathcal{A}_5(e^+e^- \to qg\bar{q})$)

Gluon splitting (cont.)

$g \to gg$:

Applying momentum conservation,

Exercise:
Work out b_0

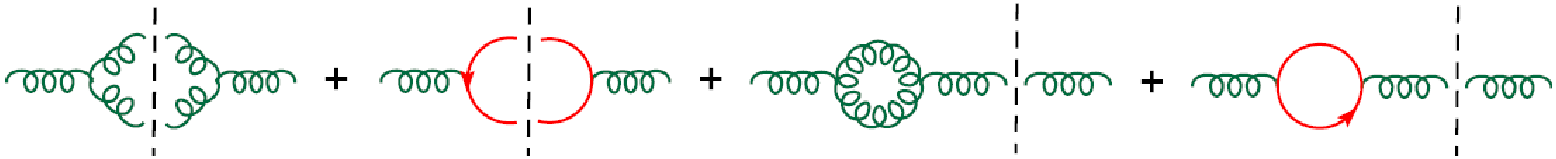
$$\int_0^1 dz\, z\, \left[P_{gg}(z) + 2n_f P_{qg}(z)\right] \;=\; 0$$

gives

$$P_{gg}(z) \;=\; 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] \;+\; b_0\,\delta(1-z)$$

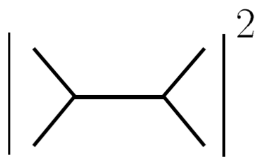
$$b_0 \;=\; \frac{11C_A - 4n_f T_R}{6}$$

Amusing that first β-function coefficient enters,
since no loops were done, except implicitly via unitarity:

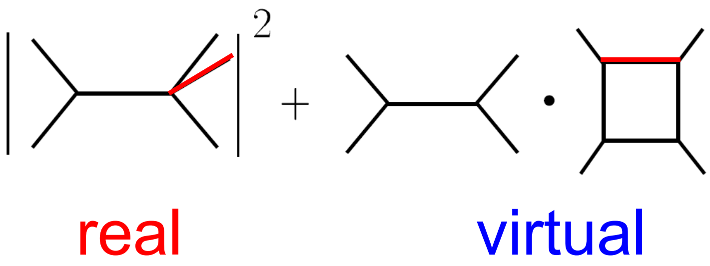


“Suitable” final state, and Infrared divergences in QCD

LO



NLO



Usually regulate dimensionally, $D = 4 - 2\epsilon$

$d^4k \;\to\; d^{4-2\epsilon}k$ in all phase-space and loop integrals

soft singularities: $k_s \to 0$

$$\sigma^{\text{real}} \sim \int \frac{dk_s^2}{k_s^{2(1+\epsilon)}}\,\sigma^{\text{LO}}(k_s = 0)$$

collinear singularities: $k_{ab}^2 \to 0 \quad (k_a \,||\, k_b)$

$$\sim \int \frac{dk_{ab}^2}{k_{ab}^{2(1+\epsilon)}}\,\sigma^{\text{LO}}(k_P)$$

virtual soft/collinear singularities:

$$\sigma^{\text{virt}} \sim \Big[-\frac{1}{\epsilon^2}\sum_i C_i - \frac{1}{\epsilon}\sum_{i,j} D_{ij}\ln\Big(\frac{\mu^2}{-s_{ij}}\Big)\Big]\,\sigma^{\text{LO}}$$

- Virtual corrections cancel real singularities, but only for quantities insensitive to soft/collinear radiation → infrared-safe **observables** ***O***

Infrared safety

Infrared-safe observables *O*:
- Behave smoothly in soft limit as any parton momentum → 0
- Behave smoothly in collinear limit as any pair of partons → parallel (||)

$$O_n(\ldots, k_s, \ldots) \;\to\; O_{n-1}(\ldots, \not{k_s}, \ldots) \qquad k_s \to 0$$

$$O_n(\ldots, k_a, k_b, \ldots) \;\to\; O_{n-1}(\ldots, k_P, \ldots) \qquad k_a \,||\, k_b$$

- Cannot predict perturbatively any infrared-unsafe quantity, such as:
  - the number of partons (hadrons) in an event
  - observables requiring no radiation in some region (rapidity gaps or overly strong isolation cuts)
  - p_T(*W*, *Z* or Higgs) precisely at $p_T = 0$