The Electroweak Standard Model

facilitating precision physics at LHC and beyond

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Bad Honnef

Lecture 1

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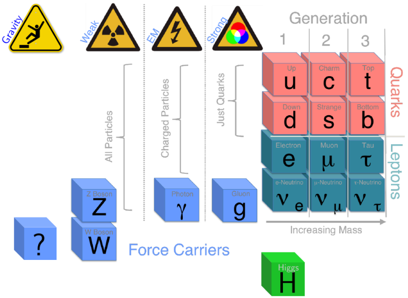
MCnet

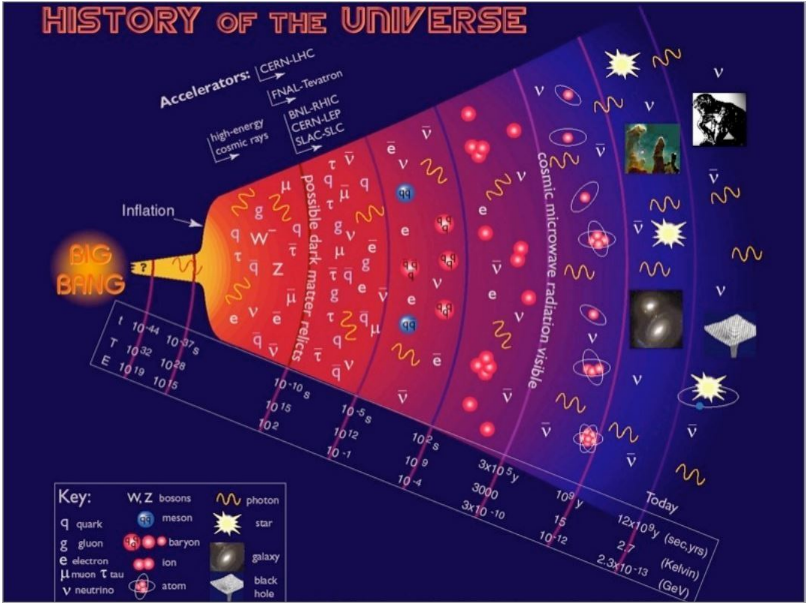
From empirical to predictive science

- SM accommodates *almost* all known particle-physics phenomena
- construction paradigms
  - theory shall be Lorentz covariant, i.e. valid in arbitrary frames
  - interactions modelled by gauge fields, gauge symmetry of Lagrangian
  - account for quantum effects, i.e. consistent Quantum Field Theory
  - theory shall be renormalizable, i.e. valid at arbitrary scales
  - Occam's razor, i.e. pick the simplest/minimal solutions
- provides means to predict outcome of new experiments/observables

  $\rightsquigarrow$ limited set of inputs, perturbative and non-perturbative calculations



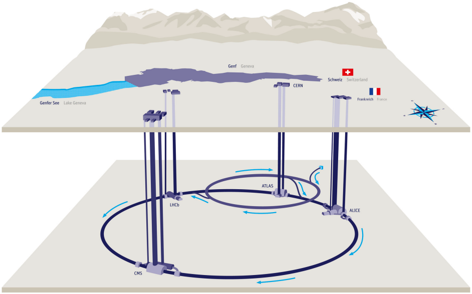
Invitation to the Standard Model: cosmological perspective



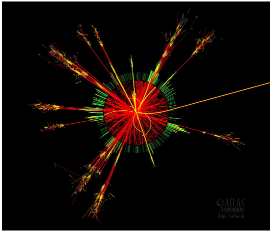
Invitation to the Standard Model: LHC perspective

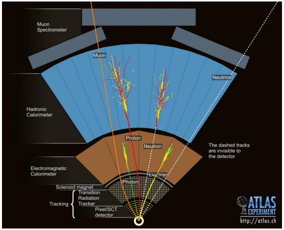
LHC experimental conditions

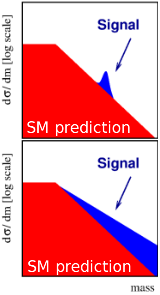
- 27 km circumference accelerator
- colliding protons of 6.8 TeV energy
- bunch crossing every 25 ns
- four interaction points
  $\hookrightarrow$ ATLAS, CMS, LHCb, ALICE

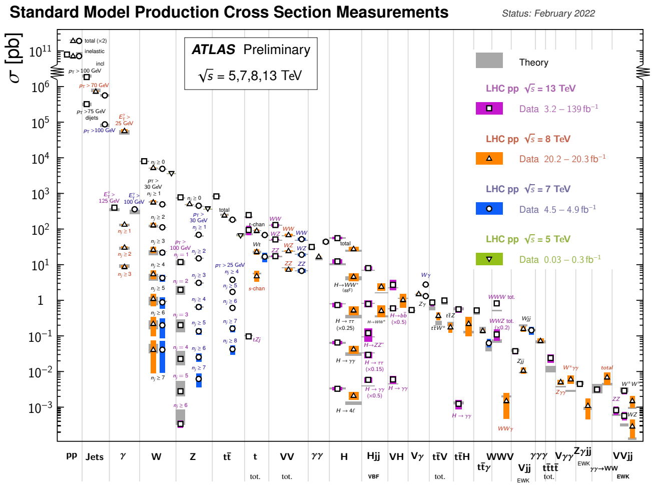


need to identify & measure final states of individual pp clashes



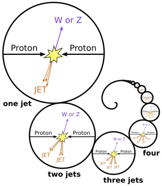


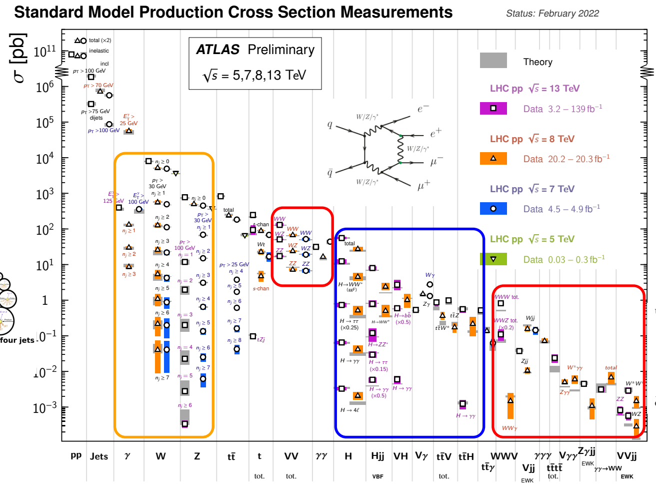


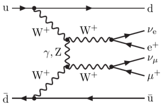


- rare & subtle electroweak signal processes probing EWSB
- triple-, quartic gauge-boson couplings, Higgs production & decay
- longitudinal polarisation modes of massive gauge bosons

Scrutinizing the Standard Model







- rare & subtle electroweak signal processes probing EWSB
- triple-, quartic gauge-boson couplings, Higgs production & decay
- longitudinal polarisation modes of massive gauge bosons

Outline of the lectures

1. **Gauge field theory basics** (Wed)
   - gauge theory of electro-magnetism
   - renormalization: a toy example
2. **The electroweak Lagrangian** (Thu)
   - chiral fermions & non-abelian gauge symmetries
   - $SU(2)_L \otimes U(1)_Y$ symmetric Lagrangian
   - SSB and the Higgs mechanism
3. **Quantum theory of EW interactions** (Fri)
   - inputs & predictions of the EW Standard Model
   - QED & EW quantum corrections
4. **Selected current topics in EW physics** (Sat)
   - simulating the Standard Model: Monte Carlo generators
   - multi-boson production processes
   - gauge boson polarizations

Getting started: QED basics

- Lagrangian for vector field $A^\mu = (\Phi, \vec{A})$ coupled to current $J^\mu = (\rho, \vec{j})$

$$\mathcal{L}_{\text{em}}(A, \partial A) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu\,, \quad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- Euler–Lagrange equations yield Maxwell theory of electro-magnetism

$$\partial_\mu \left(\frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}_{\text{em}}}{\partial A_\nu} = 0 \quad \rightsquigarrow \quad \boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

$\rightsquigarrow$ continuity equation implicit: $\partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0$

- relation to electric & magnetic fields (transverse)

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A} \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

- physical fields invariant under gauge transformation of vector potential

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi$$

Getting started: QED basics

- consider a single Dirac fermion ψ, Lorentz-trafo: $\psi \xrightarrow{\Lambda} S(\Lambda)\psi$

$$\mathcal{L}_{\text{Dirac}}(\psi, \partial\psi) = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,, \quad \text{with} \quad \partial\!\!\!/ = \gamma^\mu\partial_\mu \quad \& \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times 4}$$

- Euler–Lagrange equation yields covariant Dirac equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\psi} = 0 \quad \rightsquigarrow \quad \boxed{(i\partial\!\!\!/ - m\mathbb{1}_{4\times 4})\psi = 0}$$

- Dirac field bilinears, transforming under Lorentz-trafo Λ

$$\begin{array}{lcllcl} S & = & \bar{\psi}\psi & & : & \text{scalar} \\ A & = & \bar{\psi}\gamma_5\psi & & : & \text{pseudo-scalar} \\ J^\mu & = & \bar{\psi}\gamma^\mu\psi & & : & \text{vector} \\ J_5^\mu & = & \bar{\psi}\gamma^\mu\gamma_5\psi & & : & \text{pseudo-vector} \\ T^{\mu\nu} & = & \bar{\psi}\,\underbrace{\frac{i}{2}[\gamma^\mu, \gamma^\nu]}_{\sigma^{\mu\nu}}\,\psi & & : & \text{tensor} \end{array}$$

$\rightsquigarrow$ with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ resulting in $\{\gamma_5, \gamma^\mu\} = 0$

Getting started: QED basics

- assign fermion a charge q and minimally couple it to vector field

$$\begin{aligned}\mathcal{L}_{\mathsf{QED}}(\psi,\partial\psi,A,\partial A) &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}(i\partial\!\!\!/-m)\psi+q\bar{\psi}\gamma^\mu\psi A_\mu \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}(iD\!\!\!\!/-m)\psi\end{aligned}$$

$\rightsquigarrow$ with $D\!\!\!\!/=\gamma^\mu D_\mu$ and $D_\mu=\partial_\mu-iqA_\mu$ the *covariant derivative*

- coupled Euler–Lagrange equations of motion

$$\boxed{\begin{aligned}\partial_\mu F^{\mu\nu} &= q\bar{\psi}\gamma^\nu\psi \\ (i\partial\!\!\!/+qA\!\!\!/-m)\psi &= 0\end{aligned}}$$

Getting started: QED basics

- assign fermion a charge q and minimally couple it to vector field

$$\begin{aligned}\mathcal{L}_{\text{QED}}(\psi, \partial\psi, A, \partial A) &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi + q\bar{\psi}\gamma^{\mu}\psi A_{\mu} \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(iD\!\!\!\!/ - m)\psi\end{aligned}$$

$\rightsquigarrow$ with $D\!\!\!\!/ = \gamma^{\mu}D_{\mu}$ and $D_{\mu} = \partial_{\mu} - iqA_{\mu}$ the *covariant derivative*

- coupled Euler–Lagrange equations of motion

$$\begin{aligned}\partial_{\mu}F^{\mu\nu} &= q\bar{\psi}\gamma^{\nu}\psi \\ (i\partial\!\!\!/ + qA\!\!\!/ - m)\psi &= 0\end{aligned}$$

- consider simultaneous transformation of spinor and vector field

$$\psi \to \psi' = e^{iq\chi}\psi\,,\ \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{-iq\chi} \quad \text{and} \quad A^{\mu} \to A'^{\mu} = A^{\mu} + \partial^{\mu}\chi$$

$\rightsquigarrow$ $D_{\mu}\psi \to D'_{\mu}\psi' = e^{iq\chi}D_{\mu}\psi$, hence *covariant* derivative

$\rightsquigarrow$ leaves $\mathcal{L}_{\text{QED}}$ unchanged, i.e. $\mathcal{L}_{\text{QED}}$ is called $U(1)$ gauge invariant

$\rightsquigarrow$ successive gauge trafos commute: $e^{iq\chi_1}e^{iq\chi_2} = e^{iq\chi_2}e^{iq\chi_1} = e^{iq(\chi_1+\chi_2)}$

quantization & Feynman rules

- quantization via path integral or canonical commutation relations
- for a valid photon propagator, gauge needs to be fixed, e.g. R_ξ gauges:

$$\mathcal{L}_{\mathsf{QED}} \to \mathcal{L}_{\mathsf{QED}} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \rightsquigarrow \boxed{D^{\mu\nu}(p) = \frac{-\eta^{\mu\nu} + (1-\xi)p^\mu p^\nu/p^2}{p^2 + i\epsilon}}$$

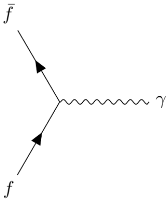
[physical results gauge independent, different choices ease calculations]

- fermion propagator given by

$$\boxed{S_{\mathsf{F}}(k,m) = \frac{\not{k} + m\mathbb{1}_{4\times 4}}{k^2 - m^2 + i\epsilon}}$$

- photon–fermion interaction vertex, e.g. $q_{e^-} = -1$

$$\boxed{V_{f\bar{f}\gamma} = iq_f e\gamma^\mu}$$



Getting started: QED basics

inputs & predictions

- universal QED coupling times fermion's charge factor q_f
- QED coupling strength set by Sommerfeld's fine-structure constant

$$\alpha(0) \stackrel{\text{SI}}{=} \frac{e^2}{4\pi\epsilon_0\hbar c} \stackrel{\text{n.u.}}{=} \frac{e^2}{4\pi} = 1/137.035\,999\,177(21)$$

[CODATA Rev Mod Phys 97 (2025) 2]

$\rightsquigarrow$ e elementary charge, fundamental physical constant

$\rightsquigarrow$ suitable as perturbative expansion parameter

- fermion masses, e.g. m_e, m_μ, m_τ, ...

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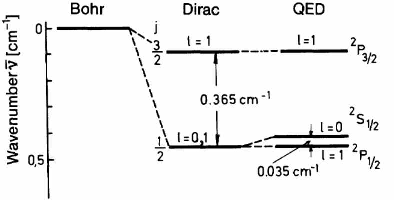
⇝ e elementary charge, fundamental physical constant

⇝ suitable as perturbative expansion parameter

- fermion masses, e.g. m_e, m_μ, m_τ, ...
- QED predictions crucial for atomic physics & HEP precision experiments

Lamb shift Hydrogen ($n = 2$)



[Haken, Wolf: Springer 2004]

QED lifts degeneracy of states with same j but different l

The muon anomalous magnetic dipole moment

- part of Lamb shift from electron anomalous magnetic moment
- for Dirac fermion ψ_f in external field it holds $(i\partial\!\!\!/ + q_f e A\!\!\!/ - m_f)^2\psi_f = 0$

$$\left((i\partial_\mu + q_f e A_\mu)^2 + \frac{q_f e}{2} F^{\mu\nu}\sigma_{\mu\nu} - m_f^2\right)\psi_f = 0$$

- $\vec{B}$ field coupling to spin magnetic dipole moment $\boxed{\vec{\mu}_{s,f} = g_f \frac{q_f e}{2m_f}\vec{S}}$

  $\rightsquigarrow$ gyromagnetic factor $g_f = 2$ in Dirac theory $(S = 1/2)$

  $\rightsquigarrow$ with 1st order QED corrections $g_f = 2 + \frac{\alpha}{\pi} \approx 2.002\,32$

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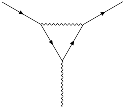

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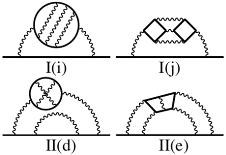
- measure deviations from Dirac value for muons by $a_\mu = \frac{g_\mu - 2}{2}$

  $a_\mu^{\mathsf{SM}} = a_\mu^{\mathsf{QED}} + a_\mu^{\mathsf{EW}} + a_\mu^{\mathsf{QCD}}$, up to five-loop QED, i.e. $\mathcal{O}(\alpha^5)$

$$\begin{aligned} a_\mu^{\mathsf{QED}} &= 116\,584\,718.8(2) \times 10^{-11} \\ a_\mu^{\mathsf{SM}} &= 116\,592\,033(62) \times 10^{-11} \\ a_\mu^{\mathsf{exp}} &= 116\,592\,059(22) \times 10^{-11} \end{aligned}$$

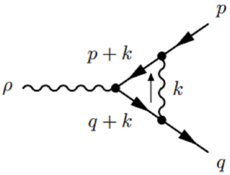
[Aliberti et al. 2505.21476]



[Aoyama et al. PRL 109 (2012) 111807]

Getting started: the need for renormalization

- quantum corrections to QED propagators & vertex exhibit UV divergences
  $\rightsquigarrow$ loop-momentum integrals divergent, e.g. 1-loop vertex correction



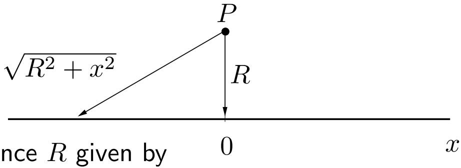
$$
\begin{aligned}
ie\Gamma^{\mu}_{\xi=1} &= i\int \frac{d^4k}{(2\pi)^4}\frac{\eta^{\nu\lambda}}{k^2} ie\gamma_\nu \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2} ie\gamma^\mu \frac{\not{q}+\not{k}+m}{(q+k)^2-m^2} ie\gamma_\lambda \\
&\propto \int^{\infty} dk^2 k^2 \frac{1}{k^2}\frac{k}{k^2}\frac{k}{k^2} \propto \ln(k\to\infty)
\end{aligned}
$$

outline for QFT regularization & renormalization program

(i) identify ways to *regularize* infinite integrals
(ii) absorb divergent terms into redefined fields & parameters (mass, coupling) typically via subtractions, called *renormalization*
(iii) check independence of physical results on regularization scheme

Renormalization: a classical example

- consider wire of constant line-charge density λ, $[\lambda] = [\text{length}]^{-1}$
- want to evaluate electric potential & electric field at point P



- potential at distance R given by

$$\Phi(R) = \lambda \int_{-\infty}^{\infty} \frac{dx}{\sqrt{R^2 + x^2}} \qquad \left(\int \frac{dx}{\sqrt{R^2 + x^2}} = \ln(x + \sqrt{R^2 + x^2}) \right)$$

$\rightsquigarrow$ logarithmically divergent, thus ill defined

$\rightsquigarrow$ but, potential is not a physical observable

- corresponding electric field well defined and finite

$$E(R) = -\frac{d\Phi(R')}{dR'}\bigg|_R = \lambda R \int_{-\infty}^{\infty} \frac{dx}{(R^2 + x^2)^{3/2}} \quad \left(\int \frac{dx}{(R^2 + x^2)^{3/2}} = \frac{x}{R^2\sqrt{R^2 + x^2}} \right)$$

Renormalization: a classical example

- attempt to give operational meaning to potential $\Phi(R)$
  $\rightsquigarrow$ regularize integral through a cut-off length Λ

$$\Phi_\Lambda(R) = \lambda \int_{-\Lambda}^{\Lambda} \frac{dx}{\sqrt{R^2 + x^2}} = \lambda \ln\left(\frac{\sqrt{R^2+\Lambda^2}+\Lambda}{\sqrt{R^2+\Lambda^2}-\Lambda}\right)$$

- obtain electric field via

$$\vec{E}(R) = \lim_{\Lambda\to\infty}\left(-\vec{\nabla}\Phi_\Lambda(R)\right) = \lim_{\Lambda\to\infty}\frac{2\lambda}{R}\frac{\Lambda}{\sqrt{R^2+\Lambda^2}}\vec{e}_r = \frac{2\lambda}{R}\vec{e}_r$$

<u>notice</u>, we just introduced dimensionful parameter Λ
$\rightsquigarrow$ regularizes potential, physical field independent

further notice

- well defined potential difference

$$\delta\Phi = \lim_{\Lambda\to\infty} \left(\Phi_\Lambda(R_2) - \Phi_\Lambda(R_1)\right) = \lambda \ln\left(\frac{R_1^2}{R_2^2}\right)$$

- allows to define potential even for $\Lambda \to \infty$ by subtracting $\Phi(R)$ at some arbitrary fixed value $R = R_0$

$$\boxed{\Phi(R) \to \Phi(R) - \Phi(R_0) = \lambda \ln\left(\frac{R_0^2}{R^2}\right)}$$

$\rightsquigarrow$ unphysical infinities in $\Phi(R)$ and $\Phi(R_0)$ cancel out

$\rightsquigarrow$ obtain finite result with non-trivial R dependence

$\rightsquigarrow$ note, we again introduced a dimensionful parameter R_0

Renormalization: a classical example

<u>last but not least</u>

- cut-off regularization breaks translational invariance,
  consider shift $x \to x' = x + c$

$$\Phi_\Lambda(R) = \lambda \int\limits_{-\Lambda+c}^{\Lambda+c} \frac{dx}{\sqrt{R^2 + x^2}} = \lambda \ln \left(\frac{\sqrt{R^2 + (\Lambda + c)^2} + (\Lambda + c)}{\sqrt{R^2 + (\Lambda - c)^2} - (\Lambda - c)} \right)$$

$\rightsquigarrow$ explicit dependence on shift c, symmetry broken

$\rightsquigarrow$ in QFT regularization preserving systems' symmetries beneficial

Renormalization: a classical example

last but not least

- cut-off regularization breaks translational invariance,
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$\rightsquigarrow$ explicit dependence on shift c, symmetry broken

$\rightsquigarrow$ in QFT regularization preserving systems' symmetries beneficial

dimensional regularization

- alternative approach: Bollini, Giambiagi & 't Hooft (1972/73)
  $\rightsquigarrow$ regulate integrals by modifying space-time dimensions
- for our toy example, work in $D = 1 - 2\epsilon < 1$ dimensions, such

$$\boxed{dx = d^1x \to d^Dx = d\Omega_{D-1}x^{D-1}dx}$$

- to ensure $[\Phi] = [\Phi_D]$ introduce auxiliary scale $[\bar{\mu}] = $ [length]

Dimensional regularization: a classical example

$$\begin{aligned}
\Phi_D(R) &= \lambda\bar{\mu}^{1-D}\int_0^\infty \int_{\Omega_{D-1}} d\Omega_{D-1} x^{D-1}\frac{dx}{\sqrt{R^2+x^2}} \\
&= \lambda\bar{\mu}^{1-D}\frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\left(\frac{1}{R^2}\right)^{-D/2}\frac{1}{2\pi^{1/2}R}\Gamma\left(\frac{1}{2}-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}\right) \\
&= \lambda\exp\left\{\epsilon\ln\left(\frac{\bar{\mu}^2}{\pi R^2}\right)\right\}\Gamma(\epsilon) \\
&\stackrel{\epsilon\to 0}{=} \lambda\left(1+\epsilon\ln\left(\frac{\bar{\mu}^2}{\pi R^2}\right)+\mathcal{O}(\epsilon^2)\right)\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right) \\
&= \lambda\left(\frac{1}{\epsilon}+\ln\left(\frac{e^{-\gamma_E}}{\pi}\right)+\ln\left(\frac{\bar{\mu}^2}{R^2}\right)\right)+\mathcal{O}(\epsilon)
\end{aligned}$$

- $\Phi_D(R)$ depends on regulator ϵ & scale $\bar{\mu}^2$, ill-defined when removing either
- can define subtracted potential, equivalent to cut-off regularization

$$\delta\Phi = \lim_{\epsilon\to 0}\left(\Phi_D(R_2)-\Phi_D(R_1)\right) = \lambda\ln\left(\frac{R_1^2}{R_2^2}\right)$$

Dimensional regularization: a classical example

- can derive electric field from $\Phi_D(R)$ via

$$\vec{E}(R) = \lim_{\epsilon\to 0}\left(-\vec{\nabla}\Phi_D(R)\right)\vec{e}_r = \lim_{\epsilon\to 0}\left(\lambda\frac{2\epsilon\bar{\mu}^{2\epsilon}}{R^{1+2\epsilon}\pi^{\epsilon}}\Gamma(\epsilon)\right)\vec{e}_r = \frac{2\lambda}{R}\vec{e}_r \quad \checkmark$$

- define (minimal) subtracted renormalized potentials

$$\Phi_D^{MS}(R) = \lambda\left(\ln\left(\frac{e^{-\gamma_E}}{\pi}\right) + \ln\left(\frac{\bar{\mu}^2}{R^2}\right) + \mathcal{O}(\epsilon)\right)$$

$$\Phi_D^{\overline{MS}}(R) = \lambda\left(\ln\left(\frac{\bar{\mu}^2}{R^2}\right) + \mathcal{O}(\epsilon)\right)$$

$\rightsquigarrow$ can safely remove regulator ϵ, however, *not* the scale $\bar{\mu}$

Summary lecture I

- QED as quantized $U(1)$ symmetric field theory, dynamical fields A^μ and ψ

$$\mathcal{L}_{\mathsf{QED},0} = -\frac{1}{4}F_0^{\mu\nu}F_{0,\mu\nu} - \frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2 + \bar{\psi}_0(i\partial\!\!\!/ + qe_0 A\!\!\!/_0 - m_0)\psi_0$$

- infinite renormalizations of bare parameters (m_0, ξ_0, e_0) and fields ($A_{0,\mu}$, ψ_0)

$$\begin{aligned}
\psi_0 &= Z_2^{1/2}\psi \\
A_{0,\mu} &= Z_3^{1/2}A_\mu \\
e_0 &= Z_1 Z_2^{-1} Z_3^{-1/2} e \\
\xi_0 &= Z_3\xi \\
m_0 &= m + \delta m
\end{aligned}$$

- divergences can be subtracted order-by-order, i.e. $Z_i^{\mathsf{MS}} = 1 + \sum_{n=1}^{\infty} \frac{c_{i,n}}{\epsilon^n}$

  $\rightsquigarrow$ bare Lagrangian can be split into physical part and counterterms

$$\mathcal{L}_{\mathsf{QED},0} = \mathcal{L}_{\mathsf{QED}} + \mathcal{L}_{\mathsf{QED},ct}$$

  $\rightsquigarrow$ $\mathcal{L}_{\mathsf{QED}}$ same operators as $\mathcal{L}_{\mathsf{QED},0}$, QED is renormalizable QFT

Backup

dimensional regularization

- alternative regularization method: Bollini, Giambiagi & 't Hooft (1972/73)
  - $\rightsquigarrow$ regulate integrals by modifying space-time dimensions
  - $\rightsquigarrow$ proto-typical Euclidean QFT integral (propagator factor)

$$I(M^2) = i\int \frac{d^4 l_E}{(2\pi)^4}\frac{1}{(l_E^2+M^2)^2}$$

(Wick rotation: $l^0 = il^0_E$, $\vec{l} = \vec{l}_E$, $d^4 l = id^4 l_E$ [suppress index E in what follows])

  - $\rightsquigarrow$ $I(M^2)$ divergent for $D=4$, however, finite for $D<4$

$$I_D(M^2) = i\bar{\mu}^{4-D}\int \frac{d^D l}{(2\pi)^D}\frac{1}{(l^2+M^2)^2}$$

  - $\rightsquigarrow$ introduced dimensionful scale $\bar{\mu}$, such that $[I]=[I_D]=0$

dimensional regularization cont'd

- in Euclidean metrics it holds

$$d^D l = d\Omega_{D-1} l^{D-1} dl \qquad \hookrightarrow \text{ solid-angle element } d\Omega_{D-1}$$

$$\begin{aligned}
\int d^D l e^{-\vec{l}^2} &= \left(\int dl e^{-l^2}\right)^D = \pi^{D/2} \\
&= \Omega_{D-1} \int dl l^{D-1} e^{-l^2} = \frac{\Omega_{D-1}}{2} \int dl^2 \left(l^2\right)^{\frac{D-2}{2}} e^{-l^2} \\
&\overset{t=l^2}{=} \frac{\Omega_{D-1}}{2} \int_0^\infty dt t^{\frac{D-2}{2}} e^{-t} = \frac{\Omega_{D-1}}{2} \Gamma(D/2)
\end{aligned}$$

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

n positive integer

$\Gamma(n) = (n-1)!$

$\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$

$\Omega_1 \overset{D=2}{=} 2\pi$

$\Omega_2 \overset{D=3}{=} 4\pi$

$\Omega_3 \overset{D=4}{=} 2\pi^2$

- back to the integral $I_D(M^2)$

$$\begin{aligned}
I_D(M^2) &= \frac{2\pi^{D/2}}{(2\pi)^D}\frac{i\bar{\mu}^{4-D}}{\Gamma(D/2)}\int_0^\infty dl\, l^{D-1}\frac{1}{(l^2+M^2)^2} \\
&= \frac{1}{(4\pi)^{D/2}}\frac{i\bar{\mu}^{4-D}}{\Gamma(D/2)}\int_0^\infty dl^2 l^{D-2}\frac{1}{(l^2+M^2)^2} \\
&\overset{t=l^2}{=} \frac{1}{(4\pi)^{D/2}}\frac{i\bar{\mu}^{4-D}}{\Gamma(D/2)}\int_0^\infty dt\, t^{\frac{D-2}{2}}\frac{1}{(t+M^2)^2} \\
&\overset{t'=t/M^2}{=} \frac{1}{(4\pi)^{D/2}}\frac{i\bar{\mu}^{4-D}}{\Gamma(D/2)}\left(M^2\right)^{D/2-2}\int_0^\infty dt'\,(t')^{\frac{D}{2}-1}\frac{1}{(t'+1)^2}
\end{aligned}$$

$$B(\alpha,\beta) = \int_0^\infty dt\,\frac{t^{\alpha-1}}{(t+1)^{\alpha+\beta}} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$ Beta-function

identify $\alpha = \frac{D}{2}$ and $\beta = 2 - \frac{D}{2}$

$$\begin{aligned} I_D(M^2) &= \frac{1}{(4\pi)^{D/2}} \frac{i\bar{\mu}^{4-D}}{\Gamma(D/2)} \left(M^2\right)^{D/2-2} \frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)} \\ &= \frac{i\bar{\mu}^{4-D}}{(4\pi)^{D/2}} \left(M^2\right)^{D/2-2} \frac{\Gamma(2-D/2)}{\Gamma(2)} \end{aligned}$$

note, $\Gamma(x)$ has isolated poles at $x = 0, -1, -2, \ldots$ (here $D = 4, 6, 8, \ldots$)
$\rightsquigarrow$ we are interested in the case $D = 4$, thus consider

$$D = 4 - 2\epsilon$$

$\rightsquigarrow$ need to understand the behaviour for $\epsilon \to 0$

$$\begin{aligned} I_D(M^2) &= \frac{i\bar{\mu}^{2\epsilon}}{(4\pi)^{2-\epsilon}} \left(M^2\right)^{-\epsilon} \Gamma(\epsilon) \\ &= \frac{i}{(4\pi)^2} \left(\frac{4\pi\bar{\mu}^2}{M^2}\right)^{\epsilon} \Gamma(\epsilon) = \frac{i}{(4\pi)^2} \exp\left\{\epsilon \ln\left(\frac{4\pi\bar{\mu}^2}{M^2}\right)\right\} \Gamma(\epsilon) \end{aligned}$$

Renormalization: dimensional regularization

interlude on the Gamma-function

- from the definition follows recurrence relation

$$\Gamma(x) = \int_0^\infty dt e^{-t} t^{x-1} \qquad \boxed{\Gamma(\epsilon) = \Gamma(1+\epsilon)/\epsilon}$$

- expansion of $\Gamma(1+\epsilon)$ yields

$$\Gamma(1+\epsilon) = 1 - \gamma_E \epsilon + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

with γ_E the Euler–Mascheroni constant

$$\gamma_E = -\int_0^\infty dt e^{-t} \ln t = 0.5772156649\ldots$$

- we thus find for $\Gamma(\epsilon)$

$$\boxed{\Gamma(\epsilon) = \Gamma(1+\epsilon)/\epsilon \overset{\epsilon\to 0}{=} \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)}$$

Renormalization: dimensional regularization

back to our desired integral ...

$$I_D(M^2) \quad = \quad \frac{i}{(4\pi)^2}\left(\frac{4\pi\bar{\mu}^2}{M^2}\right)^{\epsilon}\Gamma(\epsilon) = \frac{i}{(4\pi)^2}\exp\left\{\epsilon\ln\left(\frac{4\pi\bar{\mu}^2}{M^2}\right)\right\}\Gamma(\epsilon)$$

$\hookrightarrow$ expanding the exponential for $\epsilon \to 0$ yields

$$\exp\left\{\epsilon\ln\left(\frac{4\pi\bar{\mu}^2}{M^2}\right)\right\} \overset{\epsilon\to 0}{=} 1 + \epsilon\ln\left(\frac{4\pi\bar{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon^2)$$

- collating all our findings we finally get

$$I_D(M^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{\epsilon} - \gamma_E + \ln\left(\frac{4\pi\bar{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon)\right\}$$

- divergence successfully regulated by finite ϵ, pole for $\epsilon \to 0$, *i.e.* $D \to 4$

  $\rightsquigarrow$ two parameters appear
  - ϵ dimensionless regulator
  - $\bar{\mu}$ dimensionful scale, $[\bar{\mu}] = [\text{mass}]$

Renormalization: renormalization schemes

- can renormalize integral by subtraction (subtraction point M_0^2)

$$I(M^2) \to I(M^2) - I(M_0^2) = \frac{i}{(4\pi)^2} \ln\left(\frac{M_0^2}{M^2}\right)$$

$\rightsquigarrow$ allows to remove regulator, well defined in $D = 4$ dimensions

- alternative, simply subtract the very pole

$$\begin{aligned} I_{\mathsf{MS}}(M^2) &= \frac{i}{(4\pi)^2}\left\{-\gamma_E + \ln\left(\frac{4\pi\bar{\mu}^2}{M^2}\right)\right\} \\ &= \frac{i}{(4\pi)^2}\left\{\ln\left(4\pi e^{-\gamma_E}\right) + \ln\left(\frac{\bar{\mu}^2}{M^2}\right)\right\} \end{aligned}$$

$\rightsquigarrow$ called *Minimal Subtraction Scheme*, *i.e.* MS

- subtract pole & constants, *Modified Minimal Subtraction*, *i.e.* $\overline{\mathsf{MS}}$

$$I_{\overline{\mathsf{MS}}}(M^2) = \frac{i}{(4\pi)^2} \ln\left(\frac{\bar{\mu}^2}{M^2}\right)$$

$\rightsquigarrow$ auxiliary scale remains through dimensionless ratio $\bar{\mu}^2/M^2$