Complete NLO QCD Corrections to Z-boson Pair Production in Gluon-Fusion

Stephen Jones
IPPP Durham / Royal Society URF

Agarwal, Kerner, von Manteuffel
[2011.12325, 2404.05684]

+ Heinrich, Jahn, Langer, Magerya, Olsson, Põldaru, Schlenk, Villa
[2108.10807, 2305.19768]

Durham University

THE ROYAL SOCIETY

Outline

Motivation

Calculating the Massive 2-loop Amplitude

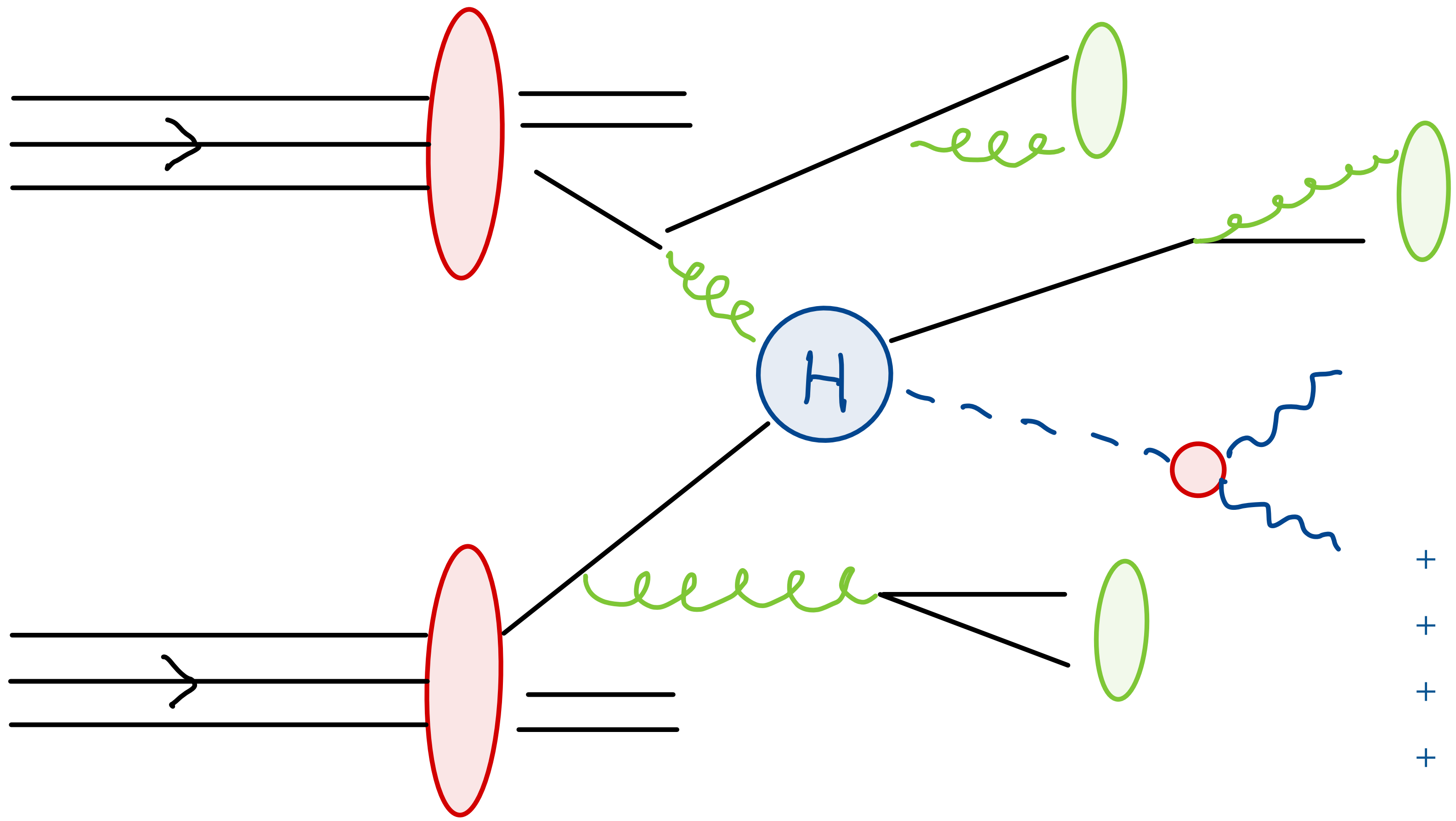
- Amplitude generation
- Integral reduction
- Numerical calculation of Feynman integrals

Results

- $gg \to ZZ$ @ NLO via massive quark loops
- $gg \to ZZ$ @ NLO massless + massive quark loops

Outlook

Collider Precision



+ Parton Shower
+ Resummation
+ Hadronisation
+ Underlying Event

$$d\sigma = \int dx_a dx_b f_a(x_a) f_b(x_b) d\hat{\sigma}_{ab}(x_a, x_b) F_J + \mathcal{O}\left((\Lambda/Q)^m\right)$$

Parton Distribution Functions (PDFs)

Hard Scattering Matrix Element

Non-perturbative effects ~ few %

Perturbation Theory

At sufficiently high energies $\alpha_s \sim 0.1$ can expand

$$\sigma = \alpha^j \alpha_s^k \left(\sigma^{LO} + \alpha_s \sigma^{\mathrm{NLO}} + \alpha_s^2 \sigma^{\mathrm{NNLO}} + \alpha_s^3 \sigma^{\mathrm{N3LO}} + \ldots + \alpha \sigma^{\mathrm{NLO\,(EW)}} + \ldots + \alpha \alpha_s \sigma^{\mathrm{NNLO\,(QCD-EW)}}\right)$$

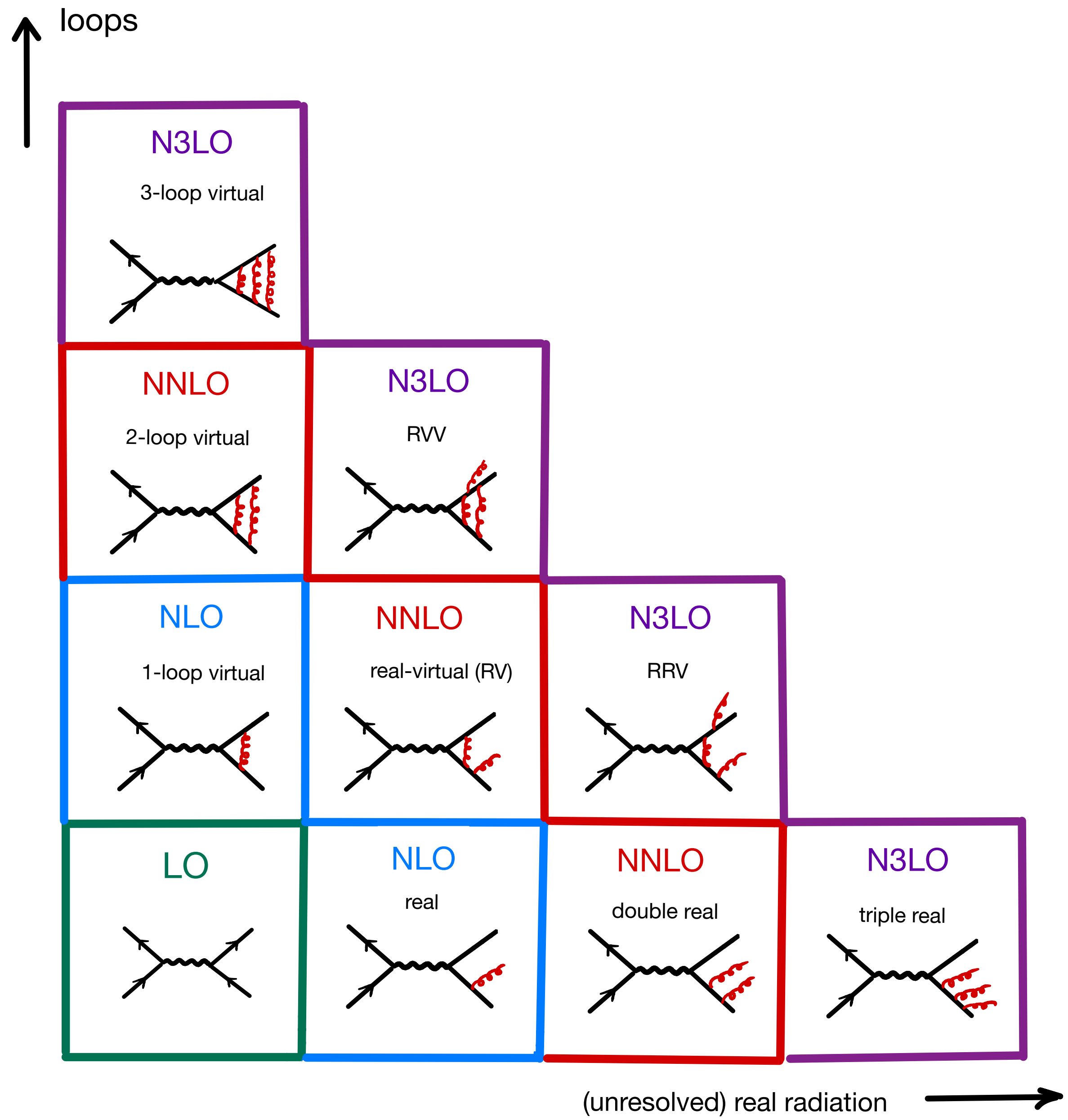


Figure ($e^+e^- \to$ jets) : G. Heinrich

$\alpha \sim 0.01$ EW effects can be enhanced

$$\sigma^{\mathrm{LO}} = \int_n \mathrm{d}\sigma^B$$

$$\sigma^{\mathrm{NLO}} = \int_n \mathrm{d}\sigma^V + \int_{n+1} \mathrm{d}\sigma^R$$

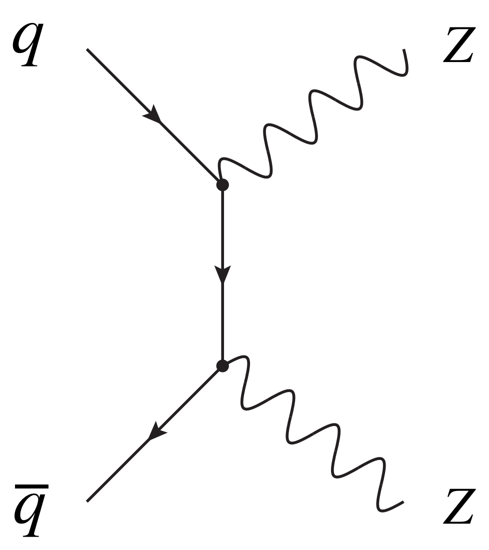
$$\sigma^{\mathrm{NNLO}} = \int_n \mathrm{d}\sigma^{VV} + \int_{n+1} \mathrm{d}\sigma^{RV} + \int_{n+2} \mathrm{d}\sigma^{RR}$$

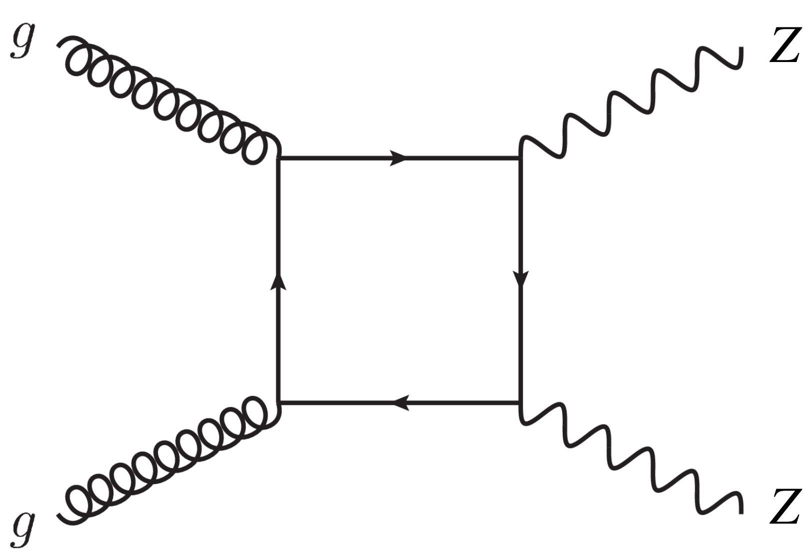
$$\vdots$$

Beyond LO each piece separately Infrared (IR) divergent in 4 dim.

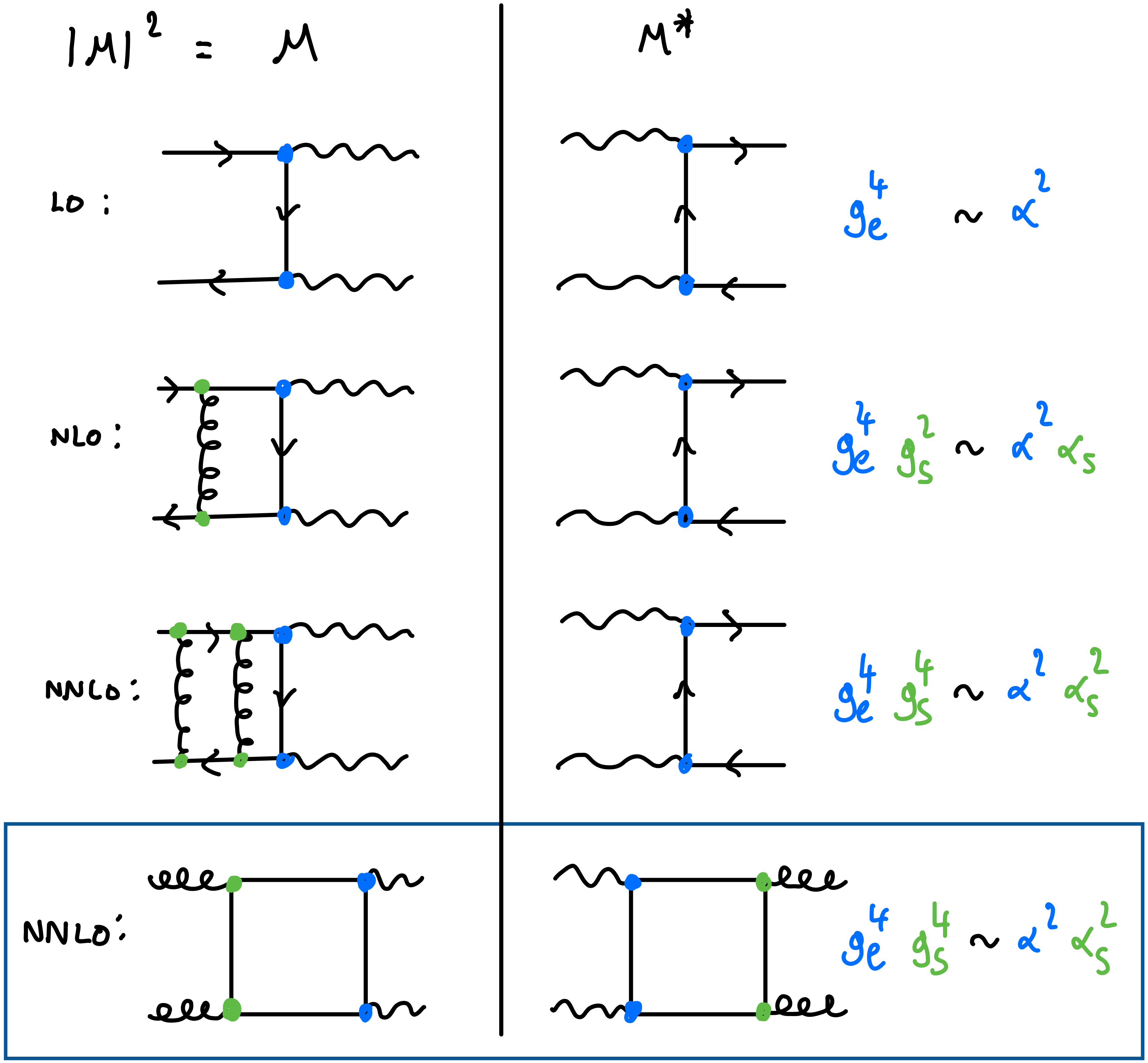
$$d = 4 - 2\epsilon$$

Motivation: ZZ Production





Overview of $pp \to ZZ$



The $gg \to ZZ$ channel contributes to $pp \to ZZ$ starting at NNLO in QCD

However due the large gluon-gluon luminosity at the LHC:

Contributes significantly to the total cross section (5-10%)

Accounts for 60% of NNLO corrections

Expected to have very large 2-loop contribution

Motivation

Why calculate Z boson pair production via gluon-fusion?

Precision measurements

Background to Higgs production through gluon fusion

CMS 18; ATLAS 20;

Higgs width

Provides indirect constraints on Higgs width via off-shell Higgs production

ATLAS 18; CMS 19;

BSM Searches

Searches for heavy diboson resonances decaying to 4 lepton final states

ATLAS 20; CMS 23;

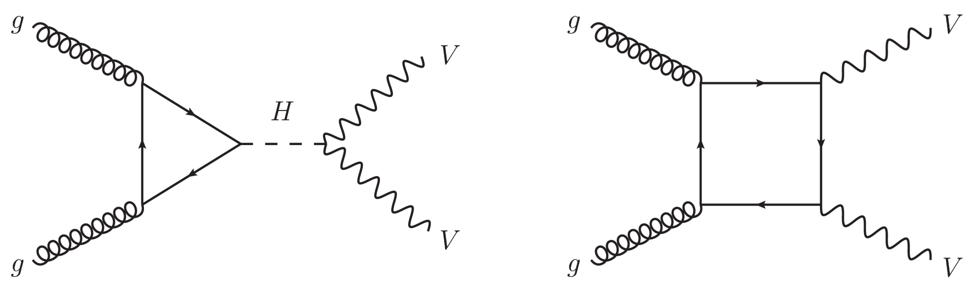
Anomalous couplings

Provides constraints on anomalous $t\bar{t}Z$ & triple gauge couplings

ATLAS 23

Higgs Width

Channel opens @ NNLO for $pp \to ZZ$, interferes with $pp \to H \to ZZ$ @ LO



Signal Prompt/Background

$$|A_{ZZ}|^2 = |A_H|^2 + |A_b|^2 + 2\mathrm{Re}[A_H A_b^*]$$

$$\to \sigma_{\mathrm{full}} = \sigma_{\mathrm{sigl}} + \sigma_{\mathrm{bkgd}} + \sigma_{\mathrm{intf}}$$

Figure: Raoul Röntsch (ICHEP 2016)

Strong destructive interference with Higgs amplitude probes unitarizing behaviour of the Higgs

Off-shell measurements can provide model-dependent indirect constraints on Higgs properties (e.g. width)

$$\sigma^{\mathrm{on-shell}}_{gg\to H\to ZZ} \sim \frac{g^2_{ggF} g^2_{HZZ}}{m_H \Gamma_H}$$

$$\sigma^{\mathrm{off-shell}}_{gg\to H\to ZZ} \sim \frac{g^2_{ggF} g^2_{HZZ}}{m^2_{ZZ}}$$

ATLAS $\Gamma_H = 4.5^{+3.3}_{-2.5}$ MeV

CERN-EP-2023-03

CMS $\Gamma_H = 3.2^{+2.4}_{-1.7}$ MeV

CERN-EP-2021-272

Kauer, Passarino 12; Caola, Melnikov 13;
Campbell, Ellis, Williams 14;
(See also: Englert, Spannowsky 14)

Anomalous Couplings

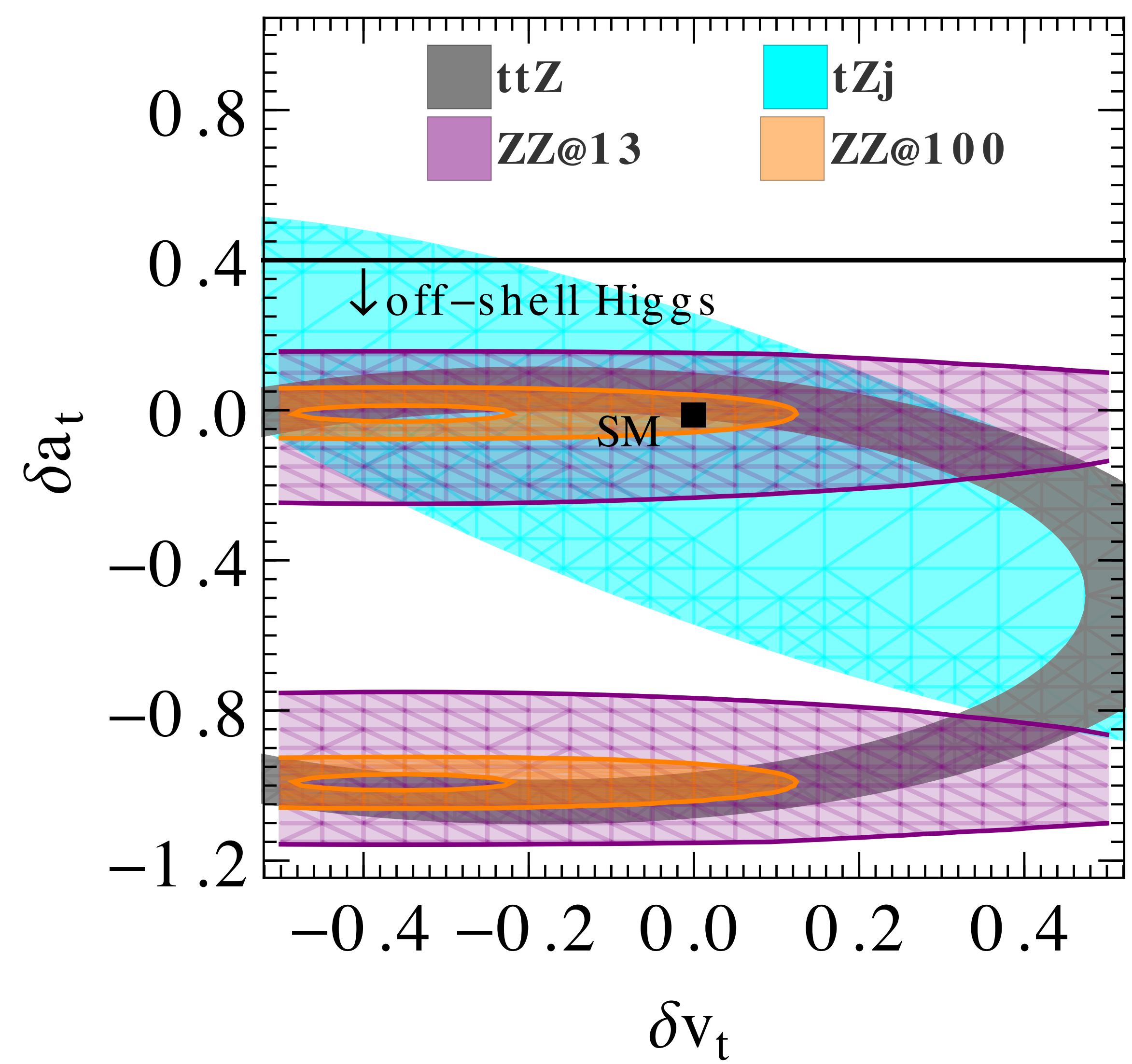
vector $\sim v_t \gamma_\mu$ axial-vector $\sim a_t \gamma_\mu \gamma_5$

$$\mathcal{V}_\mu^{Vf\bar{f}} = i \frac{e}{2 \sin\theta_W \cos\theta_W} \gamma_\mu \left(v_t + a_t \gamma_5\right)$$

For $m_q \neq 0$ production of longitudinally polarised ZZ via $gg \to ZZ$ can probe axial-vector coupling ($a_t = 1/2$ in SM)

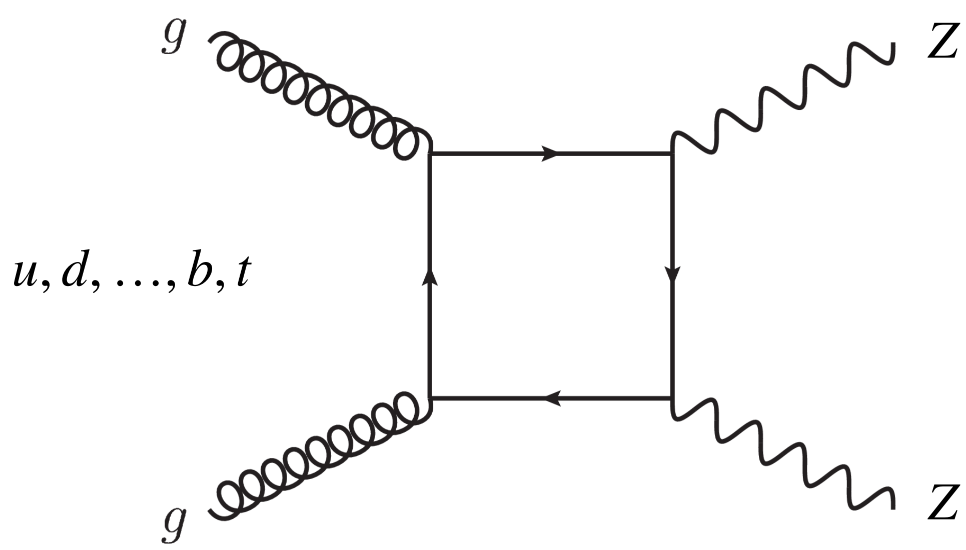
$$\mathcal{M}_{\pm\pm 00} \sim \frac{m_t^2}{m_Z^2}\left(a_t^2 - \frac{1}{4}\right)\left[\ln^2\left(\frac{s}{m_t^2}\right) - 2i\pi \ln\left(\frac{s}{m_t^2}\right)\right]$$

Complements measurements of $t\bar{t}Z, tZj$



Cao, Yan, Yuan, Zhang 20

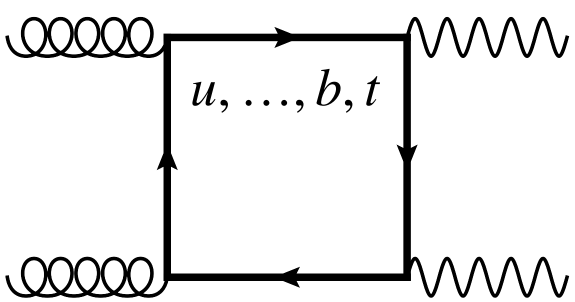
Calculation: $gg \to ZZ$



Known Amplitudes for $gg \to ZZ$

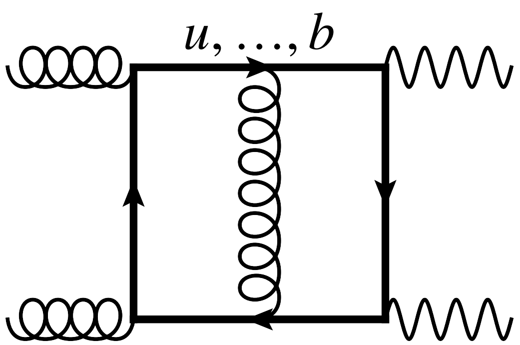
Full leading order (loop induced)

Glover, van der Bij 89



NLO (massless quark loop contribution)

von Manteuffel, Tancredi 15



NLO expansion around large top quark mass

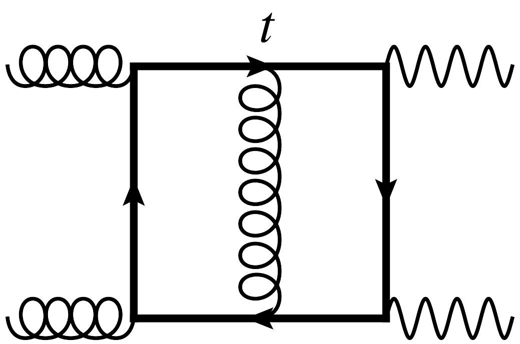
Melnikov, Dowling 15; Caola, Dowling, Melnikov, Röntsch, Tancredi 16

+ Padé approx   Campbell, Ellis, Czakon, Kirchner 16

+ Threshold (Higgs int.)   Gröber, Maier, Rauh 19

NLO expansion around small top quark mass

Davies, Mishima, Steinhauser, Wellmann 20



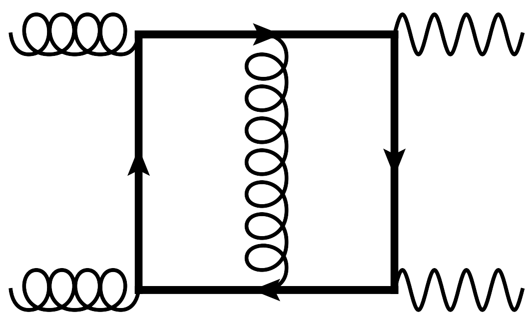
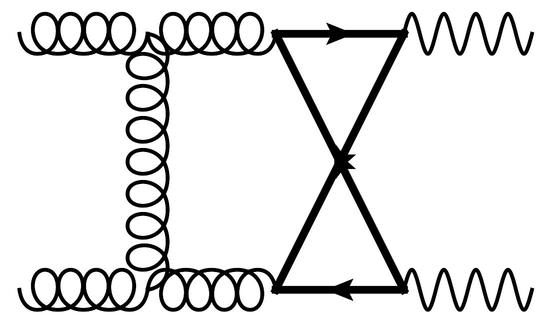
NLO amplitudes (massive quark loop) obtained via sector decomposition or series solutions of differential equations

Agarwal, SJ, von Manteuffel 20; Brønnum-Hansen, Wang 21

NLO amplitudes small-p_T + small top quark mass

Degrassi, Gröber, Vitti 24

Virtual Amplitudes

$\mathscr{M}^{\mu\nu\rho\sigma} \sim$  $+$  $+ \ldots$

$$\mathscr{M}^{\mu\nu\rho\sigma} = \sum_i A_i T_i^{\mu\nu\rho\sigma}, \qquad A_i = \sum_k C_{i,k} \; I_k$$

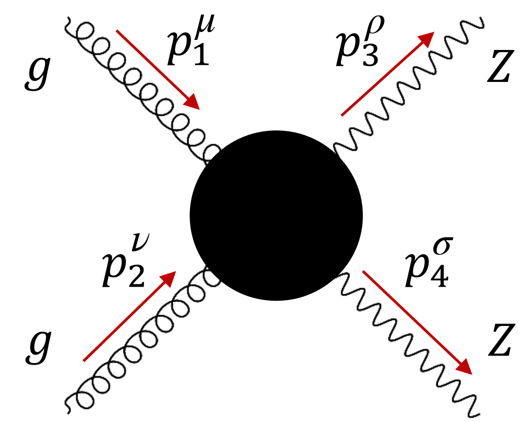
Rational functions
Large num. terms/ high degree
Handled with specialist symbolic manipulation programs

Feynman integrals
Analytically: Involved special functions (Polylogs, Elliptic…)

We compute them numerically

Decomposition

Tensor structure $\mathcal{M} = \mathcal{M}_{\mu\nu\rho\sigma}\, \epsilon_1^{\mu}(p_1)\, \epsilon_2^{\nu}(p_2)\, \epsilon_3^{*\rho}(p_3)\, \epsilon_4^{*\sigma}(p_4)$



→ 138 parity-even tensor structures

\+ transversality $\epsilon_1(p_1).p_1 = 0, \epsilon_2(p_2).p_2 = 0$

\+ gauge fixing $\epsilon_1(p_1).p_2 = 0, \epsilon_2(p_2).p_1 = 0, \epsilon_3(p_3).p_3 = 0, \epsilon_4(p_4).p_4 = 0$

→ 20 tensor structures remain

\+ Bose symmetry 9 indep. + 11 related by crossing

$$\mathcal{M}^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \sum_{i=1}^{20} A_i(s, t, m_t^2, m_Z^2)\, T_i^{\mu\nu\rho\sigma}$$

$$\begin{array}{llll}
T_1^{\rho\sigma\mu\nu} = g^{\mu\nu} g^{\rho\sigma} & T_2^{\rho\sigma\mu\nu} = g^{\mu\rho} g^{\nu\sigma} & T_3^{\rho\sigma\mu\nu} = g^{\mu\sigma} g^{\nu\rho} & T_4^{\rho\sigma\mu\nu} = p_1^{\rho}\, p_1^{\sigma}\, g^{\mu\nu} \\
T_5^{\rho\sigma\mu\nu} = p_1^{\rho}\, p_2^{\sigma}\, g^{\mu\nu} & T_6^{\rho\sigma\mu\nu} = p_1^{\sigma}\, p_2^{\rho}\, g^{\mu\nu} & T_7^{\rho\sigma\mu\nu} = p_2^{\rho}\, p_2^{\sigma}\, g^{\mu\nu} & T_8^{\rho\sigma\mu\nu} = p_1^{\sigma}\, p_3^{\nu}\, g^{\mu\rho} \\
T_9^{\rho\sigma\mu\nu} = p_2^{\sigma}\, p_3^{\nu}\, g^{\mu\rho} & T_{10}^{\rho\sigma\mu\nu} = p_1^{\rho}\, p_3^{\nu}\, g^{\mu\sigma} & T_{11}^{\rho\sigma\mu\nu} = p_2^{\rho}\, p_3^{\nu}\, g^{\mu\sigma} & T_{12}^{\rho\sigma\mu\nu} = p_1^{\sigma}\, p_3^{\mu}\, g^{\nu\rho} \\
T_{13}^{\rho\sigma\mu\nu} = p_2^{\sigma}\, p_3^{\mu}\, g^{\nu\rho} & T_{14}^{\rho\sigma\mu\nu} = p_1^{\rho}\, p_3^{\mu}\, g^{\nu\sigma} & T_{15}^{\rho\sigma\mu\nu} = p_2^{\rho}\, p_3^{\mu}\, g^{\nu\sigma} & T_{16}^{\rho\sigma\mu\nu} = p_3^{\mu}\, p_3^{\nu}\, g^{\rho\sigma} \\
T_{17}^{\rho\sigma\mu\nu} = p_1^{\rho}\, p_1^{\sigma}\, p_3^{\mu}\, p_3^{\nu} & T_{18}^{\rho\sigma\mu\nu} = p_1^{\rho}\, p_2^{\sigma}\, p_3^{\mu}\, p_3^{\nu} & T_{19}^{\rho\sigma\mu\nu} = p_2^{\rho}\, p_1^{\sigma}\, p_3^{\mu}\, p_3^{\nu} & T_{20}^{\rho\sigma\mu\nu} = p_2^{\rho}\, p_2^{\sigma}\, p_3^{\mu}\, p_3^{\nu}
\end{array}$$

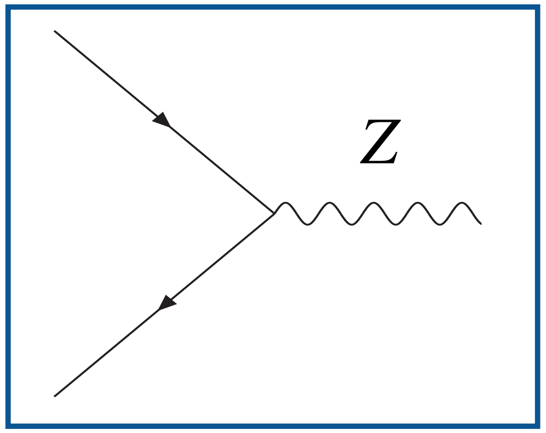
von Manteuffel, Tancredi 15

Our task is now to compute the 20 scalar form factors $A_i(s, t, m_t^2, m_Z^2)$

Dimensional Regularisation & γ_5

vector $\sim v_t \gamma_\mu$ axial-vector $\sim a_t \gamma_\mu \gamma_5$

$$\mathcal{V}_\mu^{Vf\bar{f}} = i \frac{e}{2 \sin\theta_W \cos\theta_W} \gamma_\mu \left(v_t + a_t \gamma_5\right)$$



In dim. reg. ($d = 4 - 2\epsilon$) we can't retain all properties of γ_5 in $d \neq 4$ dimensions

Larin Scheme

Sacrifice anti-commuting property of γ_5

$$J_\mu^5 = Z_{5,ns}\, J_{\mu,B}^5 = Z_{5,ns} \left[\frac{i}{3\,!} \epsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\nu \gamma^\rho \gamma^\sigma \bar{\psi}\right]$$

$$P^5 = Z_{5,p}\, P_B^5 = Z_{5,p} \left[\frac{i}{4\,!} \epsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \bar{\psi}\right]$$

Fix Ward identities/ABJ anomaly:

$$Z_{5,ns} = 1 + \alpha_s(-4C_F) + \ldots$$
$$Z_{5,p} = 1 + \alpha_s(-8C_F) + \ldots$$

Larin, Vermaseren 91; Larin 93

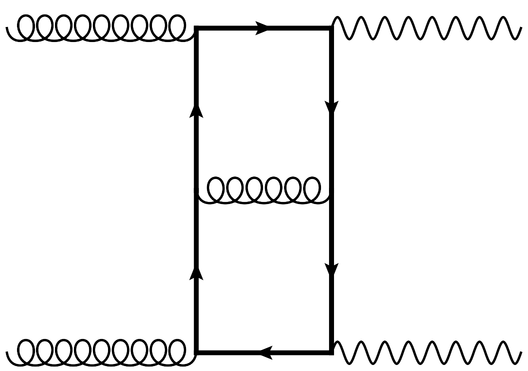
Kreimer Scheme

Retain $\{\gamma_5, \gamma^\mu\} = 0$, but, sacrifice cyclicity of traces involving γ_5

Define `reading point' and carefully manipulate all traces

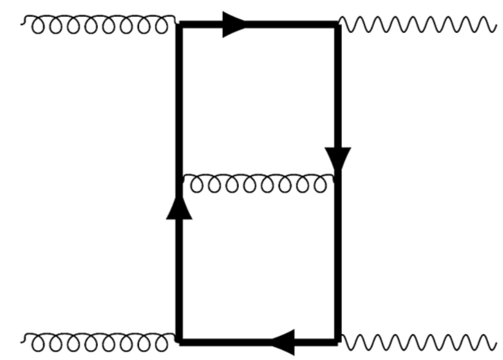
Kreimer 90; Korner, Kreimer, Schilcher 92

Diagrams



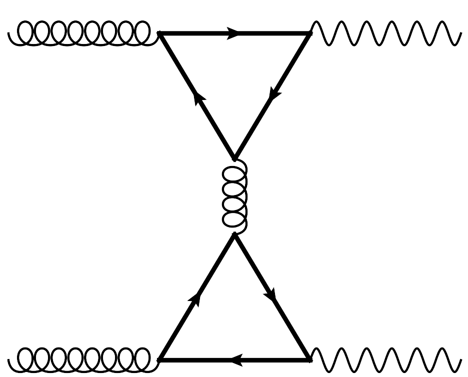
Class $\mathbf{A}_l$
Z bosons couple to same massless fermion line

von Manteuffel, Tancredi 15



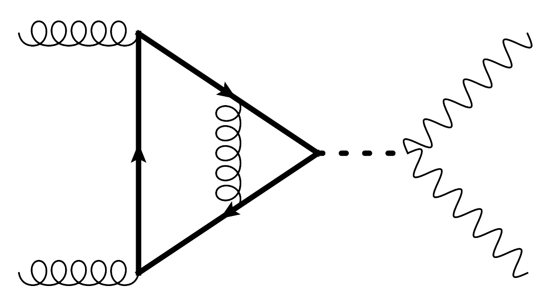
Class $\mathbf{A}_h$
Z bosons couple to same massive fermion line

Agarwal, SPJ, von Manteuffel 20



Class B
Z bosons couple to different fermion lines

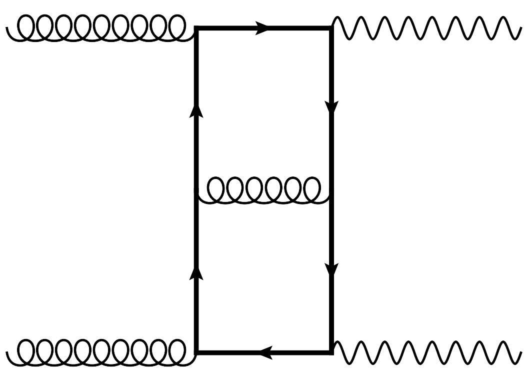
Campbell, Ellis, Czakon, Kirchner 16



Class C
Z bosons couple to Higgs boson

Djouadi, Spira, Zerwas 91

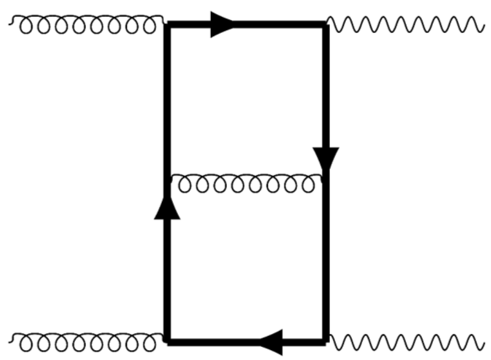
Diagrams



Class $\mathbf{A}_l$

Z bosons couple to same massless fermion line

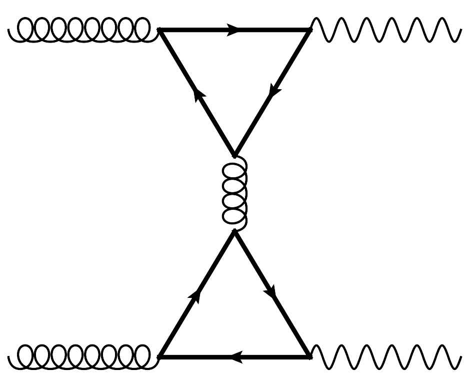
von Manteuffel, Tancredi 15



Class $\mathbf{A}_h$

Z bosons couple to same massive fermion line

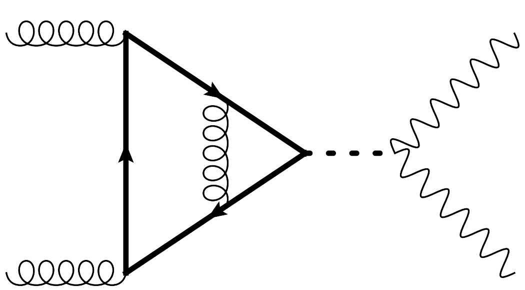
Agarwal, SPJ, von Manteuffel 20



Class B

Z bosons couple to different fermion lines

Campbell, Ellis, Czakon, Kirchner 16



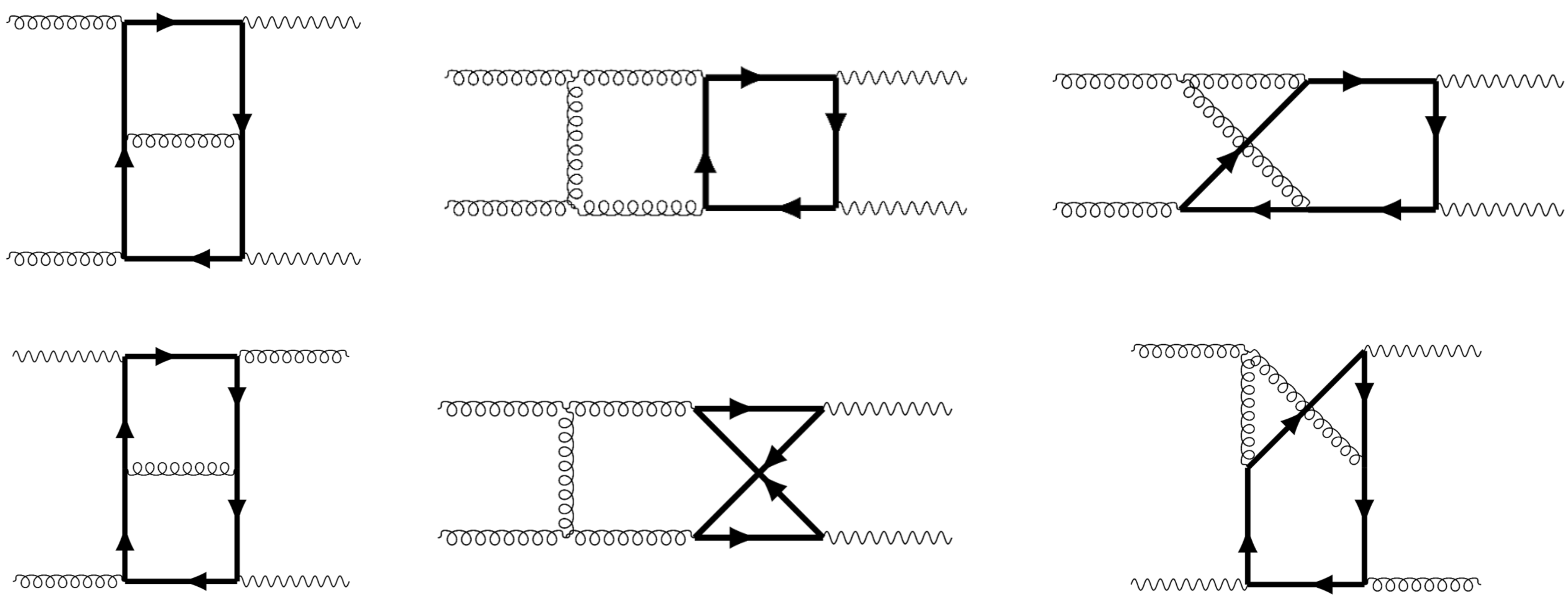
Class C

Z bosons couple to Higgs boson

Djouadi, Spira, Zerwas 91

Diagrams: $gg \to ZZ$

Class $\mathbf{A}_h$



Most challenging piece

Contains a total of 29247 scalar → 264 master Feynman Integrals

Basis choice

Choice of master integral basis is **very** important for size/performance of amplitude

Basis Choice:

1) Select finite integrals — Bern, Dixon, Kosower 92; Panzer 14; von Manteuffel, Panzer, Schabinger 14; Agarwal, SPJ, von Manteuffel 20

2) Require d- and kinematic dependence of denominators factorises — Smirnov, Smirnov 20; Usovitsch 20

$$\frac{N(s,t,d)}{D(s,t,d)} I + \ldots \rightarrow \frac{N'(s,t,d)}{D_1'(d) D_2'(s,t)} I' + \ldots$$

Avoid monstrosities like:

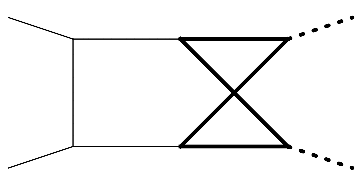
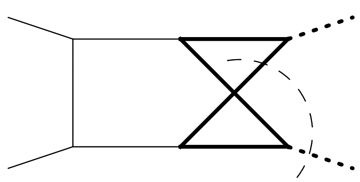
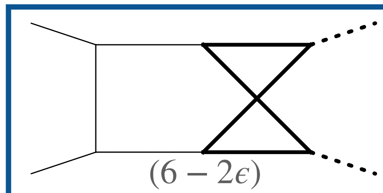
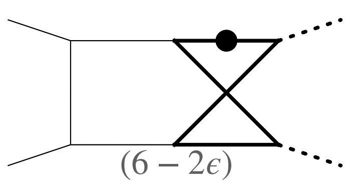
$$D(s,t,d) = \begin{array}{l} 1250 - 500\,d - 9000\,t + 3600\,d\,t + 16200\,t^2 - 6480\,d\,t^2 \\ -4050\,s + 1575\,d\,s + 19440\,s\,t - 8100\,d\,s\,t - 52488\,s\,t^2 \\ +20412\,d\,s\,t^2 - 29160\,s^2\,t + 11664\,d\,s^2\,t \end{array} \xrightarrow{d \to 4} \begin{array}{l} -125 + 375\,s + 900\,t - 2160\,s\,t \\ +2916\,s^2\,t - 1620\,t^2 + 4860\,s\,t^2 \end{array}$$

They increase expr. size and introduce spurious singularities in the amplitude

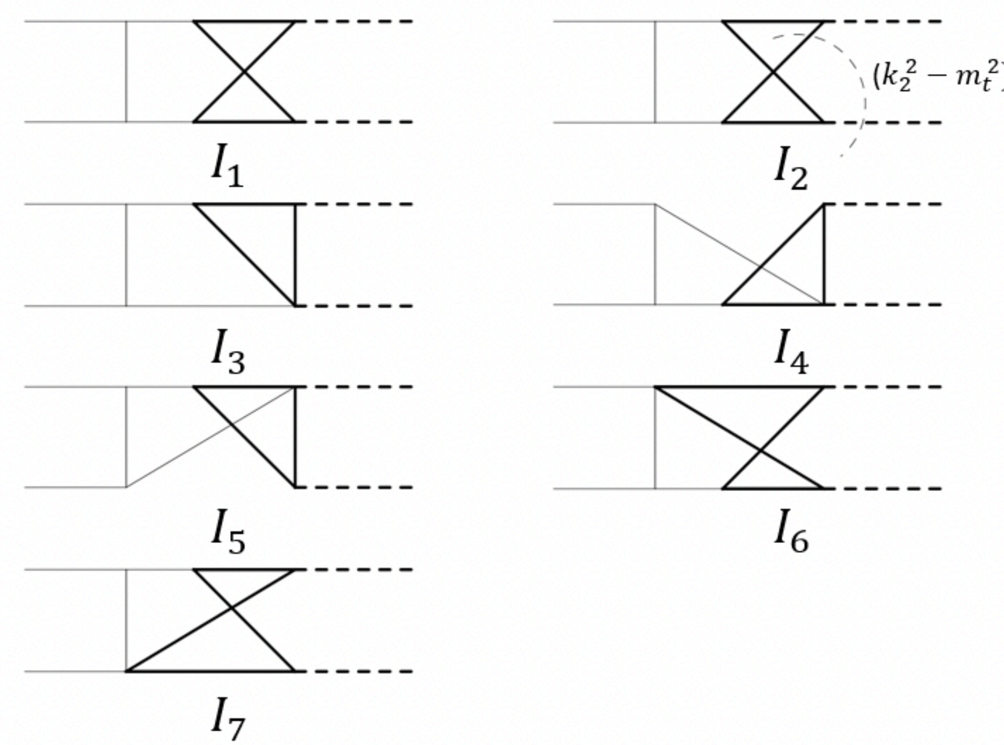
Finite Integrals

Not all finite integrals are created equal

Can take vastly different times to numerically integrate (will discuss this shortly)

Integral	Rel. err. leading term	Timing (s)	
	~2*10^-3	45	
	~4*10^-2	63	
	~8*10^-6	55	$\sim \frac{1}{\mathcal{F}}$
	~8*10^-4	60	$\sim \frac{1}{\mathcal{F}^2}$
I	~1*10^-4	18	$\sim \frac{1}{\mathcal{F}^3}$

Finite



$$I = (m_Z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_Z^2 - t)I_7$$

Can algorithmically construct finite linear combinations

Agarwal, SPJ, von Manteuffel 20
See also: Gambuti, Kosower, Novichkov, Tancredi 23

Baikov Representation

Scalar integrals can be written as Baikov 96

$$I = \mathcal{N} \int dz_1 ... dz_N \; \frac{1}{\prod_{i=1}^N \; z_i^{\nu_i}} \; P^{\frac{d-L-E-1}{2}}$$

IBPs become

$$0 = \int dz_1 ... dz_N \; \sum_{i=1}^N \frac{\partial}{\partial z_i} \left(f_i \; \frac{1}{\prod_{i=1}^N \; z_i^{\nu_i}} \; P^{\frac{d-L-E-1}{2}} \right)$$

$$0 = \int dz_1 ... dz_N \; \sum_{i=1}^N \left(\frac{\partial f_i}{\partial z_i} + \boxed{\frac{d-L-E-1}{2P} \, f_i \, \frac{\partial P}{\partial z_i}} - \boxed{\nu_i \, \frac{f_i}{z_i}} \right) \frac{1}{\prod_{i=1}^N \; z_i^{\nu_i}} \; P^{\frac{d-L-E-1}{2}}$$

Dimension shift **Dots (doubled propagators)**

IBPs generate dim shifted & dotted integrals → huge linear system to solve

Impose `Syzygy' Constraints

$$\left(\sum_{i=1}^N f_i \frac{\partial P}{\partial z_i} \right) + f_{N+1} P = 0$$ → No dimension shift

$$f_i \sim z_i$$ → No doubled props.

Larsen, Zhang 15;
Abreu, Febres Cordero, Ita, Page, Zeng 17;
Boehm, Georgoudis, Larsen, Schoenemann, Zhang 17, 18;

Reduction

IBP system now contains $\mathcal{O}(10^5)$ equations (was $\mathcal{O}(10^8)$ before syzygy constraints)

Still need tools to actually solve this system of equations…

FinRed	+	Syzygy Solver
von Mantueffel		Agarwal, von Mantueffel

Input Integrals: ~29000 integrals

After symmetry: ~1500 integrals

Master Integrals: 264 integrals

Time: $\mathcal{O}(\text{months})$ on cluster

Output IBP Tables: $\mathcal{O}(200\ \text{GB})$

Toolchain relies extensively on the use of finite fields

See e.g: von Mantueffel, Schabinger 14; Peraro 16

Even with these tools, still too difficult to obtain fully symbolic amplitudes!

Fix mass ratio: $m_z^2/m_t^2 = 5/18$

Substituting in Amplitude

Now need to substitute $\mathcal{O}(200\ \mathrm{GB})$ expression into a complicated amplitude…
Yields intermediate expressions of $\mathcal{O}(1\ \mathrm{TB})$

Multivariate partial fractioning

Pak 11; Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 19; Böhm, Wittman, Wu, Xu, Zhang 20; Bendle, Böhm, Heymann, Ma, Rahn, Ristau, Wittmann, Wu, Zhang 21; Heller, von Manteuffel 21;

1. Employ multivariate partial fractioning
2. Partial fraction in d to separate the poles
3. Expand about $d = 4 - 2\epsilon$

$$\frac{1}{(-1+d)(-3+d)^2(-4+d)(-7+2d)} = (\frac{1}{3}+\frac{2\epsilon}{9})(1+2\epsilon)^2(\frac{-1}{2\epsilon})(1+4\epsilon)$$

$\sim$ 16 terms

$$\frac{1}{3(-4+d)}+\frac{5}{4(-3+d)}+\frac{1}{2(-3+d)^2}+\frac{1}{60(-1+d)}+\frac{-16}{5(-7+2d)} = \frac{-1}{6\epsilon}+\frac{-13}{9}$$

2 terms

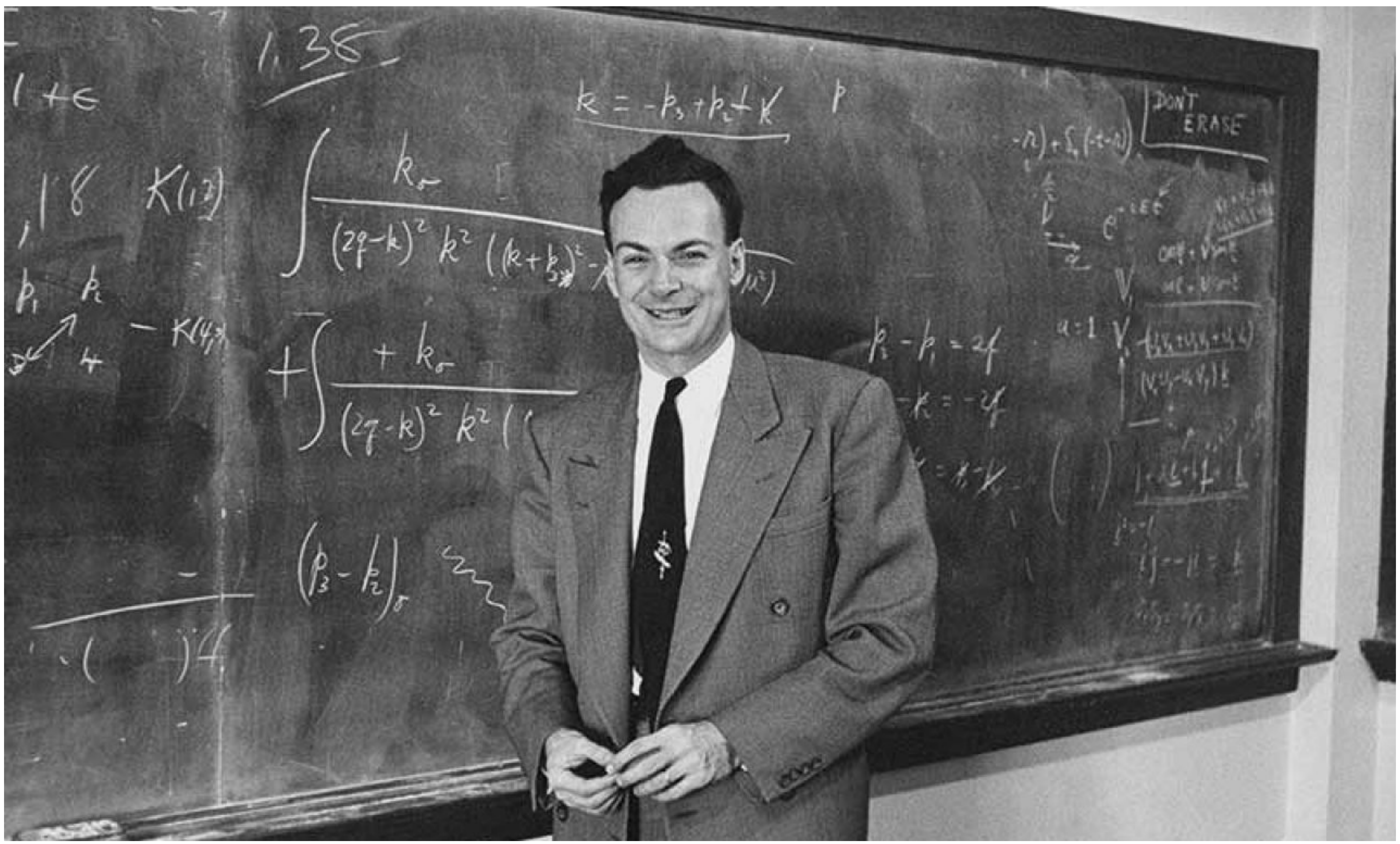
Reduces coefficients of amplitude to <$1\ \mathrm{MB}$ per coefficient
For one of the hardest coefficients:

$$\{deg(num,s)+deg(den,s), deg(num,t)+deg(den,t), deg(num,d)+deg(den,d)\} = \{107,117,38\}$$

↓

$$\{deg(num,s)+deg(den,s), deg(num,t)+deg(den,t), deg(num,d)+deg(den,d)\} = \{20,15,9\}$$

Intermezzo:
Computing Feynman Integrals



Computing Feynman Integrals

Feynman integrals can be difficult to compute analytically

Various methods to approximate/evaluate them numerically

Numerical differential equations — **ODE/PDE**

Series solutions of differential equations (AMFlow, DiffExp, Seasyde)

Taylor expansion in Feynman parameters (TayInt)

— **Series Solutions**

Numerical Mellin-Barnes (MB, Ambre)

Tropical sampling (Feyntrop)

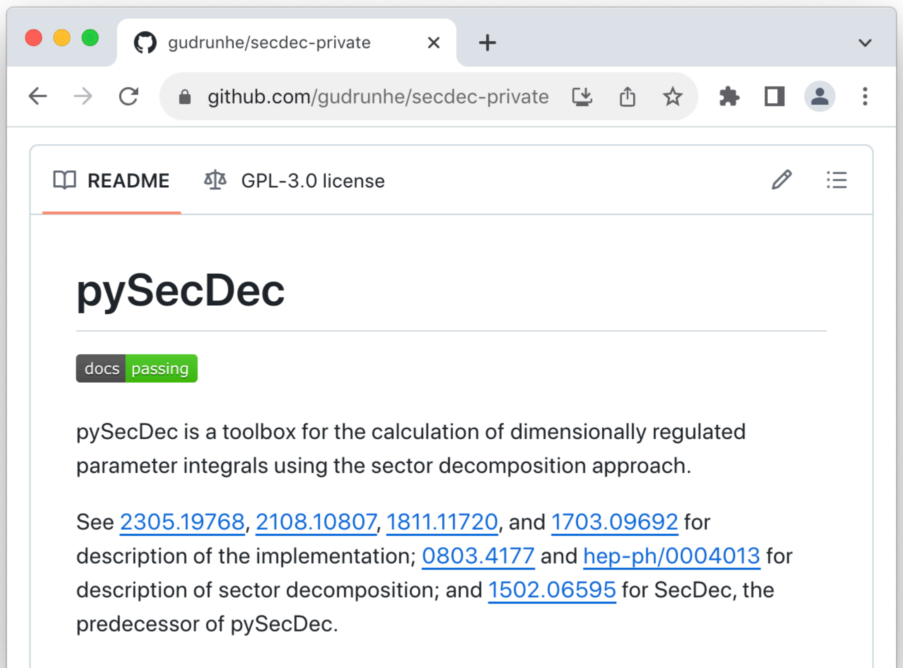
Numerical Loop-Tree Duality (cLTD, Lotty)

Sector decomposition (Sector_decomposition, FIESTA, pySecDec)

— **~Monte Carlo Integration**

pySecDec

pySecDec: a program for numerically evaluating dimensionally regulated parameter integrals on CPU or GPU



Latest Version:

Improved: Method of Regions

Improved: Amplitude Evaluation

+ disteval Integrator

+ Median QMC Rules

Heinrich, SPJ, Kerner, Magerya, Olsson, Schlenk 23

```
python3 -m pip install --user --upgrade pySecDec
```

Well known in the QCD community, often used for checking master integrals

Used to compute two-loop amplitudes for $pp \to \{HH, HJ, \gamma\gamma, ZH, ZZ, t\bar{t}H\}$

Borowka, Greiner, Heinrich, SPJ, Kerner, Schlenk, Schubert, Zirke 16; SPJ, Kerner, Luisoni 18;
Chen, Heinrich, Jahn, SPJ, Kerner, Schlenk, Yokoya 19; Chen, Heinrich, SPJ, Matthias Kerner, Klappert, Schlenk 20;
Agarwal, SPJ, von Manteuffel 20; Agarwal, Heinrich, SPJ, Kerner, Klein, Lang, Magerya, Olsson 24

Sector Decomposition in a Nutshell

Can exchange loop integrals for integrals over Feynman parameters

$$I \sim \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_l \frac{1}{\prod_{i=1}^{N} (q_i - m_i)^{\nu_i}} \leftrightarrow I \sim \int_{\mathbb{R}^{N+1}_{>0}} \left[\mathrm{d}\boldsymbol{x}\right] \boldsymbol{x}^{\boldsymbol{\nu}} \frac{[\mathcal{U}(\boldsymbol{x})]^{N-(L+1)D/2}}{[\mathcal{F}(\boldsymbol{x},\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - H(\boldsymbol{x}))$$

$\mathcal{U}, \mathcal{F}$ are *polynomials* in FP x

Singularities

1. UV/IR singularities when some $\{x\} \to 0$ simultaneously $\Longrightarrow$ Sector Decomposition
2. Thresholds when $\mathcal{F}$ vanishes inside integration region $\Longrightarrow$ $i\delta$

Sector decomposition

Find a local change of coordinates for each singularity that factorises it (blow-up)

Hepp 66; Roth, Denner 96; Binoth, Heinrich 00; Heinrich 08

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t$$

$$\mathcal{N}(I) = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

Normal vectors incident to each extremal vertex define a local change of variables*

Kaneko, Ueda 10

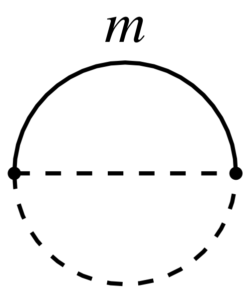
$$x_i = \prod_{f \in S_j} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_0^1 \left[\mathrm{d}\mathbf{y}_f\right] \underbrace{\prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle - t a_f}}_{\text{Singularities}} \Bigg(\underbrace{c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f}}_{\text{Finite}} \Bigg)^t$$

Singularities **Finite**

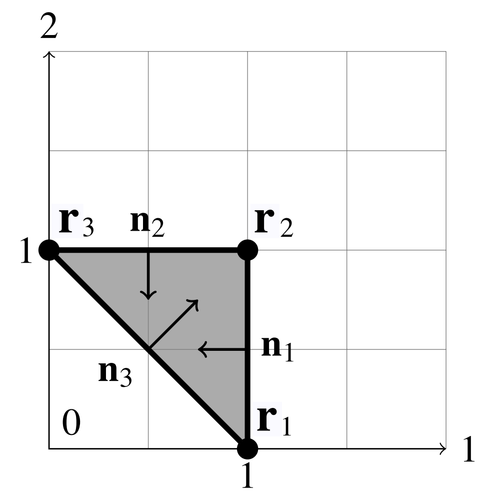
*If $|S_j| > N$, need triangulation to define variables (simplicial normal cones $\sigma \in \Delta^T_{\mathcal{N}}$)

Sector Decomposition in a Nutshell

$I =$ 

$= -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon}\int_0^\infty \frac{\mathrm{d}x_1\mathrm{d}x_2}{\left(x_1^1x_2^0 + x_1^1x_2^1 + x_1^0x_2^1\right)^{2-\varepsilon}}.$

$$\mathbf{r}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix}0\\1\end{pmatrix}$$

$\mathcal{N}(I) =$ 

$$= \quad \mathbf{n}_1 = \begin{pmatrix}-1\\0\end{pmatrix} \; \mathbf{n}_2 = \begin{pmatrix}0\\-1\end{pmatrix} \; \mathbf{n}_3 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$a_1 = 1 \quad a_2 = 1 \quad a_3 = -1$$

For each vertex make the local change of variables

e.g. $\mathbf{r}_1$: $x_1 = y_1^{-1}y_3^1,\ x_2 = y_1^0 y_3^1$, $\mathbf{r}_2 : x_1 = y_1^{-1}y_2^0,\ x_2 = y_1^0 y_2^{-1}$, $\mathbf{r}_3 : x_1 = y_2^0 y_3^1, x_2 = y_2^{-1}y_3^1$

$$I = -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon}\int_0^1 \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\,\frac{y_1^{-\varepsilon}y_2^{-\varepsilon}y_3^{-1+\varepsilon}}{(y_1+y_2+y_3)^{2-\varepsilon}}\left[\delta(1-y_2)+\delta(1-y_3)+\delta(1-y_1)\right]$$

Schlenk 2016

Performance Improvements

v1.5: Adaptive sampling of sectors, automatic contour def. adjustment

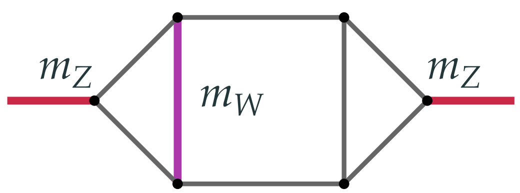
v1.5.6: Optimisations in integrand code

v1.6: New Quasi-Monte Carlo integrator ``Disteval"

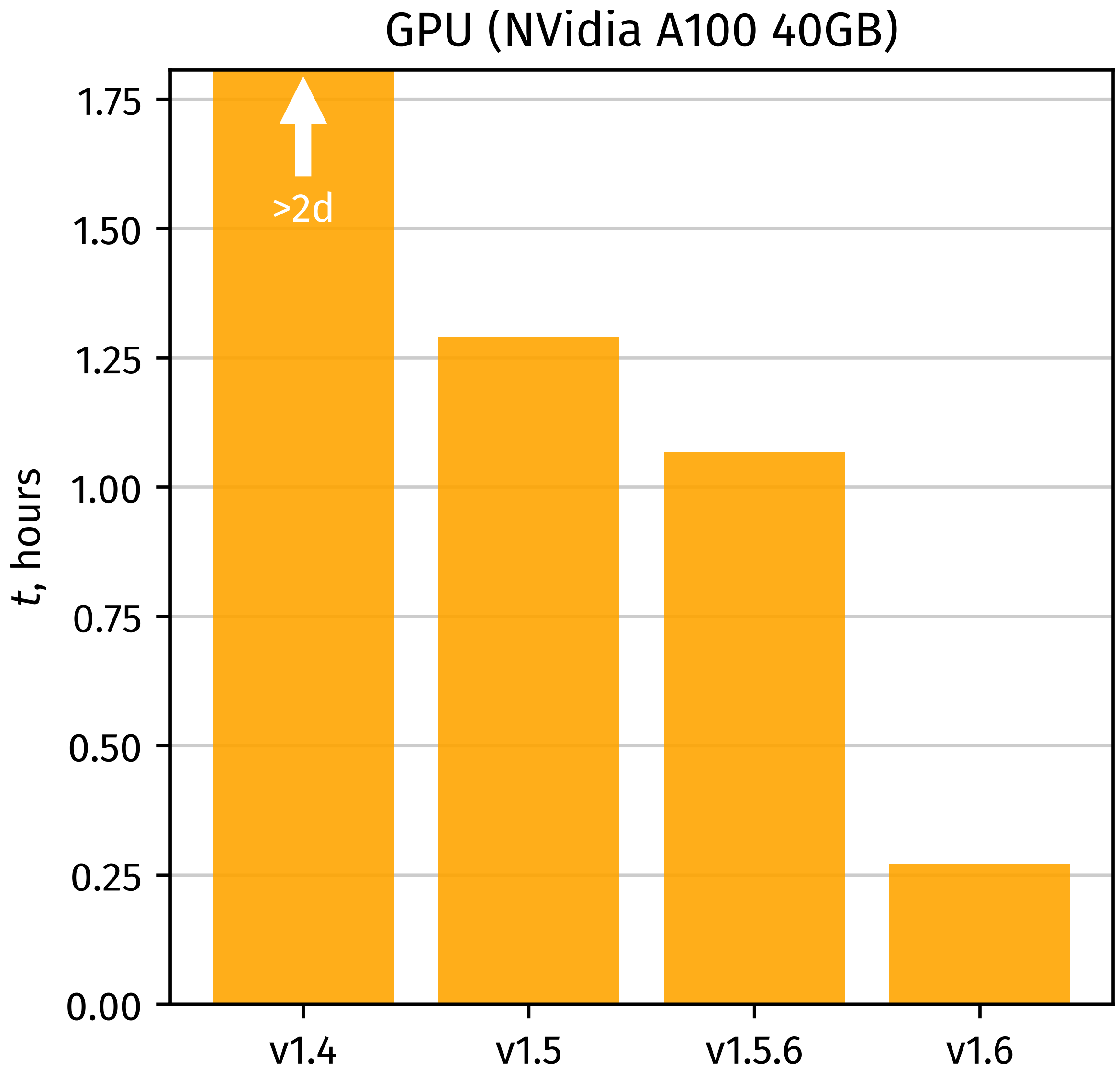
- Faster implementation of old integrator ``IntLib"
- CPU & GPU: fusion of integration/integrand code (less modular arithmetic)
- CPU: better utilisation via SIMD instructions (AVX2, FMA)
- GPU: sum result on GPU, less synchronisation
- Parse amplitude coefficients w/GiNaC (supports e.g. partial fractioned input)
- Workers can run on remote machines (via ssh)

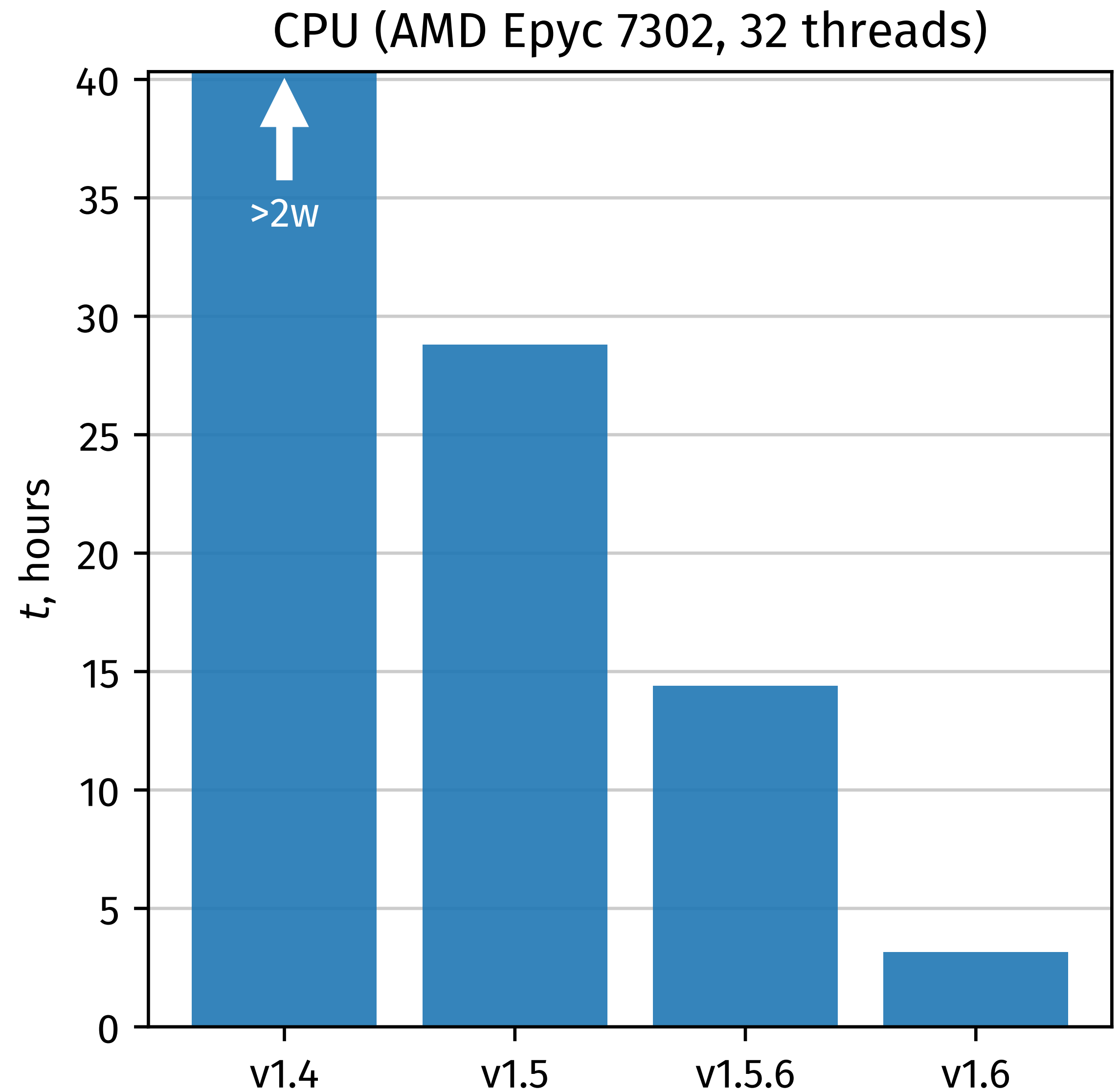
Does it help?

Performance Improvements



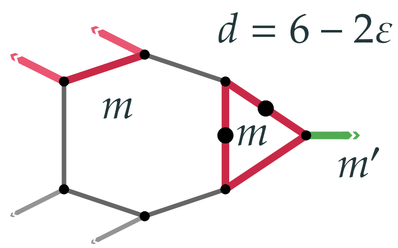
7 digits

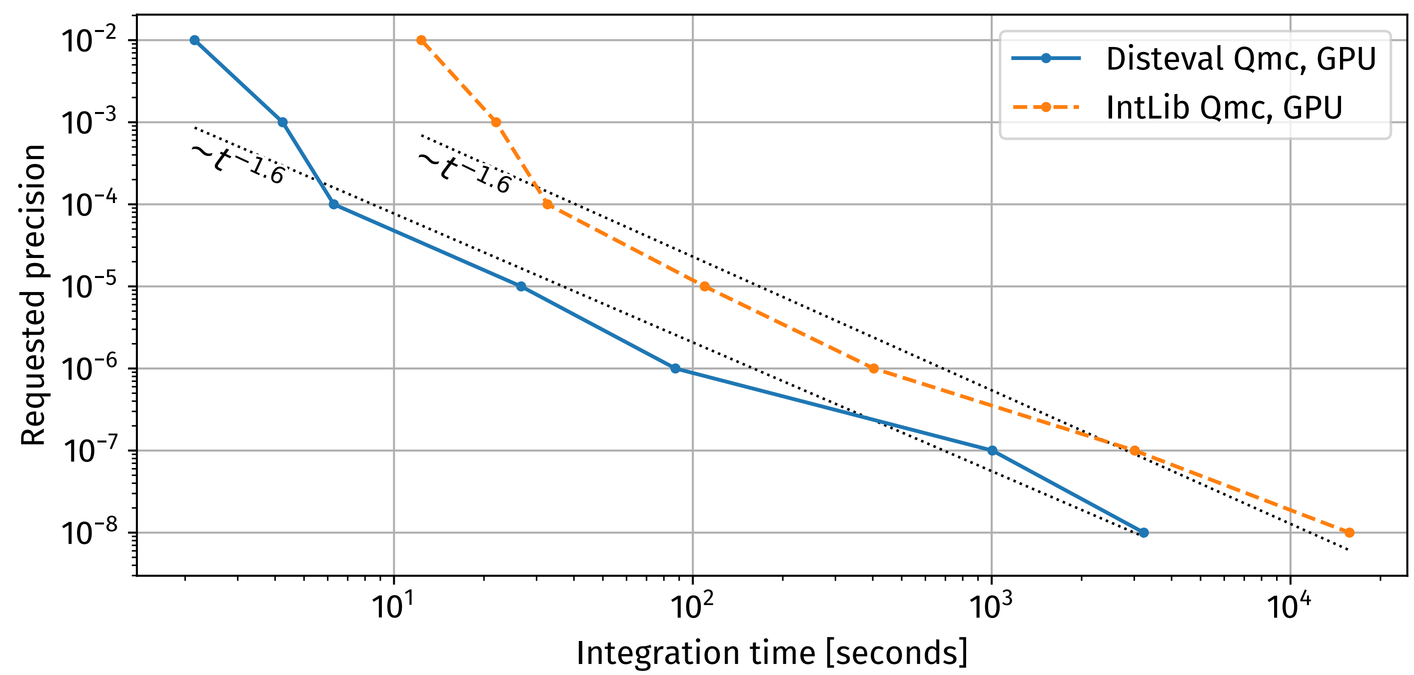




Vitaly Magerya (Radcor 2023)

Profiling



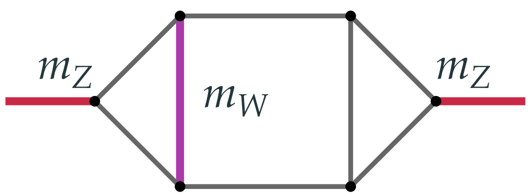
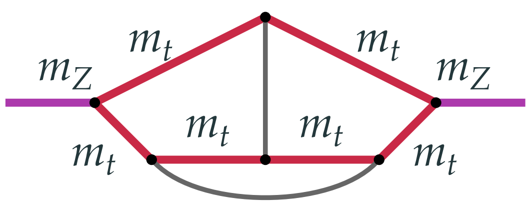
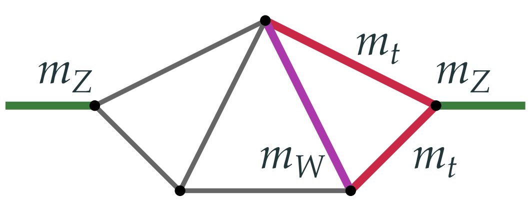


Integrator \ Accuracy		10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
GPU	DISTEVAL	4.2 s	6.3 s	27 s	1.5 m	17 m	54 m
	INTLIB	22.0 s	22.0 s	110 s	6.7 m	50 m	263 m
	Speedup	5.2	5.2	4.1	5.6	3.0	4.9
CPU	DISTEVAL	5.1 s	14 s	1.6 m	8.3 m	57 m	4.7 h
	INTLIB	20.8 s	86 s	14.2 m	62.2 m	480 m	43.1 h
	Speedup	4.1	6.1	8.7	7.5	8.4	9.2

[GPU: NVidia A100 40GB; CPU: AMD EPYC 7F32 with 32 threads]

Profiling

pySecDec Disteval *integration times* for 3-loop self-energy integrals:[3]

Diagram \ Relative precision		10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
	GPU	15s	20s	40s	200s	13m	50m
	CPU	10s	50s	400s	4000s	180m	1200m
	GPU	18s	19s	30s	20s	1.2m	2m
	CPU	5s	14s	60s	50s	12m	16m
	GPU	6s	11s	12s	30s	3m	24m
	CPU	5s	10s	50s	800s	60m	800m

[Same diagrams as in Dubovyk, Usovitsch, Grzanka '21]

In short: *seconds to minutes per integral* to achieve practical precision.

[GPU: NVidia A100 40GB; CPU: AMD Epyc 7F32 with 32 threads]

Quasi-Monte Carlo

Li, Wang, Yan, Zhao 15; de Doncker, Almulihi, Yuasa 17, 18; de Doncker, Almulihi 17; Kato, de Doncker, Ishikawa, Yuasa 18

$$Q_n^{(k)}[f] \equiv \frac{1}{n} \sum_{i=0}^{n-1} f\left(\left\{\frac{i\mathbf{z}}{n} + \boldsymbol{\Delta}_k\right\}\right) \qquad I[f] \approx \bar{Q}_{n,m}[f] \equiv \frac{1}{m} \sum_{k=0}^{m-1} Q_n^{(k)}[f],$$

$\{\}$ - Fractional part

Δ_k - Random shift vector

$\mathbf{z}$ - Generating vector

Previously:

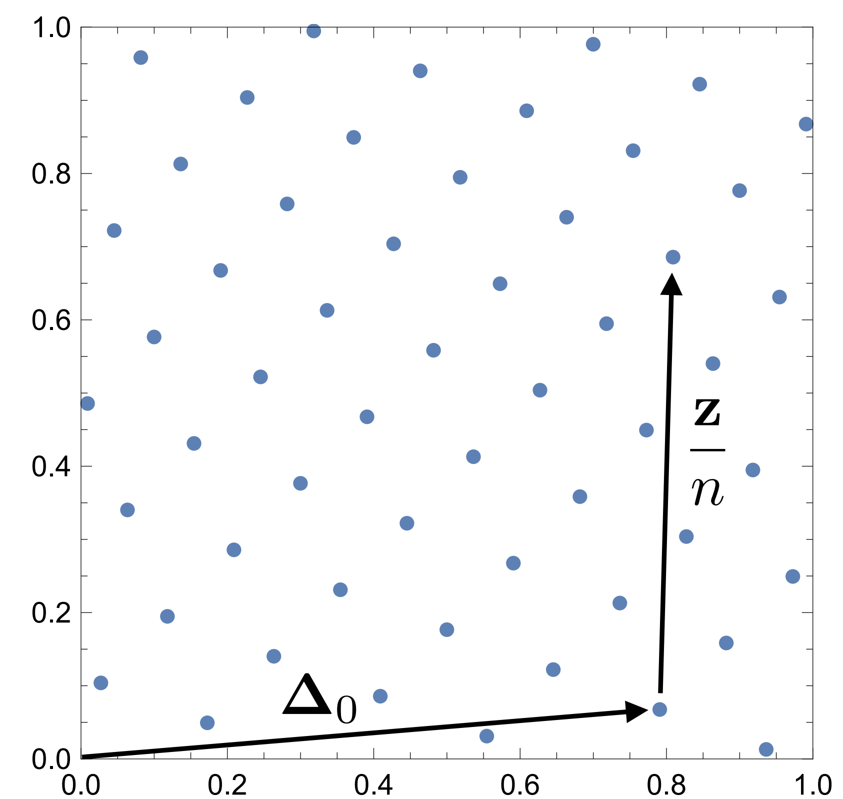
Precompute $\mathbf{z}$ with (CBC) construction

Nuyens, Cools 06

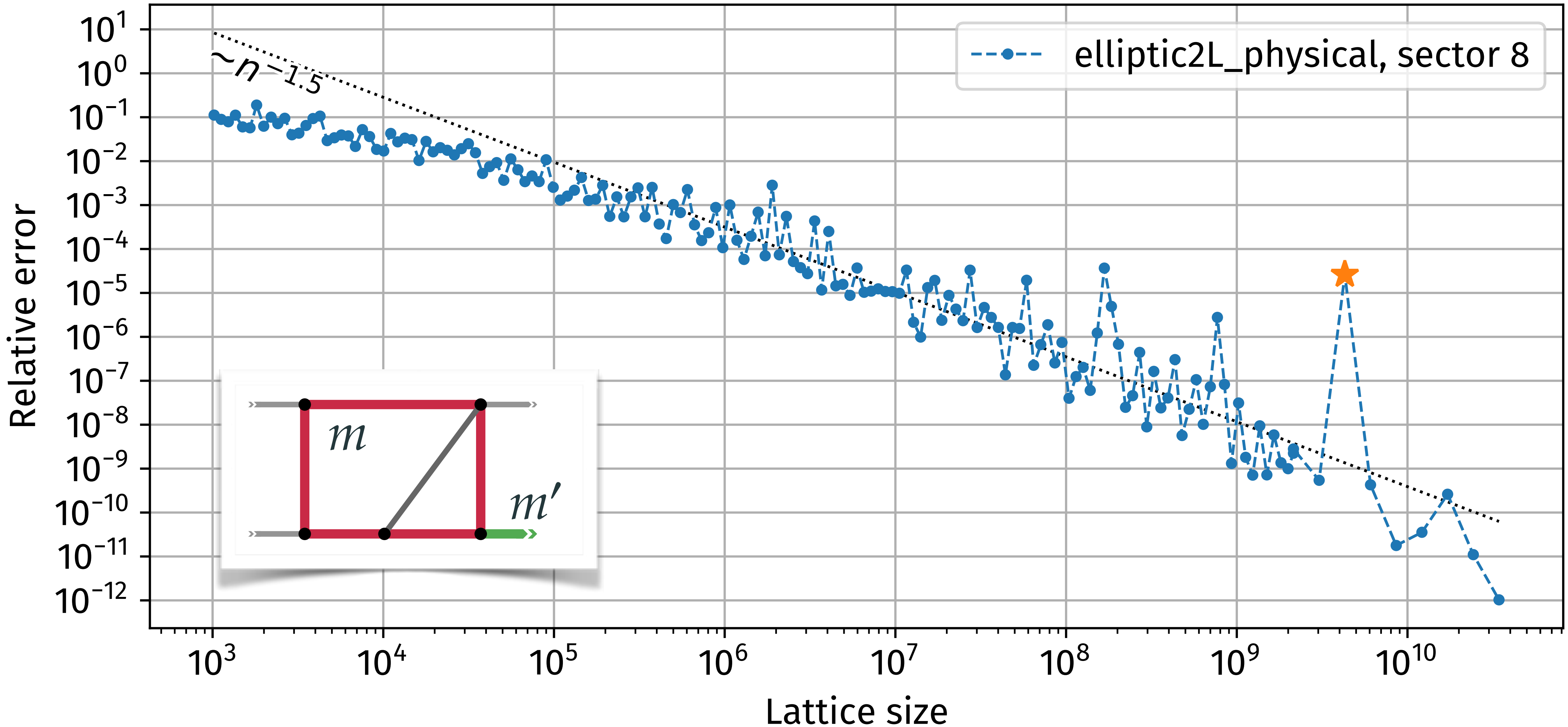
Guarantee error $\sim 1/n^{\alpha}$ if $\delta_x^{(\alpha)} I(\mathbf{x})$ is square-integrable and periodic Dick, Kuo, Sloan 13

CBC needs $\mathcal{O}(n)$ bytes memory $n \lesssim 4.10^{10}$ @ 2TB

Can encounter ``unlucky'' lattices



Quasi-Monte Carlo: Unlucky Lattices

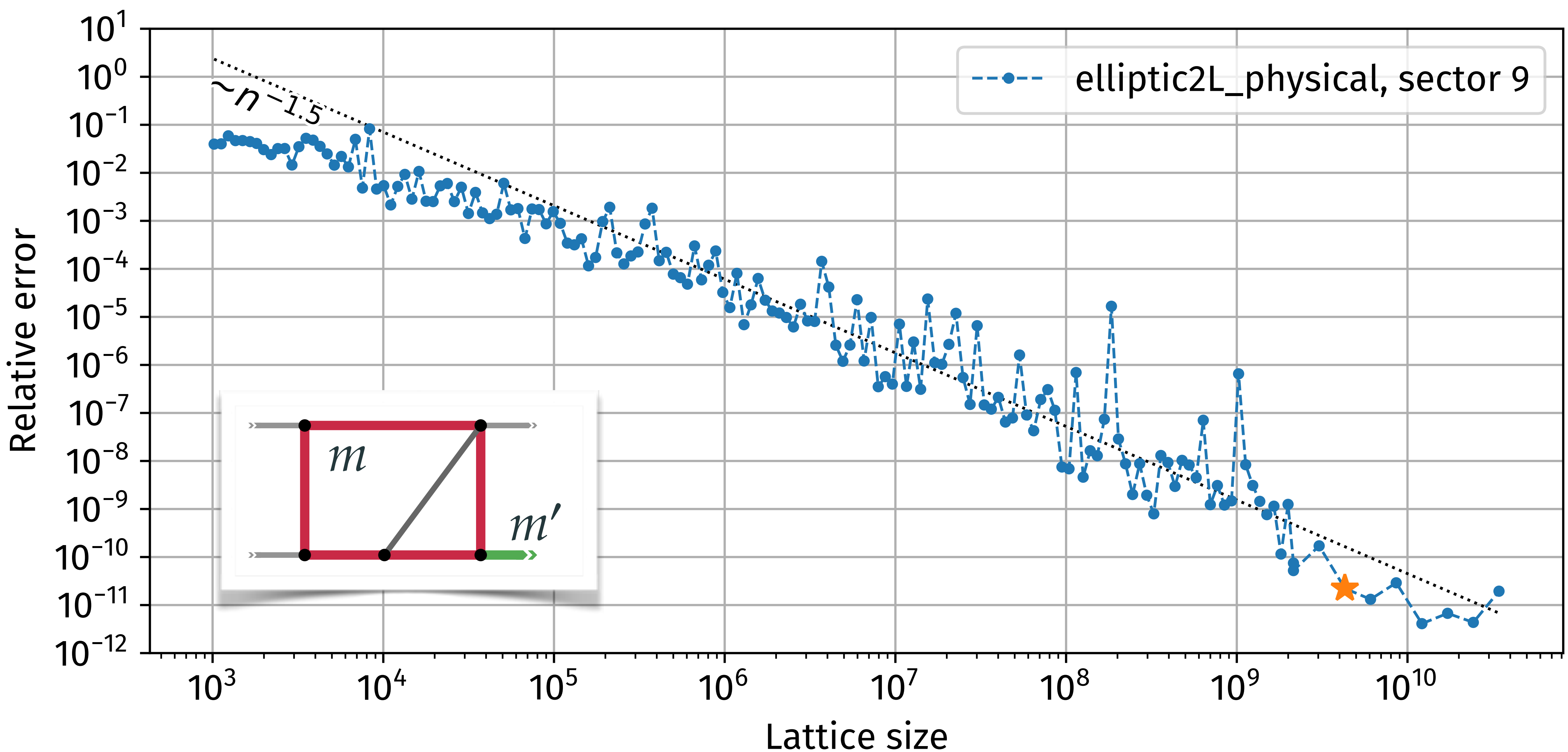


Good: Asymptotic error scaling $\sim 1/n^{1.5}$

Bad: Huge drop in precision for some "unlucky" lattices

Not consistent across integrands

Quasi-Monte Carlo: Unlucky Lattices



Good: Asymptotic error scaling $\sim 1/n^{1.5}$

Bad: Huge drop in precision for some "unlucky" lattices

Not consistent across integrands

Median Lattice Rules

Instead:

Compute $\mathbf{z}$ on-the-fly

1. Choose R random $\mathbf{z} \in \mathrm{Uniform}(0; N-1)$
2. Estimate integral on each lattice
3. Choose lattice with median integral value

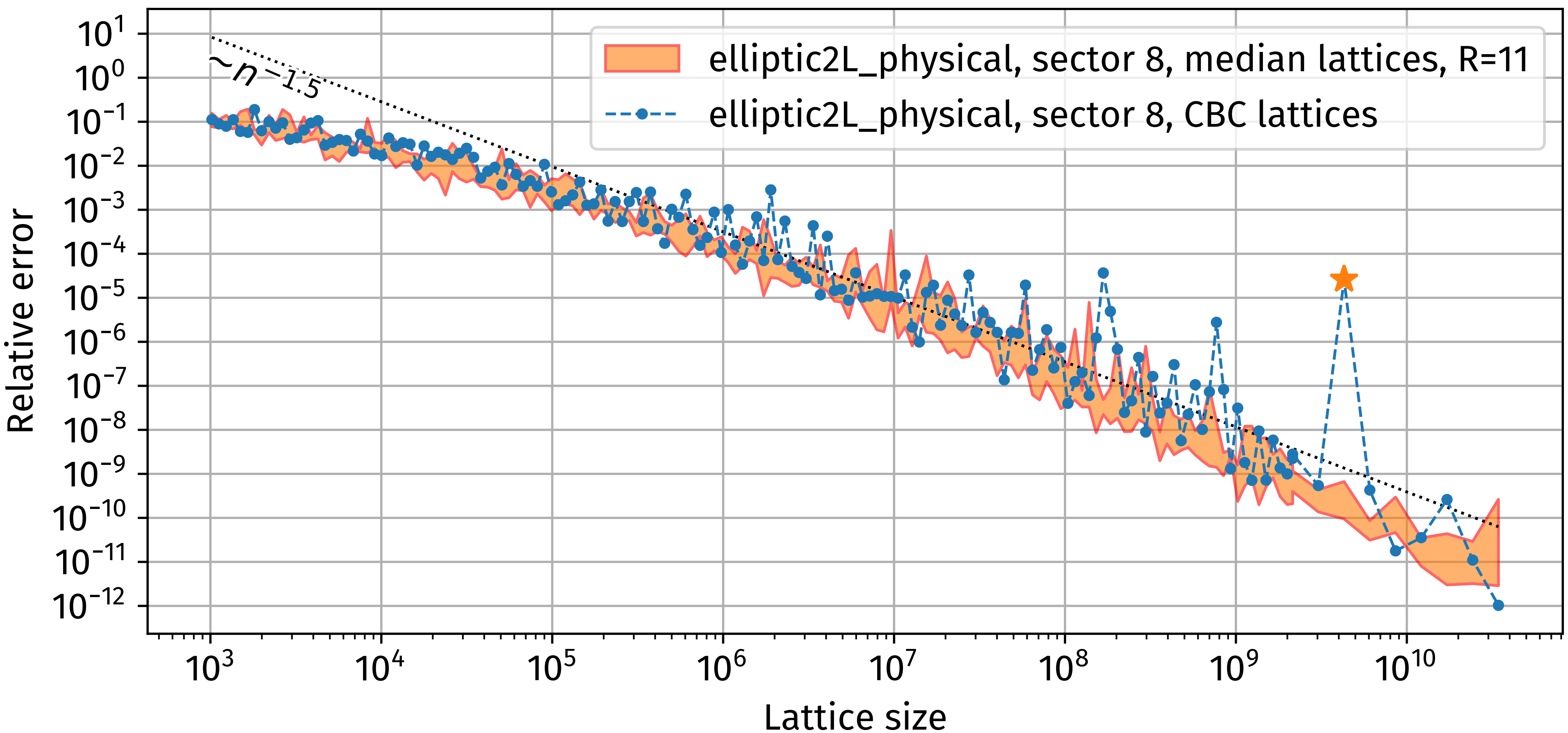
If $\delta_x^{(\alpha)} I(\mathbf{x})$ is square-integrable and periodic

Integration error: $C(\alpha, \varepsilon)/(\rho n)^{\alpha-\epsilon}$

With probability: $1 - \rho^{R+1/2}/4$

$\forall \quad 0 < \varepsilon \;\&\; 0 < \rho < 1$

Goda, L'Ecuyer 22

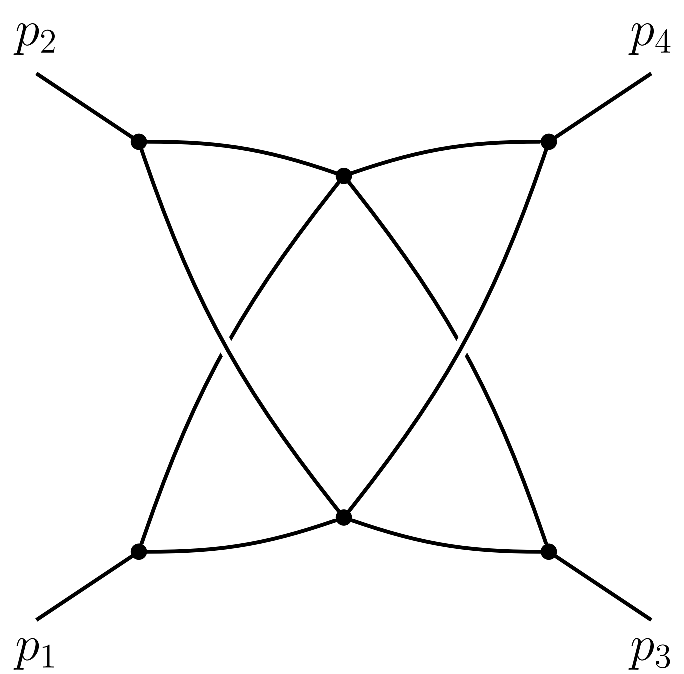


Some Future Developments

Computing integrals with leading Landau singularities inside the integration domain

w/ Gardi, Herzog, Ma (WIP)

Impossible → Possible to compute



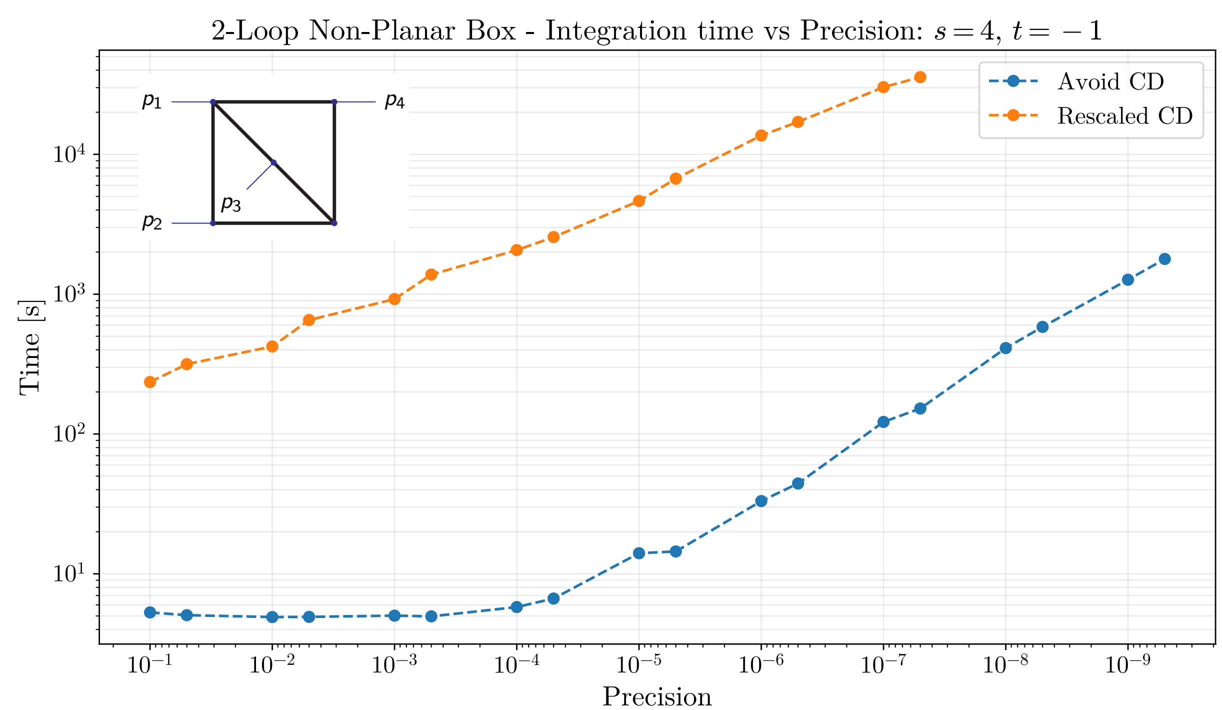
$$I = \epsilon^{-4}\left[8.3400403920\mathbf{28} - 52.35987755983\mathbf{47}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

$$I^{\text{ana.}} = \epsilon^{-4}\left[8.34004039223768 - 52.35987755984493i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

Avoiding contour deformation even in the Minkowski/physical regime

w/ Olsson, Stone (WIP)

Speedup ~100-1000x for some cases



Results

Checks

This calculation was **complicated**, before we do anything we should check it:

Symmetries between form factors (and helicity amplitudes)

Crossing relations

Pole cancellation (UV and IR)

Alternate finite basis (checks definition of our finite integrals, basis change to finite integrals, and their numerical evaluation)

Uncovered bug in nvcc (fixed in current version)

Different γ_5 schemes (Kreimer/Larin scheme)

Large top-mass and small top-mass expansions in the relevant regions

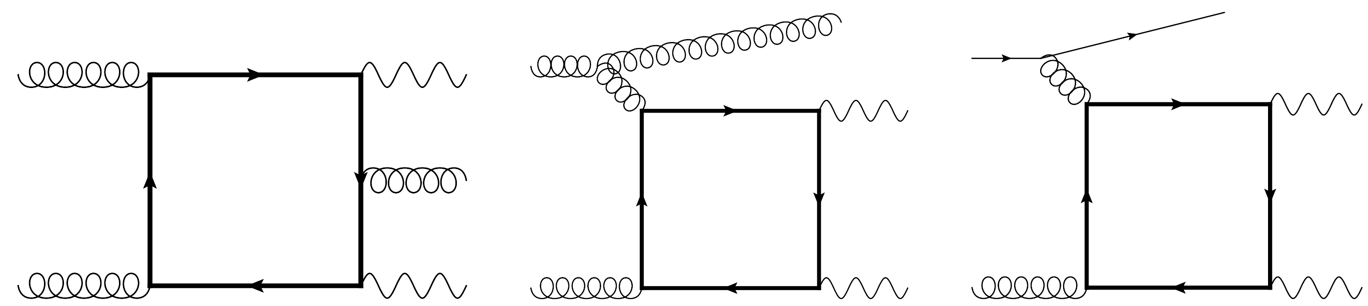
Our amplitudes agree with the later calculations using series solutions and with the small-p_T expansion in the relevant region

Brønnum-Hansen, Wang 21; Degrassi, Gröber, Vitti 24

NLO QCD Corrections

$$\hat{\sigma} = \hat{\sigma}^{\mathrm{LO}} + \hat{\sigma}^{\mathrm{NLO}}$$
$$\hat{\sigma}^{\mathrm{LO}} = \int_n \mathrm{d}\sigma^{\mathrm{B}}$$
$$\hat{\sigma}^{\mathrm{NLO}} = \int_n \mathrm{d}\sigma^{\mathrm{V}} + \int_{n+1} \mathrm{d}\sigma^{\mathrm{R}} + \int_n \mathrm{d}\sigma^{\mathrm{C}}$$

Reals



Gosam (cross-checked with MadGraph and OpenLoops)

Virtual ($\mathrm{d}\sigma^V$) and Real ($\mathrm{d}\sigma^R$) parts not separately finite for $\epsilon \to 0$

1) UV renormalize: α_s in $\overline{\mathrm{MS}}$ & top quark mass in OS scheme

2) IR structure well known @ NLO subtract divergences from V add them back to R

$$\mathcal{A}_i^{(0),\mathrm{fin}} = \mathcal{A}_i^{(0),\mathrm{UV}},$$
$$\mathcal{A}_i^{(1),\mathrm{fin}} = \mathcal{A}_i^{(1),\mathrm{UV}} - I_1 \mathcal{A}_i^{(0),\mathrm{UV}},$$

q_T -scheme:

$$I_1 = I_1^{\mathrm{soft}} + I_1^{\mathrm{coll}},$$
$$I_1^{\mathrm{soft}} = -\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{\mu_R^2}{s}\right)^{\epsilon} \left(\frac{1}{\epsilon^2} + \frac{i\pi}{\epsilon}\right) 2C_A,$$
$$I_1^{\mathrm{coll}} = -\frac{\beta_0}{\epsilon} \left(\frac{\mu_R^2}{s}\right)^{\epsilon}.$$

Catani, Cieri, de Florian, Ferrera, Grazzini 14

Amplitudes

First present results for the Born and Born-Virtual interference helicity amplitudes

Expand the helicity amplitudes in α_S

$$\mathscr{M}^{fin}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \left(\frac{\alpha_s}{2\pi}\right)\mathscr{M}^{(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4} + \left(\frac{\alpha_s}{2\pi}\right)^2\mathscr{M}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4} + \dots$$

Compute the square/interference

$$\mathscr{V}^{(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4} = |\mathscr{M}^{(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4}|^2,$$

$$\mathscr{V}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4} = 2\,\mathrm{Re}\left(\mathscr{M}^{*(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4}\mathscr{M}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4}\right)$$

We also define

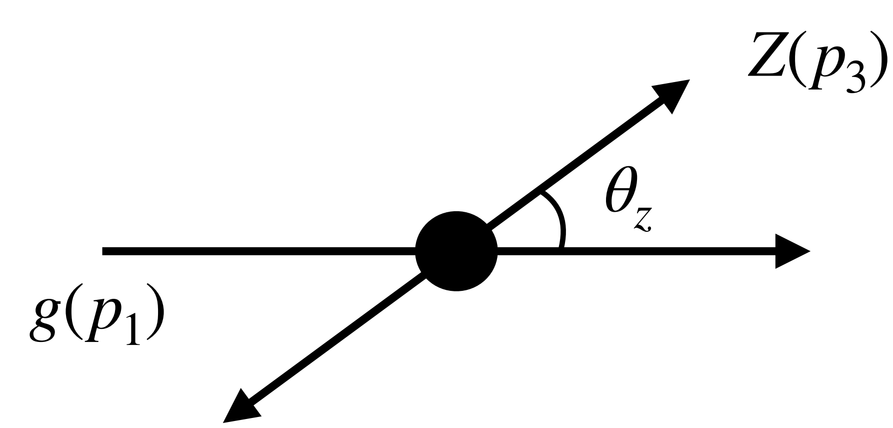
$$\mathscr{V}^{(i)}_{\lambda_3\lambda_4} = \frac{1}{4}\sum_{\lambda_1,\lambda_2}\mathscr{V}^{(i)}_{\lambda_1\lambda_2\lambda_3\lambda_4} \quad \text{and} \quad \mathscr{V}^{(i)} = \sum_{\lambda_3,\lambda_4}\mathscr{V}^{(i)}_{\lambda_3\lambda_4}$$

$2 \to 2$ amplitude depends on two kinematic variables (after fixing masses)

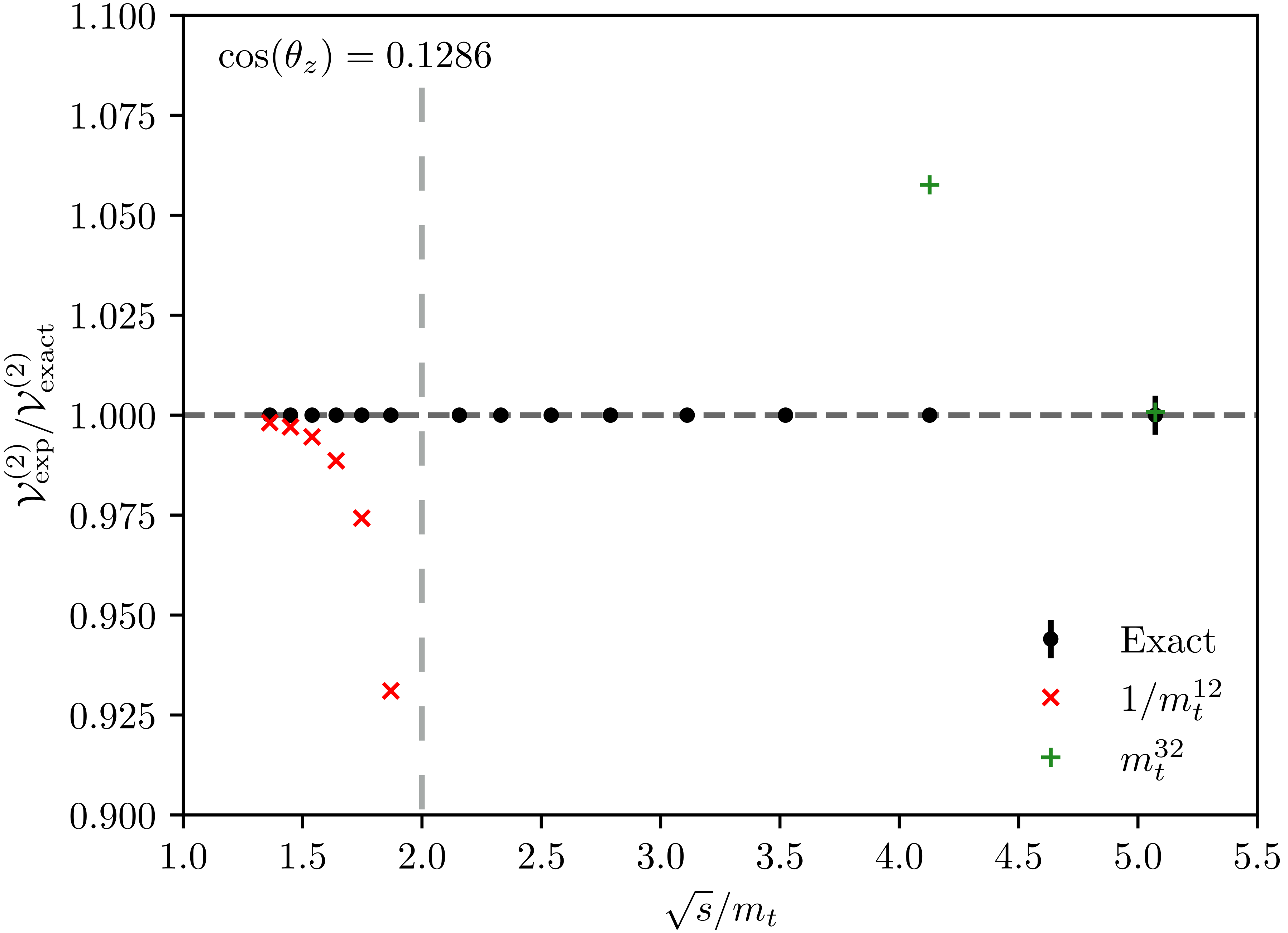
Choose:

$s = (p_1 + p_2)^2$

θ_z - angle in c.o.m frame between p_1-axis and p_3



Comparison to Expansion



Expanded results from: Davies, Mishima, Steinhauser, Wellmann 20

Expansion can be improved by fitting Padé approximants & conformal mapping, have not compared to these approaches Campbell, Ellis, Czakon, Kirchner 16; Davies, Mishima, Schönwald, Steinhauser 23

Scheme Dependence

Can change the subtraction scheme of our finite virtuals with shifts ∝ Born

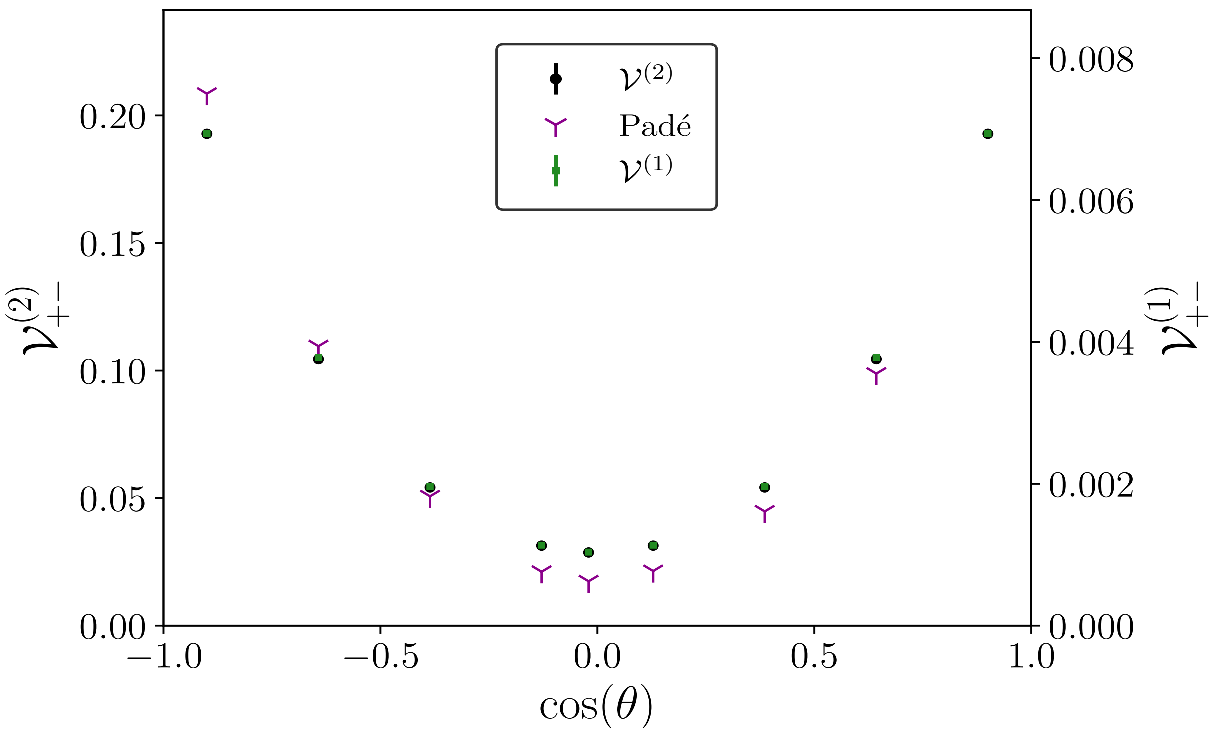
$$A_i^{(1),\text{fin},C} = A_i^{(1),\text{fin}} + \Delta \mathbf{I}_C A_i^{(0),\text{fin}}$$

$$\Delta \mathbf{I}_{\mathrm{C}} = -\frac{1}{2}\pi^2 C_A + i\pi\beta_0,$$

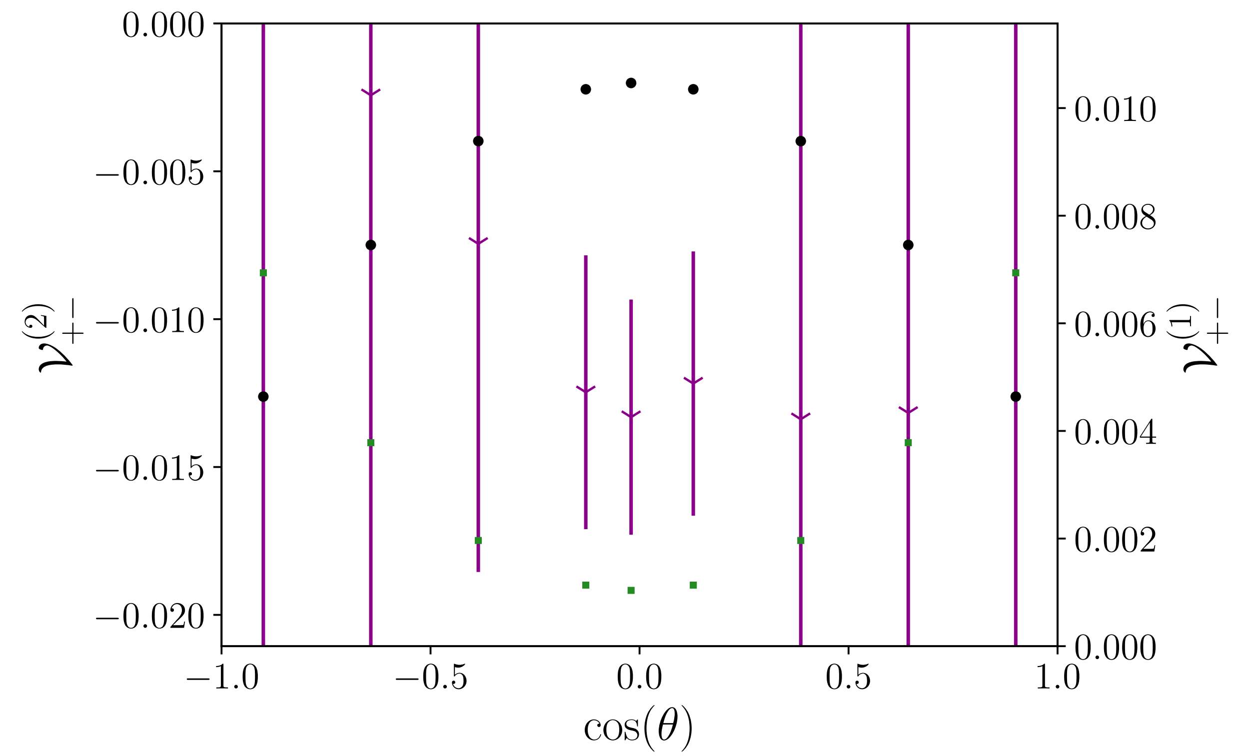
$$\Delta \mathbf{I}_{\mathrm{CS}} = -i\pi C_A \left(\frac{1}{\epsilon} + \ln\left(\frac{\mu_R^2}{s}\right)\right) - \frac{\pi^2 C_A}{3} + \beta_0 + k_g$$

This significantly impacts the level of agreement of the full virtual vs expansions

q_T **Scheme**



Catani Scheme



Observation: quality of an approximation depends strongly on the scheme

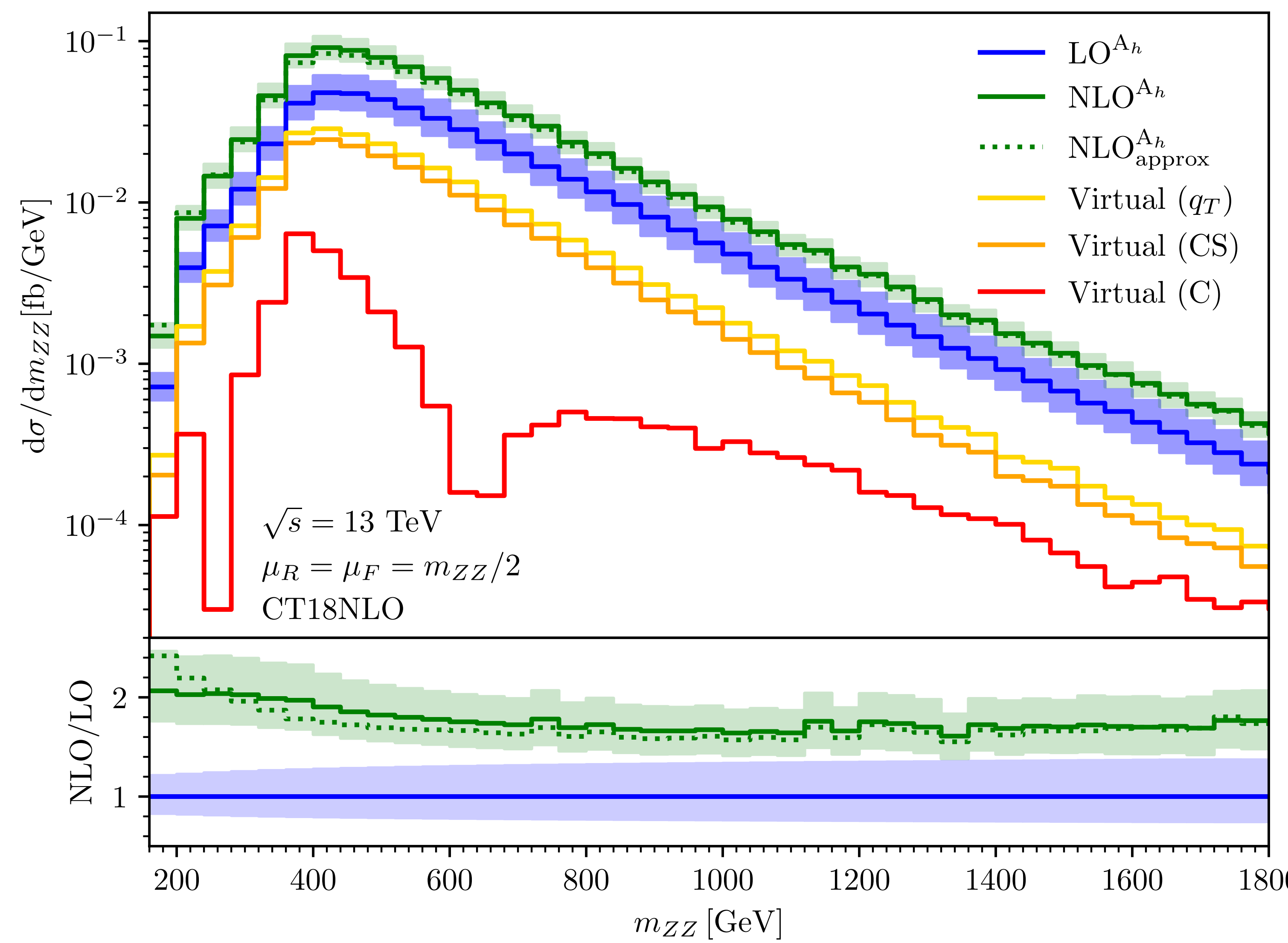
Top-Quark Only NLO QCD Corrections

We find that the NLO corrections considering only massive quarks are not small!

Size of virtual contribution varies between different schemes (NLO result is independent of scheme)

→ Some schemes are better when virtual phase-space statistics limited

Approximation obtained by applying the massless K-factor to the massive born ~agrees within scale uncertainty

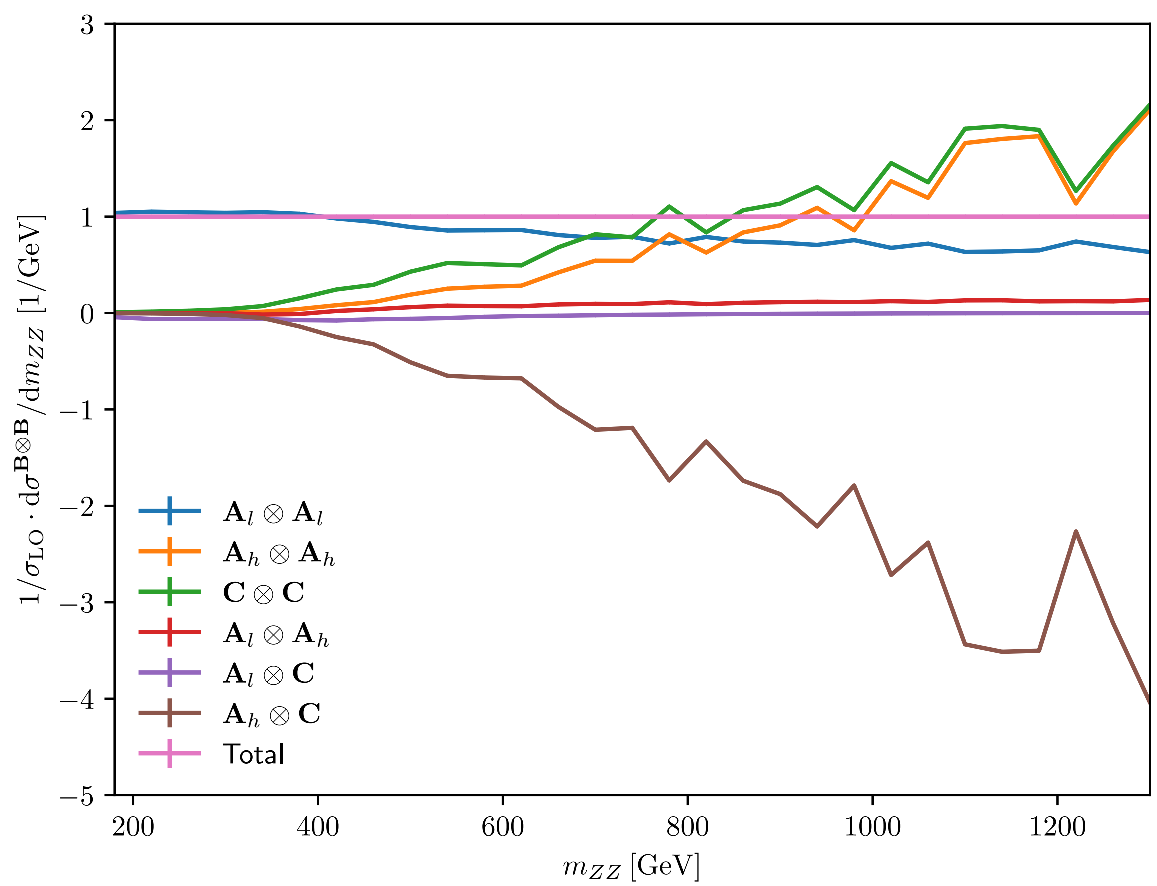


Agarwal, SPJ, Kerner, von Manteuffel 24

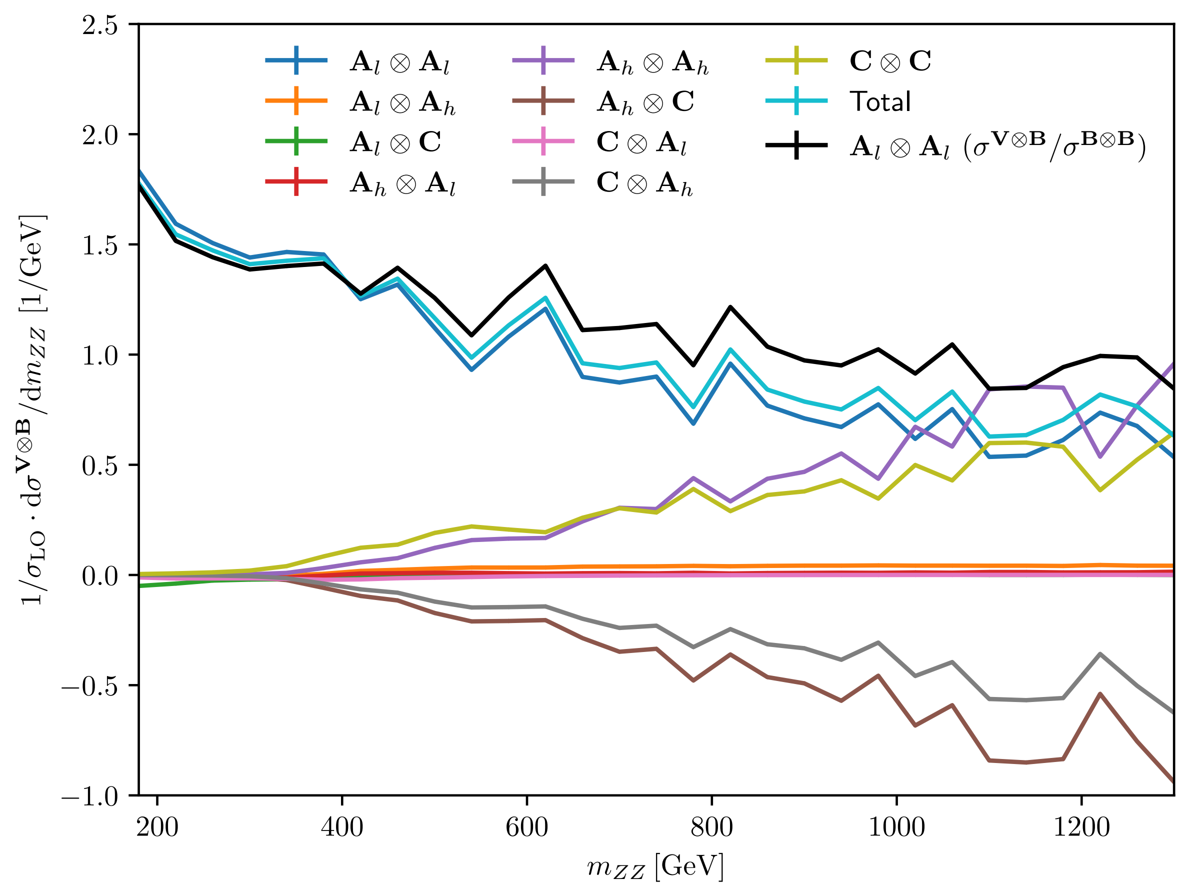
NLO QCD Channel Breakdown

We also have the complete **Class A+B+C** NLO QCD corrections implemented,
Let's examine the interference patterns

Born



Born-Virtual



Observe the unitarizing behaviour of the massive & Higgs-mediated amplitudes
Above $\sim 2m_t$ significant destructive interference between $\mathbf{A}_h$ and $\mathbf{C}$, LO and NLO

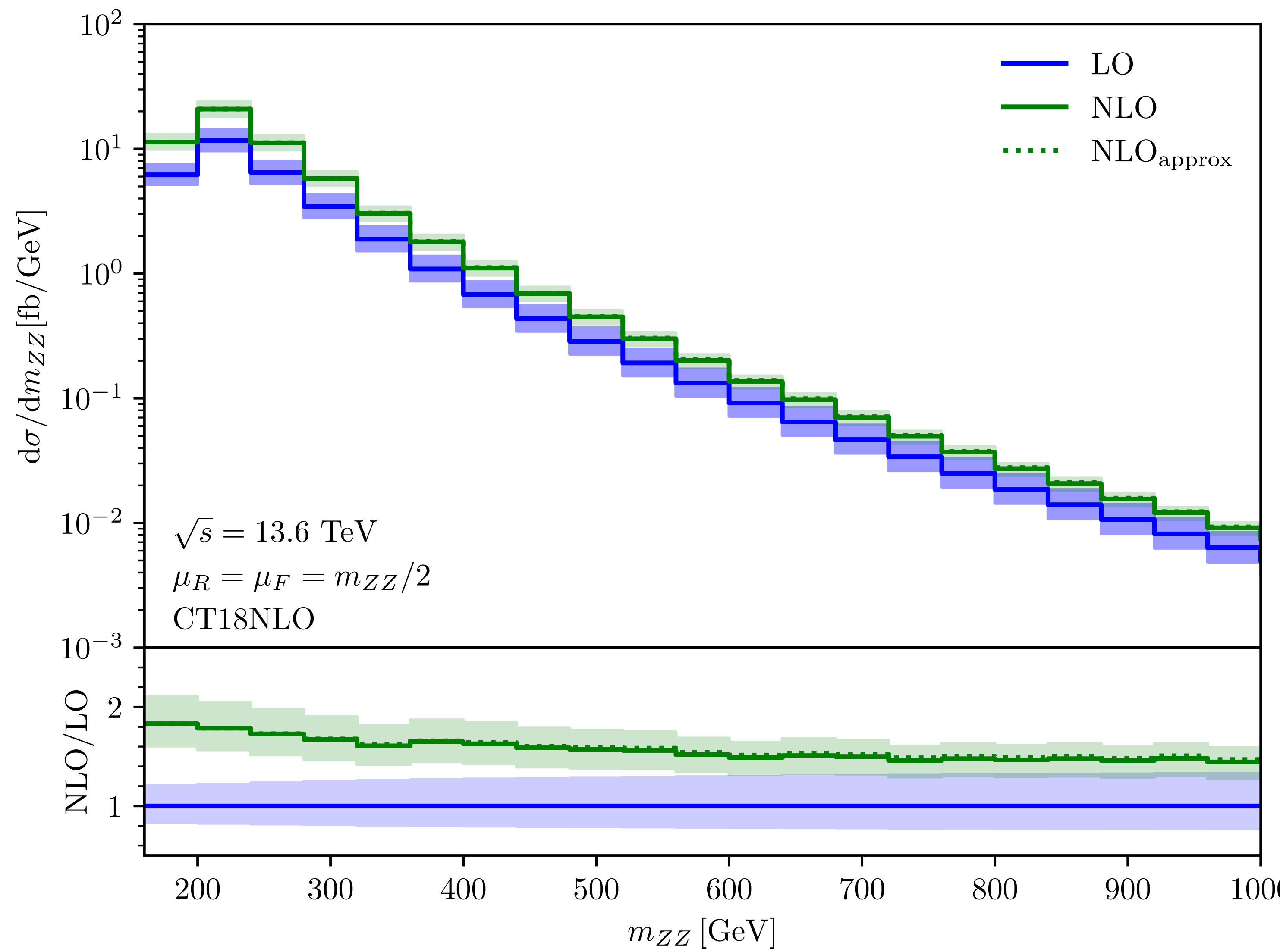
Complete NLO QCD Corrections

Putting it all together: complete NLO corrections are indeed large, K-factor ~ 1.7

NLO corrections reduce the scale uncertainties significantly

$$\sigma_{\text{LO}} = 1316^{+23.0\%}_{-18.0\%}\ \text{fb}$$
$$\sigma_{\text{NLO}} = 2275(12)^{+14.0\%}_{-12.0\%}\ \text{fb}$$

Approximation obtained by applying the massless K-factor to the massive born agrees well within the scale uncertainty, also for the invariant mass distribution



Agarwal, SPJ, Kerner, von Manteuffel 24

Conclusion

Z-boson Pair Production in Gluon-Fusion

NLO corrections are indeed very significant, also for massive amplitudes (~100% enhancement)

Size of finite remainder depends strongly on subtraction scheme

Approx. based on rescaling the massive Born by massless K-factor quite good for unpol. cross-section

Huge cancellations between Higgs and top-quark contributions, sensitive to exact SM couplings

Potential to examine p_T, longitudinal cross-section, anomalous couplings at NLO…

Sector Decomposition

New disteval integrator: ~3-10x faster than old intlib

Median lattice rules provide lattices of unlimited size, smaller fluctuations in error

Can now accept GiNaC compatible coefficient input

Work in progress: potential to avoid contour deformation in some cases provides huge speedups

Thank you for listening!

Backup

Periodizing Transforms

Sector decomposed functions are typically continuous and smooth but not periodic

Functions can be periodized by a suitable change of variables: $\mathbf{x} = \phi(\mathbf{u})$

$$I[f] \equiv \int_{[0,1]^d} \mathrm{d}\mathbf{x}\; f(\mathbf{x}) = \int_{[0,1]^d} \mathrm{d}\mathbf{u}\; \omega_d(\mathbf{u}) f(\phi(\mathbf{u}))$$

$$\phi(\mathbf{u}) = (\phi(u_1), \ldots, \phi(u_d)), \qquad \omega_d(\mathbf{u}) = \prod_{j=1}^{d} \omega(u_j) \qquad \text{and} \qquad \omega(u) = \phi'(u)$$

Korobov transform: $\omega(u) = 6u(1-u), \quad \phi(u) = 3u^2 - 2u^3$

Sidi transform: $\omega(u) = \pi/2 \sin(\pi u), \quad \phi(u) = 1/2(1 - \cos \pi t)$

Baker transform: $\phi(u) = 1 - |2u - 1|$

Weighted Function Spaces

Review: Dick, Kuo, Sloan 13

Assign weights $\gamma_{\mathfrak{u}}$ to each subset of dimension $\mathfrak{u} \subseteq \{1, \ldots, d\}$

Sobolev Space

Functions with square integrable first derivatives

Norm
$$||f||_{\gamma}^2 = \sum_{\mathfrak{u} \subseteq \{1,\ldots,d\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{d-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f(\mathbf{x})}{\partial \mathbf{x}_{\mathfrak{u}}} \mathrm{d}\mathbf{x}_{-\mathfrak{u}} \right)^2 \mathrm{d}\mathbf{x}_{\mathfrak{u}}$$

Worst-case error
$$e_{\gamma}^2 \leq \left(\frac{1}{\psi(n)} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,d\}} \gamma_{\mathfrak{u}}^{\lambda} \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}} \right)^{|\mathfrak{u}|} \right)^{\frac{1}{\lambda}}$$

$\lambda \in (1/2, 1]$

$$\varepsilon \sim \mathcal{O}(N^{-1})$$

Korobov Space

Periodic functions which are α times differentiable in each variable

$$||f||_{\gamma}^2 = \sum_{\mathbf{h} \in \mathbb{Z}^d} \frac{\prod_{j \in \mathfrak{u}(\mathbf{h})} |h_j|^{2\alpha}}{\gamma_{\mathfrak{u}(\mathbf{h})}} |\hat{f}(\mathbf{h})|^2$$

Fourier Coefficient

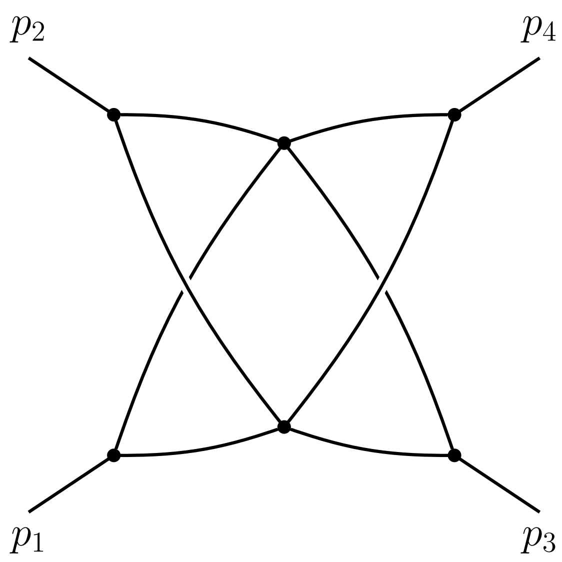
$$e_{\gamma}^2 \leq \left(\frac{1}{\psi(n)} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,d\}} \gamma_{\mathfrak{u}}^{\lambda} \left(2\zeta(2\alpha\lambda) \right)^{|\mathfrak{u}|} \right)^{\frac{1}{\lambda}}$$

$\lambda \in (1/(2\alpha), 1]$, smoothness α

$$\varepsilon \sim \mathcal{O}(N^{-\alpha})$$

Generating vector $\mathbf{z}$ precomputed for a **fixed** number of lattice points, chosen to minimise worst-case error Nuyens 07

Interesting Example



$$= \int_0^\infty \mathrm{d}x_0 \ldots \mathrm{d}x_7 \frac{\mathscr{U}(\mathbf{x})^{4\epsilon}}{\mathscr{F}(\mathbf{x};\mathbf{s})^{2+3\epsilon}} \delta(1 - x_7)$$

$$\mathscr{U}(\alpha) = \alpha_0\alpha_2\alpha_4 + \alpha_0\alpha_2\alpha_5 + \alpha_0\alpha_2\alpha_6 + (29 \text{ terms})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\ (\alpha_1\alpha_4 - \alpha_0\alpha_5)(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\ (\alpha_1\alpha_2 - \alpha_0\alpha_3)(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} = s_{12}\ \alpha_5(\alpha_3\alpha_6 - \alpha_2\alpha_7) + s_{13}\ \alpha_3(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\vdots$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} = s_{12}\ \alpha_2(\alpha_1\alpha_4 - \alpha_0\alpha_5) + s_{13}\ \alpha_4(\alpha_1\alpha_2 - \alpha_0\alpha_3)$$

Can have a leading Landau singularity with *generic kinematics* (arbitrary s_{12}, s_{13}) when each factor of $\mathscr{F}$ vanishes!

Resolution

1) Rescale parameters to *linearise* singular surfaces

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4 - \alpha_0\alpha_5)\,(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\,(\alpha_5\alpha_6 - \alpha_4\alpha_7)$$

$$\alpha_0 \to \alpha_0\alpha_1,\ \alpha_2 \to \alpha_2\alpha_3,\ \alpha_4 \to \alpha_4\alpha_5,\ \alpha_6 \to \alpha_6\alpha_7$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_4-\alpha_0)(\alpha_6-\alpha_2) - s_{13}(\alpha_2-\alpha_0)(\alpha_6-\alpha_4)\right]$$

2) Split the integral by imposing $\alpha_i \geq \alpha_j \geq \alpha_k \geq \alpha_l$

$$\begin{aligned}\alpha_0 &\to \alpha_0+\alpha_2+\alpha_4+\alpha_6,\\ \alpha_2 &\to \alpha_2+\alpha_4+\alpha_6,\\ \alpha_4 &\to \alpha_4+\alpha_6,\\ \alpha_6 &\to \alpha_6\end{aligned} \qquad +\text{perms}$$

$$\begin{aligned}\mathscr{F}_1(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_0+\alpha_2)(\alpha_2+\alpha_4) - s_{13}(\alpha_0)(\alpha_4)\right]\\ \mathscr{F}_2(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2)(\alpha_0+\alpha_2+\alpha_6) + s_{13}(\alpha_0)(\alpha_6)\right]\\ &\vdots\\ \mathscr{F}_{24}(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2+\alpha_4)(\alpha_4+\alpha_6) - s_{13}(\alpha_2)(\alpha_6)\right]\end{aligned}$$

All coefficients of s_{12}, s_{13} now have definite sign

NoCD: Avoiding Contour Deformation

Idea:

1. Construct transformations of the Feynman parameters which map the zeroes of the $\mathscr{F}$-polynomial to the boundary of integration

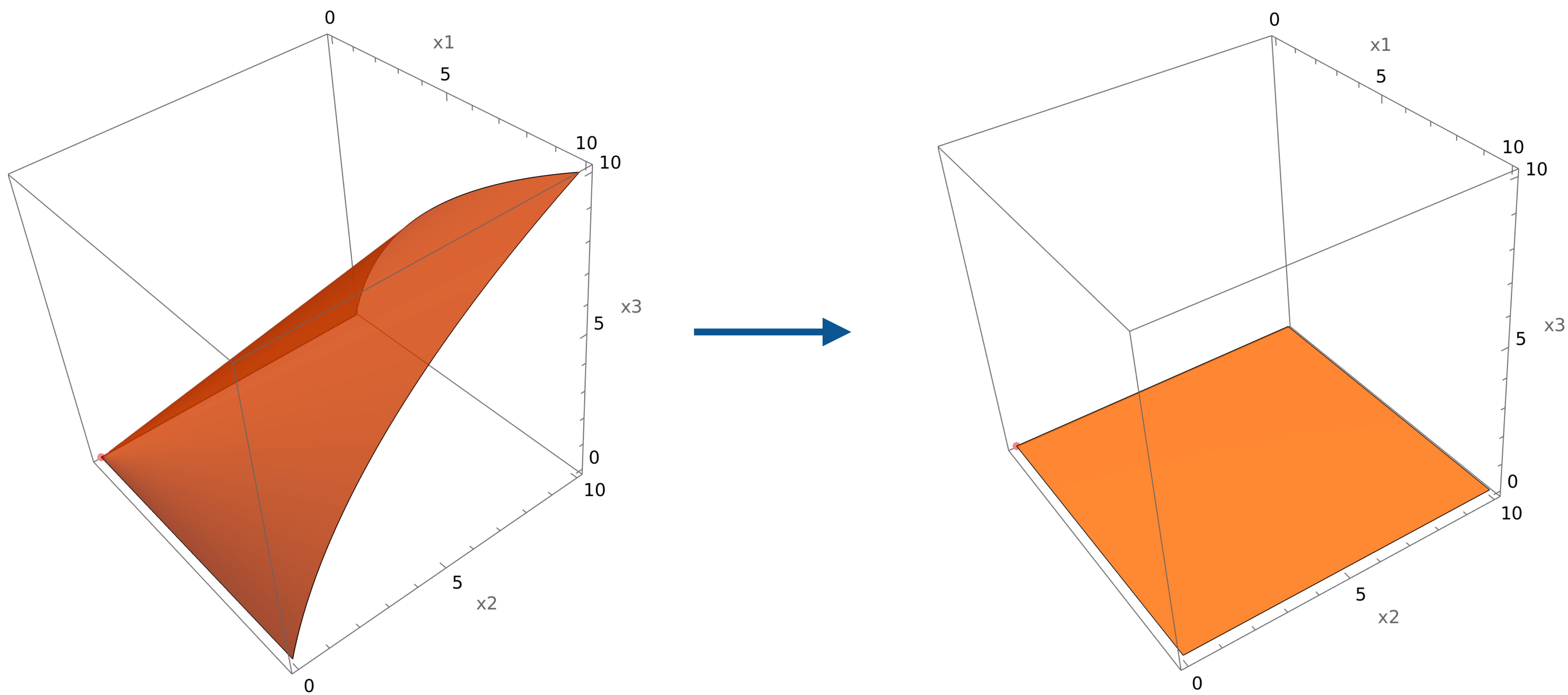


Figure: Thomas Stone

NoCD: Avoiding Contour Deformation

Idea:

2. For transformations which make $\mathscr{F}$ non-positive extract an overall minus sign (using the $i\delta$ prescription to generate the physically correct imaginary part)

3. Stitch together the resulting integrals

$$I = \sum_{n_+=1}^{N_+} I^+_{n_+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_-=1}^{N_-} I^-_{n_-}$$

The individual integrals $\{I^+_{n_+}, I^-_{n_-}\}$ have *manifestly* non-negative integrands

$\implies$ no contour deformation, trivial analytic continuation, faster to integrate

NoCD: Example 3

For $s > -t > 0$, two of the 6 independent integrals require contour deformation:

$$\mathcal{F}_3 = x_1 x_3 x_5 x_7 \left[-s x_0 x_2 + |t| \left(x_0 + x_4 \right) \left(x_2 + x_4 \right) \right]$$

$$\mathcal{F}_5 = x_1 x_3 x_5 x_7 \left[s x_6 \left(x_0 + x_2 + x_6 \right) - |t| \left(x_0 + x_6 \right) \left(x_2 + x_6 \right) \right]$$

Can express each of these in terms of 4 manifestly non-negative integrands

Putting the pieces together for the full integral:

$$I = \sum_{n_+=1}^{8} I^+_{n_+} + (-1 - i\delta)^{-2-3\varepsilon} \sum_{n_-=1}^{4} I^-_{n_-}$$

Verified result numerically against known analytic result

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21

NoCD: Example 3

Can now obtain results numerically ($s_{12} = 1,\ s_{13} = -1/5$)

$$I_3 = \epsilon^{-4}\left[(18.5195704502 - 15.707988011i) \pm (5.897 \cdot 10^{-5} + 5.897 \cdot 10^{-5} i)\right] + \ldots$$

$$I_3^{\text{NoCD}} = \epsilon^{-4}\left[(18.51948920208488 - 15.70796326794897i) \pm (4.032 \cdot 10^{-11} + 4.592 \cdot 10^{-11} i)\right] + \ldots$$

$$I_5 = \varepsilon^{-4}\left[(12.7432949988 - 23.561968275i) \pm (1.605 \cdot 10^{-5} + 1.415 \cdot 10^{-5} i)\right] + \ldots$$

$$I_5^{\text{NoCD}} = \varepsilon^{-4}\left[(12.74326269721394 - 23.5619449018131i) \pm (4.125 \cdot 10^{-11} + 6.919 \cdot 10^{-11} i)\right] + \ldots$$

Full result after a few minutes integration with pySecDec:

$$I = \epsilon^{-4}\left[8.340\mathbf{55} - 52.36\mathbf{08}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

$$I^{\text{NoCD}} = \epsilon^{-4}\left[8.3400403920\mathbf{28} - 52.3598775598\mathbf{3}\mathbf{47}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

$$I^{\text{analytic}} = \epsilon^{-4}\left[8.34004039223768 - 52.35987755984493i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

Numerics are much, much faster and more stable