Introduction to Inclusive semileptonic Decays of Heavy Hadrons

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Contents

Introduction: Why studying semileptonic heavy-quark decays?

- The Heavy Quark Expansion (HQE) opens the road to precision predictions
- Main Motivation: Determination CKM parameters, such as V_{cb} and V_{ub}
- Testing the quality of the theoretical methods
  - $b \to c\ell\bar{\nu}$ is "Heavy $\to$ Heavy": Local OPE in the timelike region
  - $b \to u\ell\bar{\nu}$ is "Heavy $\to$ Light": Local OPE, but also licht-cone OPE
  - $c \to s/d\ell\bar{\nu}$ is also "Heavy $\to$ Light"
- There is/will be a lot of data from BelleII and BESSIII

Inclusive Semileptonic Decays and the Heavy Quark Expansion (HQE)

Inclusive (Semileptonic) Heavy Hadron Decays

Effective Hamiltonian: $\mathcal{H}_{eff} = \frac{4G_F}{\sqrt{2}} V_{qQ} (\bar{q}_L \gamma_\mu Q_L)\,(\bar{\ell}_L \gamma^\mu \nu) \qquad Q = b, c \qquad q = u, d, s, c$

Optical theorem for the decay of a hadron H:

$$\Gamma \propto \sum_X (2\pi)^4 \delta^4(P_H - P_X) |\langle X | \mathcal{H}_{eff} | H(v)\rangle|^2$$

$$= \int d^4x \, \langle H(v) | \mathcal{H}_{eff}(x) \mathcal{H}_{eff}^\dagger(0) | H(v)\rangle = 2 \, \mathrm{Im} \int d^4x \, \langle H(v) | T\{\mathcal{H}_{eff}(x) \mathcal{H}_{eff}^\dagger(0)\} | H(v)\rangle$$

- Idea of the HQE: (Chay, Georgi, Grinstein, Manohar, Wise // Bigi, Shifman Uraltsev, Vainshtain, ...Neubert, ThM, ...)
  Consider a hadron H with a single heavy Q quark, moving with velocity $v = p_H / M_H$, then the (bound) heavy quark has a momentum $p_Q = m_Q v + k$, where $m_Q v$ is large compared to the residual momentum k.
  Expansion in the residual momentum k

technically: $Q(x) = \exp(-im_Q(v \cdot x))Q_v(x)$ so

$i\partial_\mu Q(x) = \exp(-im_Q(v \cdot x))(m_Q v_\mu + i\partial_\mu)Q_v(x)$

Replace $Q(x)$ by $Q_v(x)$ and define $\widetilde{\mathcal{H}}_{eff} = \frac{4G_F}{\sqrt{2}} V_{cb}(\bar{c}_L \gamma_\mu Q_{v,L})\,(\bar{\ell}_L \gamma^\mu \nu)$

$$\boxed{\Gamma \propto= 2\,\mathrm{Im} \int d^4x\, e^{-im_Q v \cdot x} \langle H(v)| T\{\widetilde{\mathcal{H}}_{eff}(x)\widetilde{\mathcal{H}}^\dagger_{eff}(0)\}|H(v)\rangle}$$

- The large scale m_Q appears in the exponent
- The remaining matrix element depends only on the residual momentum and on power suppressed terms
- As $m_Q \to \infty$: OPE of the T-product, however, $m_Q v$ is a timelike vector!

$$\boxed{\int d^4x\, e^{-im_Q v \cdot x}\, T\{\widetilde{\mathcal{H}}_{eff}(x)\widetilde{\mathcal{H}}^\dagger_{eff}(0)\} = \sum_{n=0}^{\infty} \left(\frac{1}{2m_Q}\right)^n C_{n+3}(\mu)\mathcal{O}_{n+3}}$$

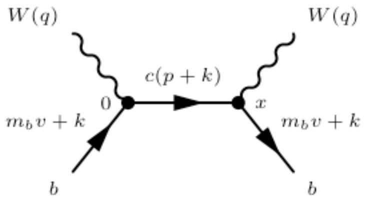
- Use optical theorem: take the forward matrix element:

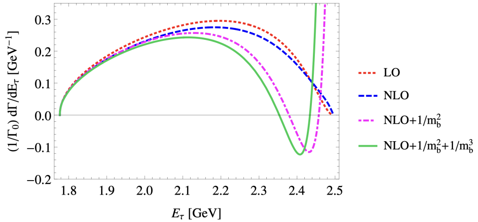
$$\Gamma \propto = 2 \,\mathrm{Im} \sum_{n=0}^{\infty} \left(\frac{1}{2m_Q} \right)^n C_{n+3}(\mu) \langle H(v) | \mathcal{O}_{n+3} | H(v) \rangle$$

- Thus the rate for the inclusive decay can be written as

$$\Gamma = \Gamma_0 + \frac{1}{m_Q}\Gamma_1 + \frac{1}{m_Q^2}\Gamma_2 + \frac{1}{m_Q^3}\Gamma_3 + \cdots$$

- The Γ_i are power series in $\alpha_s(m_Q)$: $\to$ Perturbation theory!
- In semileptonic decays:
  Leptonic Tensor can be separated, OPE for the hadronic tensor
  Differential rates can be computed!





- Endpoint region: $\rho = m_c^2/m_b^2$, $y = 2E_\ell/m_b$

$$\frac{d\Gamma}{dy} \sim \Theta(1-y-\rho)\left[2 + \frac{\lambda_1}{(m_b(1-y))^2}\left(\frac{\rho}{1-y}\right)^2\left\{3 - 4\frac{\rho}{1-y}\right\}\right]$$

- Spectra has no point-by-point interpretation
- HQE for the spectra is a series in $1/(2E_\ell - m_b)$
- HQE fails in the endpoint
- Heavy Quark Expansion for Moments of spectra

General Structure of the HQE: The Heavy $\to$ Heavy Case

$$d\Gamma = d\Gamma_0 + \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^2 d\Gamma_2 + \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^3 d\Gamma_3 + \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^4 d\Gamma_4$$
$$+ d\Gamma_5 \left(a_0 \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^5 + a_2 \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^3 \left(\frac{\Lambda_{\rm QCD}}{m_c}\right)^2 \right) + \ldots + d\Gamma_7 \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^3 \left(\frac{\Lambda_{\rm QCD}}{m_c}\right)^4$$

- $\Gamma_3 \sim \log m_c^2/m_b^2$: "Infrared" sensitivity to the charm quark mass
  (Bigi, Turczyk, Uraltsv, ThM // Breidenbach, Feldmann, Turczyk, ThM)
- At higher orders there are powers $1/m_c^n$, starting at dimension seven
- Realistic power counting $m_c^2 \sim \Lambda_{\rm QCD} m_b$

General Structure of the HQE: The Heavy $\to$ Light Case

$$d\Gamma = d\Gamma_0 + \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^2 d\Gamma_2 + \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^3 d\Gamma_3 + \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^4 d\Gamma_4 + d\Gamma_5 \left(\frac{\Lambda_{\rm QCD}}{m_b}\right)^5 + \dots$$

- Γ_3 contains additional four-quark operators, which mix with two-quark operators, so $\log m_c^2/m_b^2 \to \log \mu^2/m_b^2$
- At higher orders there are more four quark-operators
- The limit $m_q \to 0$ exists!
- Strange-quark mass is a boarderline case:
  Posssible power counting $m_s \sim \Lambda_{\rm QCD}$

HQE Parameters

- Γ_0 is the decay of a free quark (“Parton Model”)
- Γ_1 vanishes due to Heavy Quark Symmetries
- Γ_2 is expressed in terms of two parameters

$$2M_H\mu_\pi^2 = -\langle H(v)|\bar{Q}_v(iD)^2 Q_v|H(v)\rangle$$
$$2M_H\mu_G^2 = \langle H(v)|\bar{Q}_v\sigma_{\mu\nu}(iD^\mu)(iD^\nu)Q_v|H(v)\rangle$$

μ_π: Kinetic energy and μ_G: Chromomagnetic moment

- Γ_3 two more parameters from two-quark operators

$$2M_H\rho_D^3 = -\langle H(v)|\bar{Q}_v(iD_\mu)(ivD)(iD^\mu)Q_v|H(v)\rangle$$
$$2M_H\rho_{LS}^3 = \langle H(v)|\bar{Q}_v\sigma_{\mu\nu}(iD^\mu)(ivD)(iD^\nu)Q_v|H(v)\rangle$$

ρ_D: Darwin Term and ρ_{LS}: Spin-Orbit Term

$1/m_b^4$ and higher Orders (Dassinger, Feger, Turczyk, ThM, Vos, Milutin ...)

- Many new parameters, e.g.:

$\langle \vec{E}^2 \rangle$: Chromoelectric Field squared

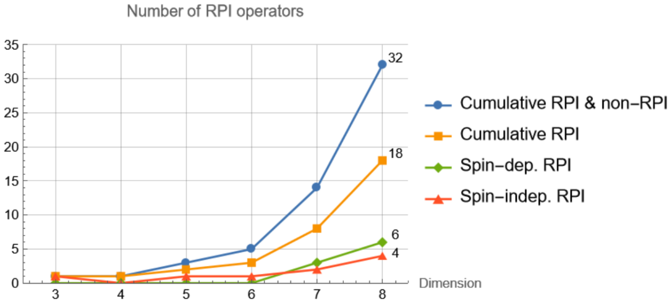
$\langle \vec{B}^2 \rangle$: Chromomagnetic Field squared

$\langle (\vec{p}^2)^2 \rangle$: Fourth power of the residual momentum

$\langle (\vec{p}^2)(\vec{\sigma} \cdot \vec{B}) \rangle$

$\langle (\vec{p} \cdot \vec{B})(\vec{\sigma} \cdot \vec{p}) \rangle$

$\dots$



(Milutin, Vos, ThM)

Dramatic (factorial) growth of the number of independent parameters

May be a hint that the HQE is asymptotic!

Device approximations to get values for the HQE parameters at higher orders

(Heinonen, ThM 1407.4384 [hep-ph])

Status of the HQE Calculation in $B \to X_c \ell \bar{\nu}$

- Tree level terms up to and including $1/m_b^5$ known
  Bigi, Zwicky, Uraltsev, Grimm, Kapustin, Turczyk, Vos, Milutin, ThM, ...
- $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^3)$ for the partonic rate and spectral moments are known
  Biswas, Melnikov, Czarnecki, Pak, Fael, Schönwald, Steinhauser ...
- $\mathcal{O}(\alpha_s)$ for $1/m_b^2$ is known for rates and spectra
  Becher, Boos, Lunghi, Gambino, Pivovarov, Rosenthal, Alberti
- $\mathcal{O}(\alpha_s)$ for $1/m_b^3$ is known for rates and spectra Pivovarov, Moreno, ThM

$$\boxed{V_{cb,incl} = (41.83 \pm 0.47) \times 10^{-3}}$$ (Finauri, Gambino: 2310.20324)

We are moving towards a TH-uncertainty of 1% in $V_{cb,incl}$!

Details start to matter: Proper definition of the quark mass, Duality Violations, ...

The quark mass definition

Total rate and spectral moments depend strongly on the quark mass:

$$\Gamma_0 = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} m_b^5 \left(1 + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^k g_k\right) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} m_b^5 \left(1 + \frac{\alpha_s}{\pi} g_1 + \cdots\right)$$

What should be inserted for m_b?

- Calculations start with the pole mass $m_b = m_b^{\mathrm{pole}}$ $\rightarrow$ This yields a large g_1
- In general: perturbative series is "asymptotic": for large order k: $g_k \sim k!$
  Renormalon Problem (of the Pole mass)
- The pole mass has an intrinsic uncertainty of $\sim \mathcal{O}(100\ \mathrm{MeV})$
- Precision requires a "short-distance" mass (Beneke, Braun, Neubert, Bigi, Uraltsev, Shifman ...)

The kinetic mass (Bigi, Shifman, Uraltsev, Vainshtein)

- Switch to the kinetic mass m_b^{kin}: Perturbative relation to the pole mass

$$m_b^{\mathrm{kin}}(\mu) = m_b^{\mathrm{pole}} \left(1 + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^k m_k(\mu) \right) = m_b^{\mathrm{pole}} \left(1 + \frac{\alpha_s}{\pi} m_1(\mu) + \cdots \right)$$

- Insert this

$$\Gamma_0 = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} (m_b^{\mathrm{kin}}(\mu))^5 \left(1 + \frac{\alpha_s}{\pi} (g_1 - 5m_1(\mu)) + \cdots \right)$$

- m_b^{kin} is is a short distance mass, much better known as the pole mass
- The perturbative series converges better: $|g_1 - 5m_1| \ll g_1$

- Kinetic mass = pole mass, but infrared gluons with $|\vec{k}| \leq \mu$ are cut out
- Definition via the HQET relation between the quark and the hadron mass:

$$M_H = m_Q \left(1 + \frac{\bar{\Lambda}}{m_Q} + \frac{\mu_\pi^2 - d_H \mu_G^2}{2m_Q^2} + \cdots \right) \to m_Q^{\rm Pole} = m_Q^{\rm kin}(\mu) \left(1 + \frac{[\bar{\Lambda}(\mu)]_{\rm pert}}{m_Q^{\rm kin}(\mu)} + \frac{[\mu_\pi^2(\mu)]_{\rm pert}}{2m_Q^{\rm kin}(\mu)^2} \right)$$

- $[\ldots]_{\rm pert}$: perturbative calculation with a gluon-momentum cut-off $|\vec{k}| \leq \mu$
- Equivalent to a perturbative calculation of the second moment of the spectral function for $b \to c\ell\bar{\nu}$ in eikonal approximation.

$$[\bar{\Lambda}(\mu)]_{\rm pert} = \frac{4}{3} C_F \frac{\alpha_s(\mu)}{\pi} \mu$$ (from first moment in eikonal approx.) $$[\mu_\pi^2(\mu)]_{\rm pert} = C_F \frac{\alpha_s(\mu)}{\pi} \mu^2$$

- If μ is large enough, the gluons with $|\vec{k}| \geq \mu$ can be treated perturbatively
- However, we still need $\mu \leq m_Q$: For bottom $\mu \sim 1$ GeV, but for charm?

Is Charm a Heavy Quark?

Look at the numbers ...

- The charm mass:

$$m_c \sim 1.2\,\mathrm{GeV} \quad \text{so} \quad \frac{\Lambda_{\mathrm{QCD}}}{m_c} \sim 0.3$$

- The nonperturbative HQE corrections are

$$\frac{\Lambda^2_{\mathrm{QCD}}}{m_c^2} \sim 0.1$$

- The strong coupling:

$$\alpha_s(m_c) \sim 0.3$$

The data do not look like this!
This was a problem in the early days of HQE ...

Lifetime Calculations for Heavy Hadrons H_Q (See recent review by Albrecht et al., 2402.04224)

- Precise data available
- Tool for Calculations: Heavy Quark Expansion:
  $m_b, m_c \to \infty$ with $\rho = m_c^2/m_b^2$ fixed, matching scale $\mu^2 \sim m_b m_c$
- Structure of the expansion

$$\Gamma(H_Q) = \Gamma_3 + \Gamma_5 \frac{\langle O_5 \rangle}{m_Q^2} + \Gamma_6 \frac{\langle O_6 \rangle}{m_Q^3} + \cdots + 16\pi^2 \left(\tilde{\Gamma}_6 \frac{\langle \tilde{O}_6 \rangle}{m_Q^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{O}_7 \rangle}{m_Q^4} + \cdots \right)$$

$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{4\pi} \Gamma_i^{(1)} + \left(\frac{\alpha_s}{4\pi} \right)^2 \Gamma_i^{(1)} + \cdots \quad \tilde{\Gamma}_i \quad \text{analogously}$$

$\langle O_i \rangle = \langle H_Q | O_i | H_Q \rangle \sim \Lambda_{\rm QCD}^{i-3}$ forward matrix elements with local operators of dimension i

Take at face value:

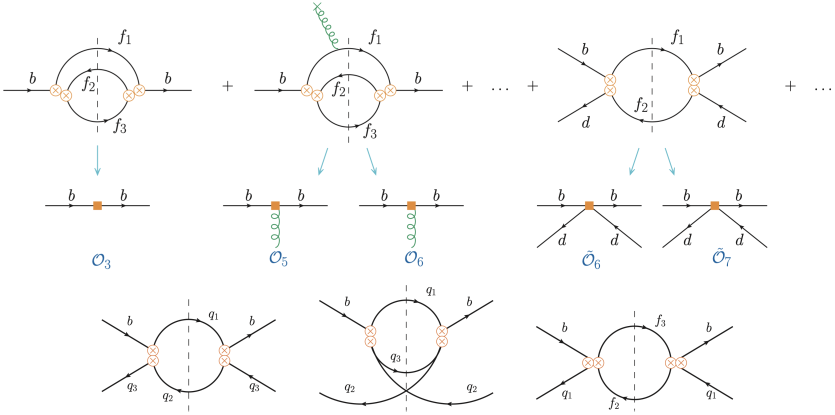
- To leading order in the HQE all H_Q hadrons have the same lifetime
- There are no terms of order $\Lambda_{\rm QCD}/m_Q$ terms
- The first non-pertrubative contribution comes at $\Lambda^2_{\rm QCD}/m_Q^2$
- Neglecting isospin breaking in the HQE parameters, lifetime differences between different H_Q hadrons come in at $\Lambda^3_{\rm QCD}/m_Q^3$

Confronting this with the data:

$$\left.\frac{\tau(D^\pm)}{\tau(D^0)}\right|^{\rm exp} = 2.563 \pm 0.017\,, \quad \left.\frac{\tau(D_s)}{\tau(D^0)}\right|^{\rm exp} = 1.219 \pm 0.017\,, \quad \left.\frac{\tau(D^\pm)}{\tau(\Lambda_c)}\right|^{\rm exp} = 5.123 \pm 0.014$$

$$\left.\frac{\tau(B_s)}{\tau(B_d)}\right|^{\rm exp} = 0.998 \pm 0.004\,, \quad \left.\frac{\tau(B^+)}{\tau(B_d)}\right|^{\rm exp} = 1.076 \pm 0.004 \quad \left.\frac{\tau(\Lambda_b)}{\tau(B^+)}\right|^{\rm exp} = 0.969 \pm 0.006$$

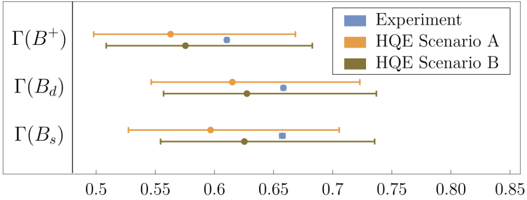
The $16\pi^2$ story:

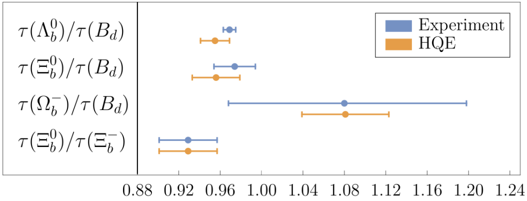


$$16\pi^2 \frac{\Lambda_{\rm QCD}^3}{m_b^3} \sim 0.2 \quad \text{but} \quad 16\pi^2 \frac{\Lambda_{\rm QCD}^3}{m_c^3} \sim 6$$

Bottom Lifetimes

The HQE works!





- Full α_s and α_s^2 known
- Full α_s/m^2 known
- Tree level complete dimension seven in the pipeline
- Lifetime rations do not have the m_Q^5 dependence

Inclusive semileptonic charm decays (Gambino, Kamenik, Fael, Vos, ThM)

- The $16\pi^2$ are as large as the leading term for the lifetimes (Pauli Interference)
- In semileptonic decays, there is no Pauli Interference, only annihilation

Relevant piece of the hadronic tensor at order $1/m_c^3$:

$$d\Gamma_{sl} \sim (g_{\mu\nu}q^2 - q_\mu q_\nu)\langle H_c(q)|\bar{c}\gamma^\mu(1-\gamma_5)q\bar{q}\gamma^\nu(1-\gamma_5)c|H_c(q)\rangle$$

where in general

$$\langle H_c(q)|\bar{c}\gamma^\mu(1-\gamma_5)q\bar{q}\gamma^\nu(1-\gamma_5)c|H_c(q)\rangle = A(q^2)q^\mu q^\nu + B(q^2)g^{\mu\nu}$$

and so

$$d\Gamma_{sl} \sim 3B(q^2)q^2$$

in naive vacuum insertion we get $B(q^2) = 0$

$$\langle H_c(q)|\bar{c}\gamma^\mu(1-\gamma_5)q\bar{q}\gamma^\nu(1-\gamma_5)c|H_c(q)\rangle \sim \langle H_c(q)|\bar{c}\gamma^\mu(1-\gamma_5)q|0\rangle \langle 0|\bar{q}\gamma^\nu(1-\gamma_5)c|H_c(q)\rangle = f_H^2 q^\mu q^\nu$$

Compare to the data:

$$\frac{\tau(D^0)\,\mathrm{BR}(D^+ \to e^+ + \text{anything})}{\tau(D^+)\,\mathrm{BR}(D^0 \to e^+ + \text{anything})} = 0.985 \pm 0.015 \pm 0.024$$

$$\frac{\tau(D^0)\,\mathrm{BR}(D_s \to e^+ + \text{anything})}{\tau(D_s)\,\mathrm{BR}(D^0 \to e^+ + \text{anything})} = 0.828 \pm 0.051 \pm 0.025$$

$$\frac{\tau(D^0)\,\mathrm{BR}(\Lambda_c \to e^+ + \text{anything})}{\tau(\Lambda_c)\,\mathrm{BR}(D^0 \to e^+ + \text{anything})} = 1.27 \pm 0.06$$

- No significant $16\pi^2$ enhancement for the partial rates!
- Nature seems to be close to naive vacuum insertion...

Is there a HQE for inclusive semileptonic charm decays?

The $16\pi^2$ enhancement seems to play a role only in the nonleptonic channels

Try the HQE for charm fo the semileptonic rates:

- Start from the Heavy $\to$ Light case with massless final-state quark
- Add in power supressed terms $m_s/m_c \sim \Lambda_{\rm QCD}/m_c$
- Include the four-quark operators, thus no infrared sensitive terms related $\log m_s$ can appear
- Employ vacuum insertion to account for the phenomenological absence of $(16\pi^2)$ enhanced terms
- Use the same machinery as in B Decays: Moments etc.
- Remaining issue: Define a short-distance charm mass for scales $\mu \leq m_c$

HQE parameters for charm

- We have: $\mu_\pi^2(B) = \mu_\pi^2(D) + \mathcal{O}(\Lambda_{\rm QCD}/m_c)\,, \mu_G^2(B) = \mu_G^2(D) + \mathcal{O}(\Lambda_{\rm QCD}/m_c)$
- ρ_D mixes with the four quark operators, so we define new HQE parameters

$$\tau_0 = 128\pi^2(T_1 - T_2) + 8\ln\left(\frac{\mu^2}{m_c^2}\right)\rho_D$$

with

$$2M_D T_1 = \langle D|(\bar{c}_v \slashed{v} P_L s)(\bar{s}\slashed{v} P_L c_v) \quad 2M_D T_2 = \langle D|(\bar{c}_v \gamma_\mu P_L s)(\bar{s}\gamma^\mu P_L c_v)$$

- ρ_D depends on the renormalization scale, τ_0 is RG invariant.
- ρ_D extracted in $b \to c\ell\bar{\nu}$ is actually $\rho_D(m_c)$.

Summary and Conclusion

- The HQE for inclusive bottom-hadron decays is a mature precison tool not only for the semileptonic case
- The HQE for charm needs to be scrutinized further
  - The HQE for lifetimes suffers from large terms formmally of higher order
  - The HQE for semileptonic decay seems to be protected against such terms
    - ... is this accidential?
    - Can we come up with a suitable charm-mass definition?

Is there a way to do also precision charm physics?