Hadronic matrix elements for decay rate differences using sum rules

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published as JHEP 04 (2025) 081

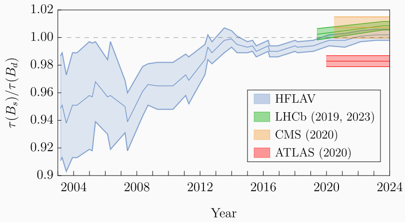
24 September 2025

Theoretical Particle Physics (TP1), U Siegen

Introduction

Motivation

- B (and D) mesons live long enough to be resolved with modern detectors.
- We live in a time of experiments (accelerators) designed for their study: Belle II (SuperKEKB), LHCb $\Rightarrow$ data is (will be) plentiful.
- Their decays provide vital input for the scrutiny of the Standard Model, in particular the CKM mechanism.
- Their (total) lifetimes are experimentally determined to excellent precision.
- For some observables theory has achieved a reasonable level of precision as well.



[taken from Albrecht, Bernlochner, Lenz, Rusov, '24]

Lifetimes: Status, early 2024

Experiment: HFLAV, PDG

$$\Gamma\left(B^{+}\right) = 0.611(2)\,\mathrm{ps}^{-1}$$

$$\Gamma\left(B_d\right) = 0.658(2)\,\mathrm{ps}^{-1}$$

$$\Gamma\left(B_s\right) = 0.657(2)\,\mathrm{ps}^{-1}$$

$$\frac{\tau\left(B^{+}\right)}{\tau\left(B_d\right)} = 1.076(4)$$

$$\frac{\tau\left(B_s\right)}{\tau\left(B_d\right)} = 1.002(4)$$

Theory: Albrecht, Bernlochner, Lenz, Rusov, '24, Lenz, Piscopo, Rusov, '22

$$\Gamma\left(B^{+}\right) = \left(0.58^{+0.11}_{-0.07}\right)\mathrm{ps}^{-1}$$

$$\Gamma\left(B_d\right) = \left(0.63^{+0.11}_{-0.07}\right)\mathrm{ps}^{-1}$$

Kay's, Maria Laura's talks

$$\Gamma\left(B_s\right) = \left(0.63^{+0.11}_{-0.07}\right)\mathrm{ps}^{-1}$$

$$\frac{\tau\left(B^{+}\right)}{\tau\left(B_d\right)} = 1.086(22)$$

this talk

$$\frac{\tau\left(B_s\right)}{\tau\left(B_d\right)} = 1.003(6)$$ small Darwin, small $\cancel{SU(3)_F}$

Lifetimes: Theory

We integrate out the W boson to obtain the **effective** $|\Delta B| = 1$ Hamiltonian (QCD)

$$\mathcal{H}_{\text{eff}}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}} V_{q_1 b}^{*} V_{q_2 q_3} \sum_i C_i^{(\prime)} \mathcal{Q}_i^{(\prime)} \,.$$

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The **optical theorem** then yields the well-known form for the **total decay rate** of a B_q meson

$$\begin{aligned}\Gamma\left(B_q\right) &= \mathrm{Im}\,\langle B_b | \mathcal{T} | B_b \rangle \\ &= \frac{1}{2M_{B_q}} \mathrm{Im}\left[\left\langle B_q \middle| \mathrm{i} \int \mathrm{d}^4x\, \mathbf{T}\left\{\mathcal{H}_{\mathrm{eff}}^{|\Delta B|=1}(x)\, \mathcal{H}_{\mathrm{eff}}^{|\Delta B|=1}(0)\right\} \middle| B_q \right\rangle\right] .\end{aligned}$$

Heavy-quark expansion

The operator-product expansion (OPE) of the transition operator $\mathcal{T}$ is the **heavy-quark expansion** (HQE)

$$\Gamma\left(B_q\right) = \boxed{\Gamma_3 + \Gamma_5 \frac{\langle O_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle O_6 \rangle}{m_b^3} + \dots} + \boxed{16\pi^2 \left(\tilde{\Gamma}_6 \frac{\left\langle \tilde{O}_6 \right\rangle}{m_b^3} + \tilde{\Gamma}_7 \frac{\left\langle \tilde{O}_7 \right\rangle}{m_b^4} + \dots \right)},$$

with $\left\langle \tilde{O}_i \right\rangle \equiv \left\langle B_q \middle| \tilde{O}_i \middle| B_q \right\rangle$,

We distinguish between *non-spectator effects* and *spectator effects*.

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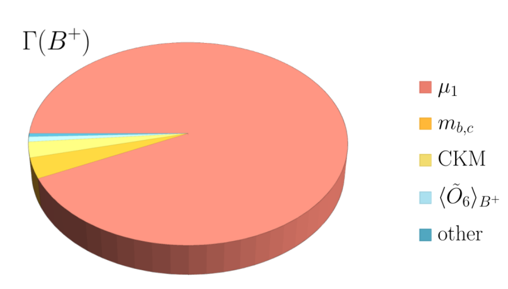
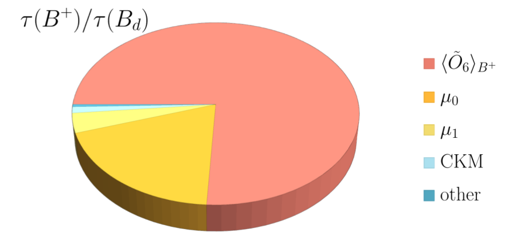
We distinguish between *non-spectator effects* and *spectator effects*.

Our focus is on the spectator effects at mass dimension 6 ($\mathcal{O}\left(1/m_b^3\right)$).

For the SM, the hadronic matrix elements were computed in [Kirk, Lenz, Rauh, 2017].

Theoretical uncertainties, early 2024





[taken from Albrecht, Bernlochner, Lenz, Rusov, '24]

- Total decay rates are dominated by the scale uncertainty μ_1 of Γ_3.
- For $\tau\left(B^+\right)/\tau\left(B_d\right)$, the HQET sum rules determination is still the dominant source of uncertainty. But lattice results are imminent! Matthew's talk

BSM contributions to the dimension-6 spectator effects

- Spectator effects at dimension-6 are the main reason for $\Gamma\left(B_d\right) \neq \Gamma\left(B^+\right)$.
- New physics in the $|\Delta B| = 1$ Hamiltonian would lead to additional $\Delta B = 0$ operators in the HQE $\quad\Rightarrow\quad$ **What are their matrix elements?**
- We compute them via heavy-quark effective theory (HQET) sum rules.

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Why sum rules (and not lattice QCD)?...

... because sum rules are an independent systematic approach, computationally cheaper and rather flexible.

... but they are intrinsically rather imprecise and it is non-trivial to systematically refine them.

Lifetimes: theory

The HQE yields at dimension 6 a number of 20 physical operators defined in terms of **QCD** fields.

Effective $\Delta B = 0$ operators in QCD

$$
\begin{aligned}
O_1^q &\equiv Q_1^q \equiv \bar{b}\gamma_\mu\,(1-\gamma_5)\,q\,\bar{q}\gamma^\mu\,(1-\gamma_5)\,b\,, &\quad O_2^q &\equiv Q_2^q \equiv \bar{b}\,(1-\gamma_5)\,q\,\bar{q}\,(1+\gamma_5)\,b\,,\\
O_3^q &\equiv T_1^q \equiv \bar{b}\gamma_\mu\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^\mu\,(1-\gamma_5)\,T^a b\,, &\quad O_4^q &\equiv T_2^q \equiv \bar{b}\,(1-\gamma_5)\,T^a q\,\bar{q}\,(1+\gamma_5)\,T^a b\,,\\
O_5^q &\equiv Q_3 \equiv \bar{b}\gamma_\mu\,(1-\gamma_5)\,q\,\bar{q}\gamma^\mu\,(1+\gamma_5)\,b\,, &\quad O_6^q &\equiv Q_4 \equiv \bar{b}\,(1-\gamma_5)\,q\,\bar{q}\,(1-\gamma_5)\,b\,,\\
O_7^q &\equiv T_3 \equiv \bar{b}\gamma_\mu\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^\mu\,(1+\gamma_5)\,T^a b\,, &\quad O_8^q &\equiv T_4 \equiv \bar{b}\,(1-\gamma_5)\,T^a q\,\bar{q}\,(1-\gamma_5)\,T^a b\,,\\
O_9^q &\equiv Q_5 \equiv \bar{b}\sigma_{\mu\nu}\,(1-\gamma_5)\,q\,\bar{q}\sigma^{\mu\nu}\,(1-\gamma_5)\,b\,, &\quad O_{10}^q &\equiv T_5 \equiv \bar{b}\sigma_{\mu\nu}\,(1-\gamma_5)\,T^a q\,\bar{q}\sigma^{\mu\nu}\,(1-\gamma_5)\,T^a b\,,
\end{aligned}
$$

$$
O_i'^{\,q} \equiv O_i^q\big|_{1\mp\gamma_5\to 1\pm\gamma_5}\,, \qquad i = 1,\dots,10\,.
$$

Lifetimes: theory

Effective $\Delta B = 0$ operators in HQET

$$\tilde{O}_1^q \equiv \tilde{Q}_1^q \equiv \bar{h}\gamma_\mu\,(1-\gamma_5)\,q\,\bar{q}\gamma^\mu\,(1-\gamma_5)\,h\,, \qquad \tilde{O}_2^q \equiv \tilde{Q}_2^q \equiv \bar{h}\,(1-\gamma_5)\,q\,\bar{q}\,(1+\gamma_5)\,h\,,$$
$$\tilde{O}_3^q \equiv \tilde{T}_1^q \equiv \bar{h}\gamma_\mu\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^\mu\,(1-\gamma_5)\,T^a h\,, \qquad \tilde{O}_4^q \equiv \tilde{T}_2^q \equiv \bar{h}\,(1-\gamma_5)\,T^a q\,\bar{q}\,(1+\gamma_5)\,T^a h\,,$$
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$$\tilde{O}_i^{\prime\, q} \equiv \tilde{O}_i^q\Big|_{1\mp\gamma_5 \leftrightarrow 1\pm\gamma_5}\,, \qquad i = 1,\ldots,8\,.$$

There are no tensor operators in HQET owing to [Lenz, Müller, Piscopo, Rusov, 2022]

$$\bar{h}\sigma_{\mu\nu}P_L q\,\bar{q}\sigma^{\mu\nu}P_L h = -4\left[\bar{h}P_L q\,\bar{q}P_L h - \bar{h}\gamma_\mu P_L q\,\bar{q}\gamma^\mu P_R h\right] + \mathcal{O}\left(\frac{1}{m_Q}\right)\,.$$

Bag parameters

Vacuum saturation/insertion approximation (VSA/VIA)

Take for example the QCD operator $Q_1 = 4\bar{b}\gamma_\mu P_L q\,\bar{q}\gamma^\mu P_L b$ and $\mathbb{1} = \sum_X |X\rangle\langle X|$, then

$$\begin{aligned}
\langle B|Q_1|B\rangle &= 4\sum_X \left\langle B\middle|\bar{b}\gamma_\mu P_L q\middle|X\right\rangle \langle X|\bar{q}\gamma^\mu P_L b|B\rangle \\
&= 4\left\langle B\middle|\bar{b}\gamma_\mu P_L q\middle|0\right\rangle \langle 0|\bar{q}\gamma^\mu P_L b|B\rangle + 4\sum_{|X\rangle\neq|0\rangle} \left\langle B\middle|\bar{b}\gamma_\mu P_L q\middle|X\right\rangle \langle X|\bar{q}\gamma^\mu P_L b|B\rangle \\
&= f_B^2 M_B^2 + 4\sum_{|X\rangle\neq|0\rangle} \left\langle B\middle|\bar{b}\gamma_\mu P_L q\middle|X\right\rangle \langle X|\bar{q}\gamma^\mu P_L b|B\rangle \equiv f_B^2 M_B^2\, B_1(\mu)\,.
\end{aligned}$$

The bag parameters $B(\mu)$ incorporate the deviations from the VIA!

QCD: $\left\langle 0\middle|\bar{b}\gamma_\mu\gamma_5 q\middle|B(p)\right\rangle = -\mathrm{i} f_B p_\mu$, HQET: $\left\langle 0\middle|\bar{h}\gamma_\mu\gamma_5 q\middle|B(v)\right\rangle = -\mathrm{i}F(\mu)\, v_\mu$.

Lifetime ratios

$$\frac{\Gamma\left(B_d\right)}{\Gamma\left(B^+\right)} = 1 + \tau(B^+)\frac{16\pi^2}{m_b^3}\sum_{q=u,d,s,c}\sum_i \tilde{\Gamma}^q_{6,i}\left(\frac{\overline{\left\langle O_i^q\right\rangle}_{B_d}}{2M_{B_d}} - \frac{\overline{\left\langle O_i^q\right\rangle}_{B^+}}{2M_{B^+}}\right) + \mathcal{O}\left(\frac{1}{m_b^4}\right)\,.$$

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Using **isospin symmetry** we can relate the matrix elements between the different external states and express the decay rate ratio in terms of matrix elements between a **single** external state:

Lifetime ratio at dimension 6

$$\frac{\Gamma\left(B_d\right)}{\Gamma\left(B^+\right)} = 1 + \tau(B^+)\frac{16\pi^2}{m_b^3}\sum_i \left(\tilde{\Gamma}^d_{6,i} - \tilde{\Gamma}^u_{6,i}\right)\frac{\overline{\left\langle O_i^u - O_i^d\right\rangle}_{B^+}}{2M_{B^+}} + \mathcal{O}\left(\frac{1}{m_b^4}\right).$$

VIA approximation

One can express the decay rate in terms of **isospin-breaking combinations** of the **HQET** operators,

$$\tilde{O}_i \equiv \tilde{O}_i^u - \tilde{O}_i^d \,,$$

whose matrix elements we parameterise in terms of **bag parameters** and the HQET meson decay constant $F(\mu)$,

$$\overline{\left\langle \tilde{Q}_i \right\rangle}_{B^+}(\mu) = \tilde{A}_i F^2(\mu)\, \tilde{B}_i(\mu) \,,$$
$$\overline{\left\langle \tilde{T}_i \right\rangle}_{B^+}(\mu) = \tilde{A}_i F^2(\mu)\, \tilde{\epsilon}_i(\mu) \,,$$

with $\tilde{A}_{1,2} = +1$ and $\tilde{A}_{3,4} = -1$ in HQET. In the VIA, the HQET bag parameters read

$$\tilde{B}_i(\mu) = 1 \,, \qquad \tilde{\epsilon}_i(\mu) = 0 \,.$$

The HQET sum rule

HQET sum rules for bag parameters

Consider the three-point correlator for an HQET operator $\tilde{Q}$

$$K_{\tilde{Q}}(\omega_1, \omega_2) = \int \mathrm{d}^d x_1 \mathrm{d}^d x_2 \mathrm{e}^{\mathrm{i}p_1 \cdot x_1 - \mathrm{i}p_2 \cdot x_2} \left\langle 0 \middle| \mathbf{T}\left\{ \tilde{j}^\dagger(x_2)\, \tilde{Q}(0)\, \tilde{j}(x_1) \right\} \middle| 0 \right\rangle ,$$

with $\tilde{j}_5 \equiv \bar{h}\gamma_5 q$ and $\omega_{1,2} = p_{1,2} \cdot v$.

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with $\tilde{j}_5 \equiv \bar{h} \gamma_5 q$ and $\omega_{1,2} = p_{1,2} \cdot v$. K_Q is analytic in $\omega_{1,2}$ except for discontinuities at positive real $\omega_{1,2}$ $\quad \Rightarrow \quad$ **dispersion relation** for the **hadronic** correlator:

$$K_{\tilde{Q}}(\omega_1, \omega_2) = \int_0^\infty \mathrm{d}\eta_1 \mathrm{d}\eta_2 \frac{\rho^{\mathrm{had}}(\omega_1, \omega_2)}{(\eta_1 - \omega_1)(\eta_2 - \omega_2)} + \text{ subtraction terms} ,$$

where

$$\rho^{\mathrm{had}}(\omega_1, \omega_2) = F^2(\mu) \left\langle \tilde{Q}(\mu) \right\rangle \delta\left(\omega_1 - \bar{\Lambda}\right) \delta\left(\omega_2 - \bar{\Lambda}\right) + \rho^{\mathrm{cont}}(\omega_1, \omega_2) , \qquad \bar{\Lambda} = M_B - m_b .$$

HQET sum rules: OPE

At large negative $\omega_{1,2}$ compute $K_{\tilde{Q}}$ through an OPE:

$$K_{\tilde{Q}}^{\mathrm{OPE}}(\omega_1,\omega_2) = K_{\tilde{Q}}^{\mathrm{pert}}(\omega_1,\omega_2) + K_{\tilde{Q}}^{\langle\bar{q}q\rangle}(\omega_1,\omega_2)\,\langle\bar{q}q\rangle + K_{\tilde{Q}}^{\langle\alpha_s G^2\rangle}(\omega_1,\omega_2)\,\langle\alpha_s G^2\rangle + \dots$$

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Quark-hadron duality allows us to equate the OPE correlator and the hadronic correlator.

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Quark-hadron duality allows us to equate the OPE correlator and the hadronic correlator. In order to suppress the dependence on the subtraction terms and the continuum part of the spectral density, we perform a **double Borel transform**

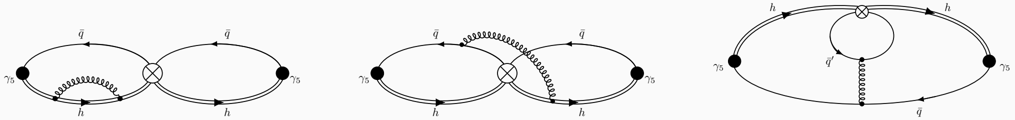
$$\int_0^\infty \mathrm{e}^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}}\rho^{\mathrm{OPE}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\mathrm{d}\omega_2 = \int_0^\infty \mathrm{e}^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}}\rho^{\mathrm{had}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\mathrm{d}\omega_2\,.$$

Thus, we can simply cut off the continuum part of the spectral density:

$$F^2(\mu)\left\langle\tilde{Q}(\mu)\right\rangle \mathrm{e}^{-\frac{\bar{\Lambda}}{t_1}+\frac{\bar{\Lambda}}{t_2}} = \int_0^\infty \mathrm{e}^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}}\rho^{\mathrm{OPE}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\mathrm{d}\omega_2\,.$$

Perturbative calculation

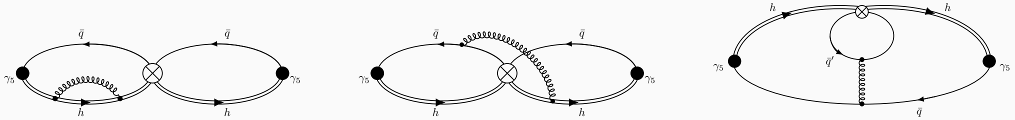
$$K_{\tilde{Q}}^{\mathrm{OPE}}(\omega_1,\omega_2) = \boxed{K_{\tilde{Q}}^{\mathrm{pert}}(\omega_1,\omega_2)} + K_{\tilde{Q}}^{\langle\bar{q}q\rangle}(\omega_1,\omega_2)\langle\bar{q}q\rangle + K_{\tilde{Q}}^{\langle\alpha_s G^2\rangle}(\omega_1,\omega_2)\langle\alpha_s G^2\rangle + \ldots$$



- Only **non-factorisable** diagrams contribute to the deviation $B(\mu) - B_{\mathrm{VIA}}(\mu)$.
- Eye diagrams do not contribute to the lifetime difference (up to tiny isospin-breaking corrections).

Perturbative calculation

$$K^{\mathrm{OPE}}_{\tilde{Q}}(\omega_1, \omega_2) = \boxed{K^{\mathrm{pert}}_{\tilde{Q}}(\omega_1, \omega_2)} + K^{\langle \bar{q}q \rangle}_{\tilde{Q}}(\omega_1, \omega_2) \langle \bar{q}q \rangle + K^{\langle \alpha_s G^2 \rangle}_{\tilde{Q}}(\omega_1, \omega_2) \langle \alpha_s G^2 \rangle + \ldots$$



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- Eye diagrams do not contribute to the lifetime difference (up to tiny isospin-breaking corrections).

Hence we only need to compute diagrams of the second type! Master integrals have been computed before [Grozin, Lee, 2009; Grozin et al., 2016].

Tools used for the perturbative calculation

Method 1

- FeynCalc [Mertig et al., 1990; 2016; Shtabovenko et al., 2020]; LiteRed [Lee, 2012; Lee, 2013]; FIRE [Smirnov, 2008; Smirnov, Chukharev, 2019]
- Dirac algebra in the BMHV scheme (FeynCalc)

Method 2

- qgraf [Nogueira, 1993]; tapir [Gerlach, Herren, Lang, 2022]; FORM [Vermaseren et al, 2001; 2012; 2017]
- Dirac algebra using TRACER [Jamin, Lautenbacher, 1993] (BMHV)
- integrals treated using Kira [Maierhöfer et al., 2018; 2021] (IBP) and HypExp [Huber, Maitre, 2006; 2007] (expansion of master integrals)

Renormalisation

The renormalisation procedure involves—as usual—the physical as well as some *evanescent* operators

$$\boxed{K^{(1)}_{\tilde{O}_i} = K^{(1),\mathrm{bare}}_{\tilde{O}_i} + \frac{\alpha_s}{4\pi}\frac{1}{2\epsilon}\left[\left(\hat{\tilde{\gamma}}^{(0)}_{\tilde{O}_i\tilde{O}_j} - 2\hat{\tilde{\gamma}}^{(0)}_{\tilde{j}}\delta_{ij}\right)K^{(0)}_{\tilde{O}_j} + \hat{\tilde{\gamma}}^{(0)}_{\tilde{O}_i\tilde{E}_j}K^{(0)}_{\tilde{E}_j}\right].}$$

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Note:

- The “renormalised” correlator still has $1/\epsilon^3$ poles.
- However, upon taking the double discontinuity, the resulting $1/\epsilon^1$ poles from both terms precisely cancel.
- This is an important check because they have very different origins.

Analytical results for the perturbative contribution

Multiplying by a suitable weight function, one can avoid the double Borel integral.

Analytical results

$$r_{\tilde{T}_1}(x, L_\nu) = -7 + \frac{a_1}{8} + \frac{2\pi^2}{3} - \frac{3}{2}L_\nu - \frac{1}{4}\phi(x) \,,$$

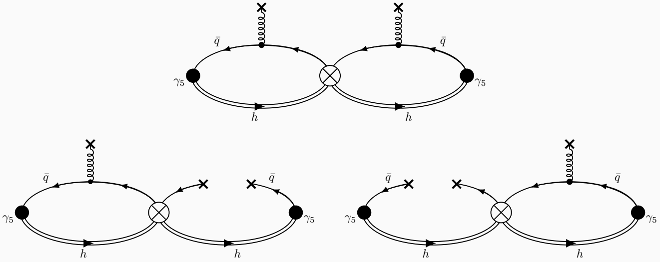
$$r_{\tilde{T}_2}(x, L_\nu) = -\frac{29}{4} + \frac{a_2}{8} + \frac{2\pi^2}{3} - \frac{3}{2}L_\nu - \frac{1}{4}\phi(x) \,, \quad L_\nu = \log\left(\frac{\mu^2}{4\nu_1\nu_2}\right) ,$$

$$r_{\tilde{T}_3}(x, L_\nu) = 1 - \frac{a_3}{8} - \frac{2\pi^2}{3} \,,$$

$$r_{\tilde{T}_4}(x, L_\nu) = \frac{15}{2} - \frac{2\pi^2}{3} + \frac{3}{2}L_\nu + \frac{1}{4}\phi(x) \,, \quad \phi(x) = \begin{cases} x^2 - 8x + 6\log(x) \,, x \le 1 \\ \frac{1}{x^2} - \frac{8}{x} - 6\log(x) \,, x > 1 \end{cases}$$

$$\Delta\tilde{B}^{\text{pert}}_{\tilde{O}_i}(\mu_\rho) = \frac{C_F}{N_c \tilde{A}_{\tilde{O}_i}} \frac{\alpha_s(\mu_\rho)}{4\pi} \, r_{\tilde{O}_i}\left(1, \log\frac{\mu_\rho^2}{4\bar{\Lambda}^2}\right) .$$

Condensate contributions



Condensate contributions (up to mass dimension 5) occur because of either…

- … soft gluons emitted from light quarks, forming a condensate $\langle G_{\mu\nu} G^{\mu\nu} \rangle$.
- … uncontracted light-quark fields, forming a condensate $\langle \bar{q} \sigma_{\mu\nu} G^{\mu\nu} q \rangle$.

Previous results for the SM were **contradictory**, so we computed them again.

Condensate contributions

Results:

$$\Delta\rho^{\text{cond}}_{\tilde{O}_i} = 0, \qquad i = 1, 2, 4, 5, 6, 8,$$

$$\Delta\rho^{\text{cond}}_{\tilde{T}_1} = \Delta\rho^{\text{cond}}_{\tilde{T}_3} = -\frac{\langle \frac{\alpha_s}{\pi} GG \rangle}{64\pi^2} + \frac{N_c \langle g_s \bar{q} \sigma_{\mu\nu} G^{\mu\nu} q \rangle}{192\pi^2} \left[\delta(\nu_1) + \delta(\nu_2)\right].$$

This time we have to do the double Borel integral to obtain the contribution to the bag parameter

$$\Delta\tilde{B}_{\tilde{O}_i} = \frac{1}{\tilde{A}_{\tilde{O}_i}} \frac{\int_0^{\omega_c} \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \, e^{-\frac{\omega_1}{t_1} - \frac{\omega_2}{t_2}} \Delta\rho_{\tilde{O}_i}(\omega_1, \omega_2)}{\left[\int_0^{\omega_c} \mathrm{d}\omega_1 \, e^{-\frac{\omega_1}{t_1}} \rho_\Pi(\omega_1)\right] \left[\int_0^{\omega_c} \mathrm{d}\omega_2 \, e^{-\frac{\omega_2}{t_2}} \rho_\Pi(\omega_2)\right]}.$$

Results

Results

Numerical results (HQET)

$$
\begin{aligned}
\tilde{B}_1\,(1.5\,\text{GeV}) &= 1.0000 \pm 0.0201 &&= 1.0000\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0000}_{-0.0000}\,(\bar{\Lambda})\,{}^{+0.002}_{-0.002}\,(\text{cond.})\,{}^{+0.0002}_{-0.0009}\,(\mu_\rho)\,,\\
\tilde{B}_2\,(1.5\,\text{GeV}) &= 1.0000 \pm 0.0201 &&= 1.0000\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0000}_{-0.0000}\,(\bar{\Lambda})\,{}^{+0.002}_{-0.002}\,(\text{cond.})\,{}^{+0.0000}_{-0.0011}\,(\mu_\rho)\,,\\
\tilde{\epsilon}_1\,(1.5\,\text{GeV}) &= -0.0053^{+0.0220}_{-0.0224} &&= -0.0053\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0067}_{-0.0082}\,(\bar{\Lambda})\,{}^{+0.0057}_{-0.0057}\,(\text{cond.})\,{}^{+0.0029}_{-0.0018}\,(\mu_\rho)\,,\\
\tilde{\epsilon}_2\,(1.5\,\text{GeV}) &= -0.0017^{+0.0216}_{-0.0221} &&= -0.0017\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0067}_{-0.0082}\,(\bar{\Lambda})\,{}^{+0.002}_{-0.002}\,(\text{cond.})\,{}^{+0.0042}_{-0.0041}\,(\mu_\rho)\,,\\
\tilde{B}_3\,(1.5\,\text{GeV}) &= 1.0000 \pm 0.0201 &&= 1.0000\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0000}_{-0.0000}\,(\bar{\Lambda})\,{}^{+0.002}_{-0.002}\,(\text{cond.})\,{}^{+0.0000}_{-0.0000}\,(\mu_\rho)\,,\\
\tilde{B}_4\,(1.5\,\text{GeV}) &= 1.0000^{+0.0206}_{-0.0204} &&= 1.0000\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0000}_{-0.0000}\,(\bar{\Lambda})\,{}^{+0.002}_{-0.002}\,(\text{cond.})\,{}^{+0.0046}_{-0.0034}\,(\mu_\rho)\,,\\
\tilde{\epsilon}_3\,(1.5\,\text{GeV}) &= 0.0747^{+0.0437}_{-0.0275} &&= 0.0747\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0000}_{-0.0000}\,(\bar{\Lambda})\,{}^{+0.0057}_{-0.0057}\,(\text{cond.})\,{}^{+0.0384}_{-0.0179}\,(\mu_\rho)\,,\\
\tilde{\epsilon}_4\,(1.5\,\text{GeV}) &= -0.0047^{+0.0212}_{-0.0217} &&= -0.0047\,{}^{+0.02}_{-0.02}\,(\text{intr.})\,{}^{+0.0067}_{-0.0082}\,(\bar{\Lambda})\,{}^{+0.002}_{-0.002}\,(\text{cond.})\,{}^{+0.0016}_{-0.0003}\,(\mu_\rho)\,.
\end{aligned}
$$

- Reminder: $\tilde{B}_i$ = colour singlet operators, $\tilde{\epsilon}_i$ = colour octet operators.
- All uncertainties (except $\tilde{\epsilon}_3$) are completely dominated by the intrinsic sum rule uncertainty.

Results: Discussion

Some observations:

- All colour-singlet operators are exactly equal to their VIA value at the scale of the sum rule ($\sim (\mathrm{Tr}\,(T^a))^2 = 0$).
- Both SM colour-octet operators $\tilde{\epsilon}_{1,2}$ have values different from their previous determination [Kirk, Lenz, Rauh, 2017], due to ...
  - ... updated perturbative **and** condensate results ($\tilde{\epsilon}_3$).
  - ... updated condensate results ($\tilde{\epsilon}_4$).
- With the exception of $\tilde{\epsilon}_3$ all operators obey their VIA values within uncertainties of roughly 0.02.

Summary

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- We have computed the bag parameters of $\Delta B = 0$ four-quark operators in the framework of HQET sum rules, within and beyond the SM...
  - ... updating the existing results concerning the SM operators.
  - ... providing entirely new results for the BSM operators.
- Deviations from the VIA are very small for almost all operators!
- These new sum rules results have been used for the recent theory determination of [Egner et al., 2025] $\boxed{\frac{\tau(B^+)}{\tau(B_d)} = 1.081^{+0.014}_{-0.016}}$. Maria Laura's talk
- Our results allow the study of NP effects entering the $\Delta B = 0$ Hamiltonian.

Summary

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- Our results allow the study of NP effects entering the $\Delta B = 0$ Hamiltonian.

The next step will be a sum rules determination of the **dimension-7** four-quark operator matrix elements, for **mixing and lifetimes.**

Thank you!

Backup

Evanescent operators

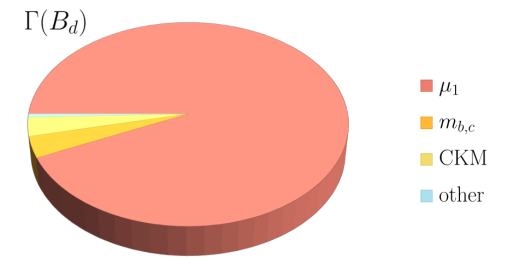
Evanescent operators in HQET:

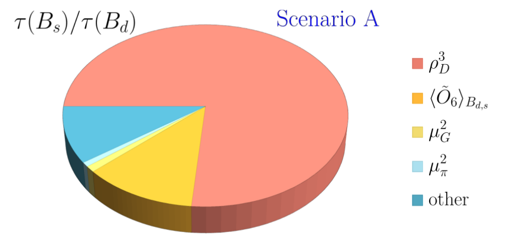
$$\tilde{E}_1^q \equiv \bar{h}\gamma_{\mu\nu\rho}\,(1-\gamma_5)\,q\,\bar{q}\gamma^{\rho\nu\mu}\,(1-\gamma_5)\,h \quad -(4+a_1\epsilon)\,\tilde{O}_1^q\,,$$
$$\tilde{E}_2^q \equiv \bar{h}\gamma_{\mu\nu}\,(1-\gamma_5)\,q\,\bar{q}\gamma^{\nu\mu}\,(1+\gamma_5)\,h \quad -(4+a_2\epsilon)\,\tilde{O}_2^q\,,$$
$$\tilde{E}_3^q \equiv \bar{h}\gamma_{\mu\nu\rho}\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^{\rho\nu\mu}\,(1-\gamma_5)\,T^a h \quad -(4+a_1\epsilon)\,\tilde{O}_3^q\,,$$
$$\tilde{E}_4^q \equiv \bar{h}\gamma_{\mu\nu}\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^{\nu\mu}\,(1+\gamma_5)\,T^a h \quad -(4+a_2\epsilon)\,\tilde{O}_4^q\,,$$
$$\tilde{E}_5^q \equiv \bar{h}\gamma_{\mu\nu\rho}\,(1-\gamma_5)\,q\,\bar{q}\gamma^{\rho\nu\mu}\,(1+\gamma_5)\,h \quad -(16+a_3\epsilon)\,\tilde{O}_5^q\,,$$
$$\tilde{E}_6^q \equiv \bar{h}\gamma_{\mu\nu\rho}\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^{\rho\nu\mu}\,(1+\gamma_5)\,T^a h \quad -(16+a_3\epsilon)\,\tilde{O}_7^q\,,$$

In QCD, additionally:

$$E_7^q \equiv \bar{b}\sigma_{\mu\nu}\gamma_{\rho\sigma}\,(1-\gamma_5)\,q\,\bar{q}\gamma^{\sigma\rho}\sigma^{\mu\nu}\,(1-\gamma_5)\,b$$
$$-(48-80\epsilon)\,O_6^q \quad -(12-14\epsilon)\,O_9^q\,,$$
$$E_8^q \equiv \bar{b}\sigma_{\mu\nu}\gamma_{\rho\sigma}\,(1-\gamma_5)\,T^a q\,\bar{q}\gamma^{\sigma\rho}\sigma^{\mu\nu}\,(1-\gamma_5)\,T^a b$$
$$-(48-80\epsilon)\,O_8^q \quad -(12-14\epsilon)\,O_{10}^q$$

Uncertainties, early 2024





[taken from Albrecht, Bernlochner, Lenz, Rusov, '24]

The sum rule

From Cauchy's integral formula:

$$\Pi(\omega) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\Pi(s)}{s-\omega}\mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon\to 0} \int_0^\infty \frac{\Pi(s+\mathrm{i}\epsilon) - \Pi(s-\mathrm{i}\epsilon)}{s-\omega}\mathrm{d}s + \frac{1}{2\pi\mathrm{i}} \oint_{R\to\infty} \frac{\Pi(s)}{s-\omega}\mathrm{d}s$$

$$= \int_0^\infty \frac{\rho(s)}{s-\omega}\mathrm{d}s + [\text{subtraction terms}]\ .$$

Analogously,

$$K_{\tilde{Q}}(\omega_1,\omega_2) = \int_0^\infty \mathrm{d}\eta_1\mathrm{d}\eta_2 \frac{\tilde{\rho}(\omega_1,\omega_2)}{(\eta_1-\omega_1)(\eta_2-\omega_2)} + \text{ subtraction terms}\,.$$

The **hadronic** spectral function is given by

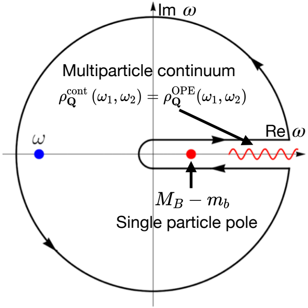
$$\rho^{\mathrm{had}}(\omega_1,\omega_2) = F^2(\mu)\left\langle \tilde{Q}(\mu)\right\rangle \delta(\omega_1-\bar{\Lambda})\,\delta(\omega_2-\bar{\Lambda}) + \rho^{\mathrm{cont}}(\omega_1,\omega_2)\ , \qquad \bar{\Lambda} = M_B - m_b\,.$$

The sum rule

Since the double Borel transform suppresses the continuum part of the spectral density, we can cut off this part by

$$\rho^{\mathrm{cont}}(\omega_1, \omega_2) = \rho^{\mathrm{OPE}}(\omega_1, \omega_2)\left[1 - \theta(\omega_c - \omega_1)\,\theta(\omega_c - \omega_2)\right] .$$

Dispersion relation (figure: Z. Wüthrich)



Two-point correlator

The **two-point correlator** is defined as

$$\Pi(\omega) = \mathrm{i} \int \mathrm{d}^d x \mathrm{e}^{\mathrm{i} p \cdot x} \left\langle 0 \middle| \mathbf{T} \tilde{j}^{\dagger}(x) \tilde{j}(0) \middle| 0 \right\rangle .$$

The corresponding spectral density is given by

$$\begin{aligned} \rho_{\Pi}(\omega) &= \lim_{\epsilon \to 0} \frac{\Pi(\omega + \mathrm{i}\epsilon) - \Pi(\omega - \mathrm{i}\epsilon)}{2\pi \mathrm{i}} \\ &= \frac{N_c \omega^2}{2\pi^2} \left[1 + \frac{\alpha_s C_F}{4\pi} \left(17 + \frac{4\pi^2}{3} + 3 \log \frac{\mu^2}{4\omega^2} \right) + \mathcal{O}\left(\alpha_s^2\right) \right] . \end{aligned}$$

Sum rule for the bag parameter

$$F^2(\mu)\,\mathrm{e}^{-\frac{\bar\Lambda}{t}} = \int\limits_0^{\omega_c} \mathrm{e}^{-\frac{\bar\Lambda}{t}}\,\tilde\rho_\Pi(\omega) + \dots$$

$$\Delta\tilde B_{\tilde O_i} = \frac{1}{\tilde A_{\tilde O_i} F^4(\mu)} \int\limits_0^{\omega_c} \mathrm{e}^{\frac{\bar\Lambda-\omega_1}{t_1}+\frac{\bar\Lambda-\omega_2}{t_2}}\,\Delta\rho_{\tilde O_i}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\mathrm{d}\omega_2$$

$$= \frac{1}{\tilde A_{\tilde O_i}} \frac{\int_0^{\omega_c}\mathrm{d}\omega_1\,\mathrm{d}\omega_2\, e^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}}\,\Delta\rho_{\tilde O_i}(\omega_1,\omega_2)}{\left[\int_0^{\omega_c}\mathrm{d}\omega_1\, e^{-\frac{\omega_1}{t_1}}\rho_\Pi(\omega_1)\right]\left[\int_0^{\omega_c}\mathrm{d}\omega_2\, e^{-\frac{\omega_2}{t_2}}\rho_\Pi(\omega_2)\right]}\,.$$

Weight function

$$F^4(\mu)\,\mathrm{e}^{-\frac{\bar\Lambda}{t_1}-\frac{\bar\Lambda}{t_2}} w_i\left(\bar\Lambda,\bar\Lambda\right) = \int_0^{\omega_c} \mathrm{e}^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}} w_i\left(\omega_1,\omega_2\right)\rho_\Pi\left(\omega_1\right)\rho_\Pi\left(\omega_2\right) + \dots$$

$$w_{\tilde O_i}\left(\omega_1,\omega_2\right) := \frac{\Delta\rho_{\tilde O_i}\left(\omega_1,\omega_2\right)}{\tilde\rho^{\mathrm{pert}}\left(\omega_1\right)\tilde\rho^{\mathrm{pert}}\left(\omega_2\right)} = \frac{C_F}{N_c}\frac{\alpha_s}{4\pi} r_{\tilde O_i}\left(x, L_w\right)\,.$$

$$\Delta\tilde B_{\tilde O_i}\left(\mu_\rho\right) = \frac{C_F}{N_c}\tilde A_{\tilde O_i}\frac{\alpha_s\left(\mu_\rho\right)}{4\pi} r_{\tilde O_i}\left(1, \log\frac{\mu_\rho^2}{4\bar\Lambda^2}\right)\,.$$

Stability of the sum rule: Condensates