Heavy Meson Lifetimes from Lattice QCD

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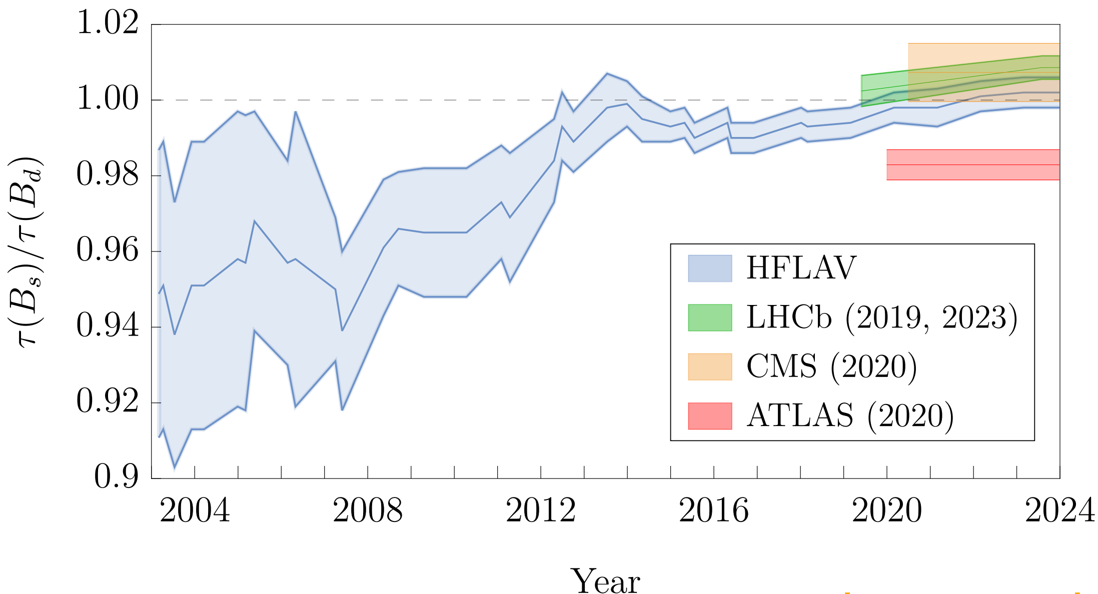
Introduction

- B-meson lifetimes are measured experimentally to high precision
  - ➥ Key observables for probing New Physics ➡ **high precision in theory needed!**

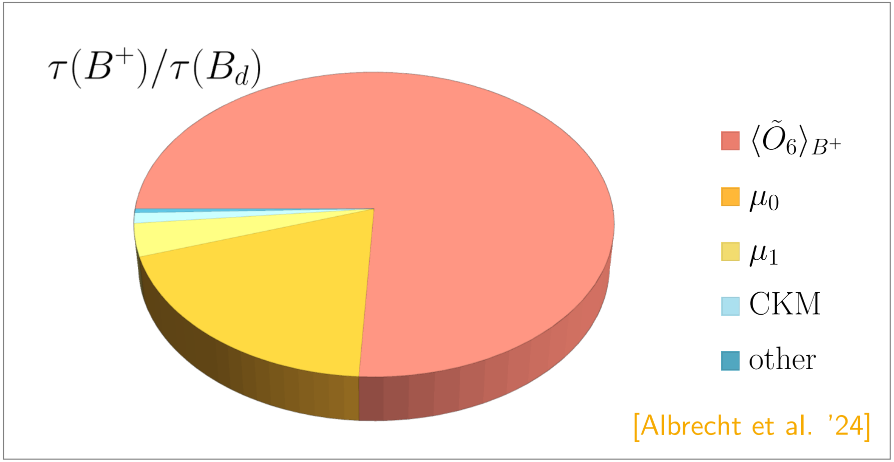


[Albrecht et al. '24]

- For B lifetimes, we use the **Heavy Quark Expansion**

$$\Gamma(H_Q) = \Gamma_3\langle\mathcal{O}_3\rangle + \Gamma_5\frac{\langle\mathcal{O}_5\rangle}{m_Q^2} + \Gamma_6\frac{\langle\mathcal{O}_6\rangle}{m_Q^3} + \ldots + 16\pi^2\left[\tilde{\Gamma}_6\frac{\langle\tilde{\mathcal{O}}_6\rangle}{m_Q^3} + \tilde{\Gamma}_7\frac{\langle\tilde{\mathcal{O}}_7\rangle}{m_Q^4} + \ldots\right]$$

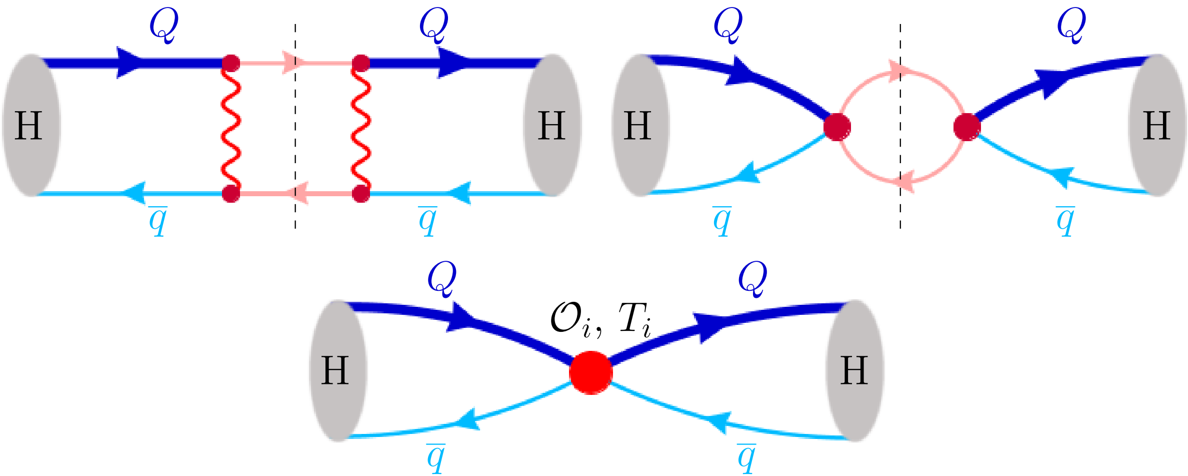
- $\langle\tilde{\mathcal{O}}_6\rangle$ are leading uncertainties for B lifetime differences



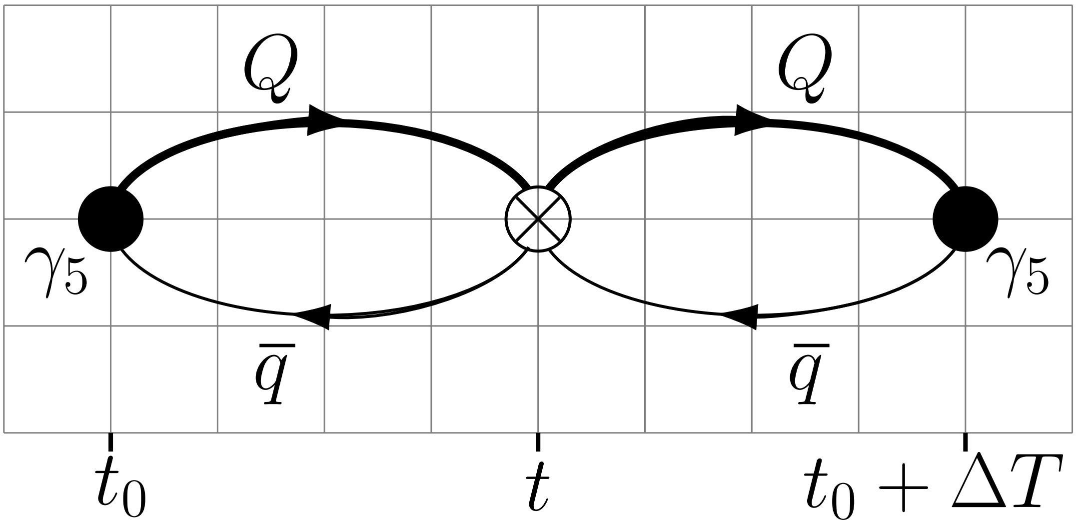
- Matrix elements of four-quark operators can be determined from lattice QCD simulations

- $\Delta Q = 2$ well-studied by several groups ➡ precision increasing
  - ➥ Preliminary $\Delta K = 2$ for Kaon mixing with gradient flow [Suzuki et al. '20], [Taniguchi, Lattice '19]

- $\Delta Q = 0$ ➡ exploratory studies from ~20 years ago



- Start of calculation follows similar to operators for neutral meson mixing
  - ➥ Well-established on lattice!



H_Q ➡ $\langle \mathcal{O}_6 \rangle$ ➡ H_Q

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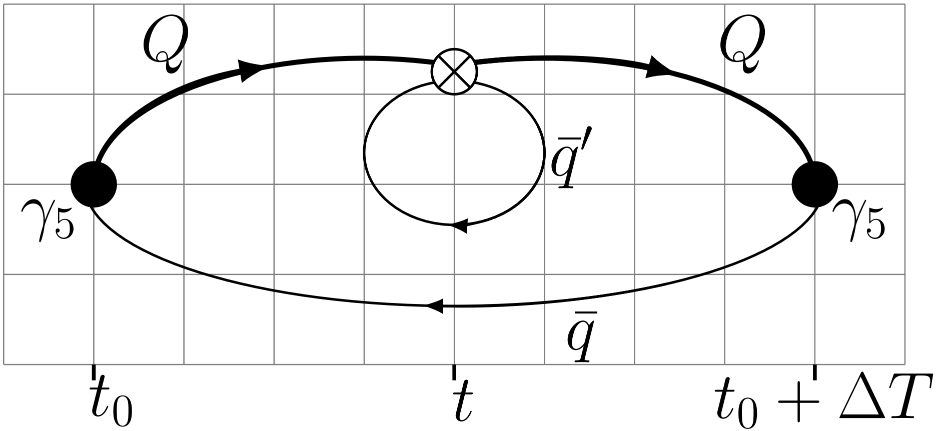
But

- Gluon-disconnected diagrams
- 'Eye' diagrams

→ Statistically very noisy

➥ Modern computing speed and algorithms can help! ✔

Not yet included in lattice simulation or matching



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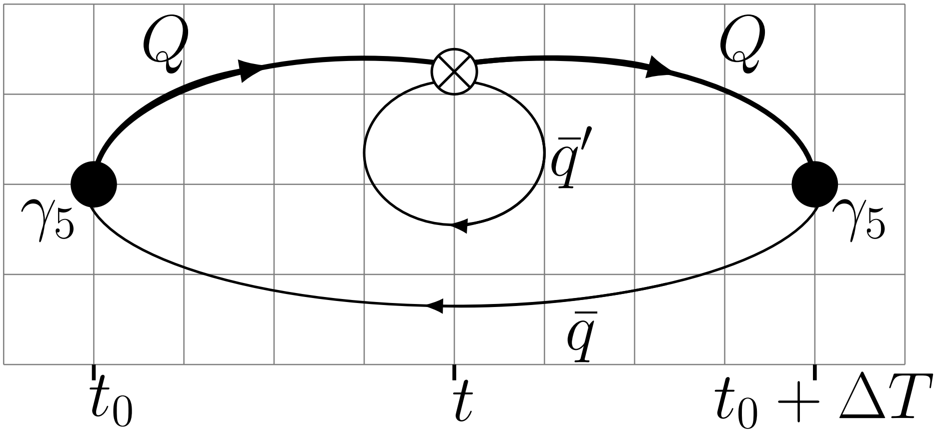
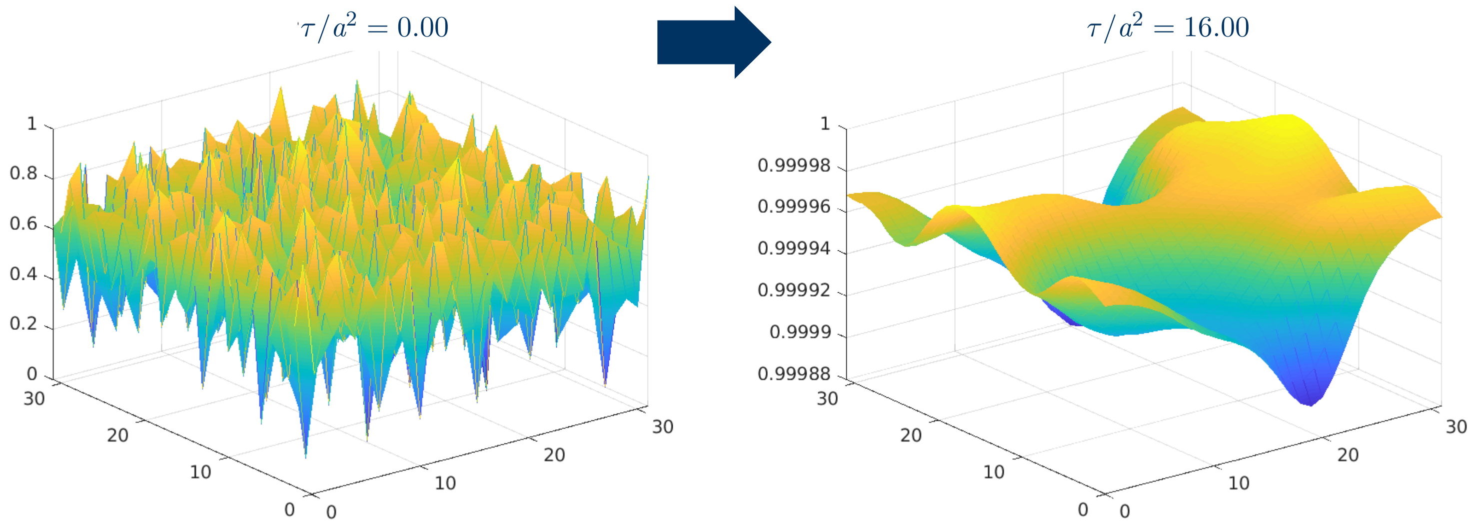


- Mixing with lower-dimensional operators in renormalisation
  - ➥ Power divergent ➥ Notoriously challenging

→ **How can we tackle this?**

Gradient Flow

- Introduced by [Narayanan, Neuberger '06] [Lüscher '10] [Lüscher '13]
  - ➥ Scale setting ($\sqrt{8t_0}$), RG β-function, Λ parameter
- Introduce auxiliary dimension, **flow time** τ as a way to regularise the UV
- Well-defined damping of UV fluctuations



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  - Scale setting ($\sqrt{8t_0}$), RG β-function, Λ parameter
- Introduce auxiliary dimension, **flow time** τ as a way to regularise the UV
- Well-defined damping of UV fluctuations
- Extend gauge and fermion fields in flow time and express dependence with first-order differential equations:

$$\begin{aligned} \partial_t B_\mu(\tau,x) &= \mathcal{D}_\nu(\tau)\,G_{\nu\mu}(\tau,x), \quad & B_\mu(0,x) &= A_\mu(x), \\ \partial_t \chi(\tau,x) &= \mathcal{D}^2(\tau)\chi(\tau,x), & \chi(0,x) &= q(x). \end{aligned}$$

- For use in renormalisation, there are two concepts:
  - Gradient flow as an RG transformation [Carosso et al. '18] [Harlander et al. '20] [Hasenfratz et al. '22]
  - Short-flow-time expansion [Lüscher, Weisz '11] [Suzuki '13], [Lüscher '13]

- Well-studied for e.g. energy-momentum tensor [Makino, Suzuki '14] [Harlander, Kluth, Lange '18]
  quark masses [Takaura et al. '25] [Black et al. '25]

- Re-express effective Hamiltonian in terms of 'flowed' operators:

$$\mathcal{H}_{\text{eff}} = \sum_n C_n \mathcal{O}_n = \sum_n \widetilde{C}_n(\tau)\widetilde{\mathcal{O}}_n(\tau).$$

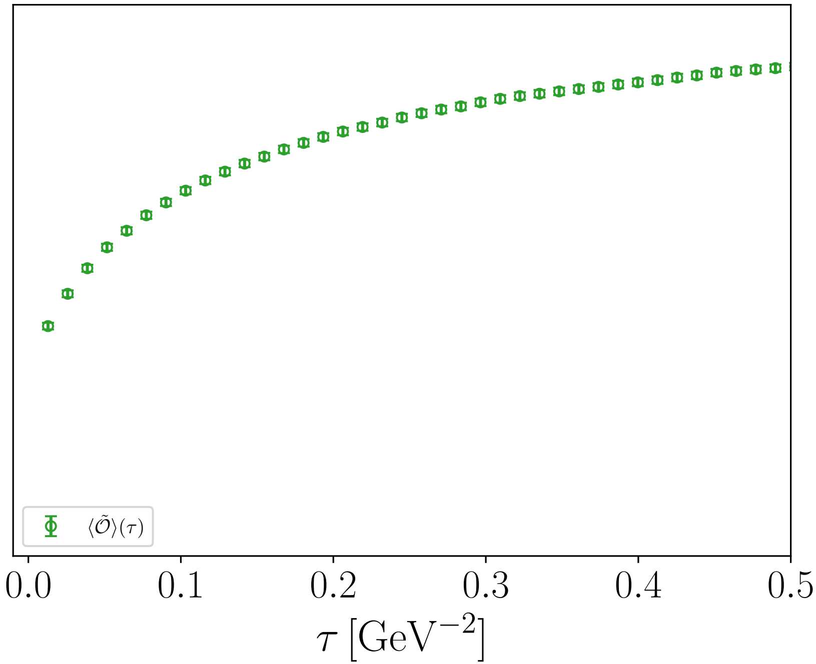
- Relate to regular operators in 'short-flow-time expansion':

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

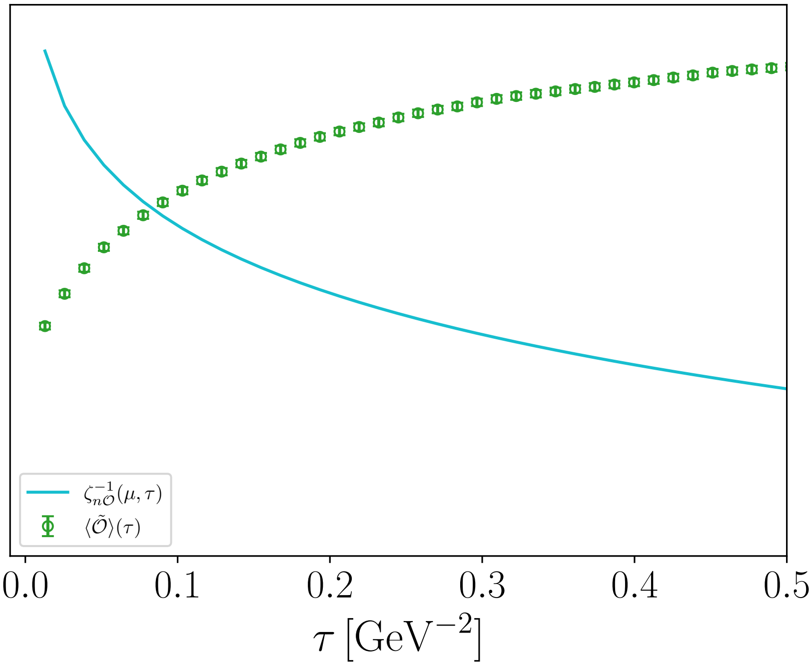
'flowed' MEs calculated on lattice
renormalised along flow time

matching matrix
calculated perturbatively

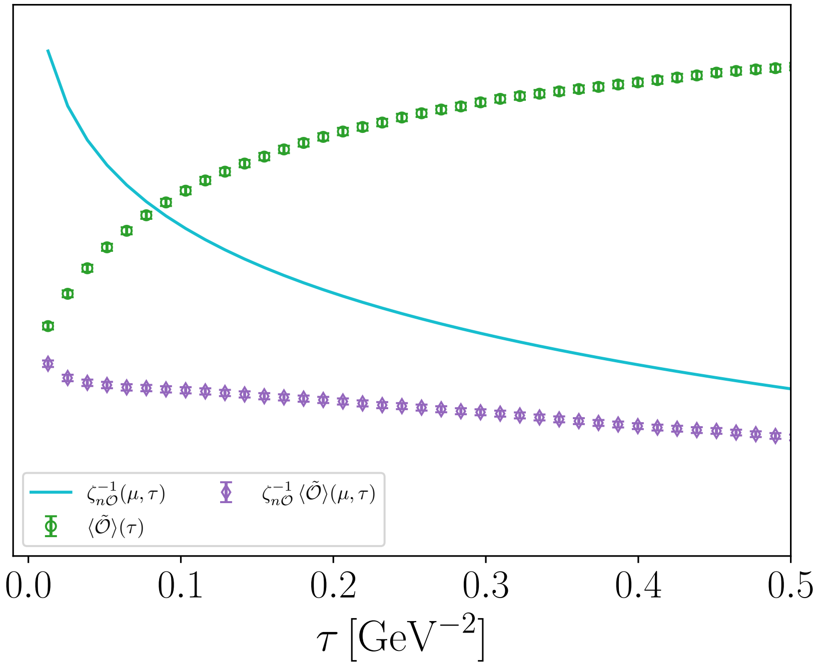
$$\sum_n \zeta^{-1}_{nm}(\mu,\tau)\langle\widetilde{\mathcal{O}}_n\rangle(\tau) = \langle\mathcal{O}_m\rangle(\mu)$$



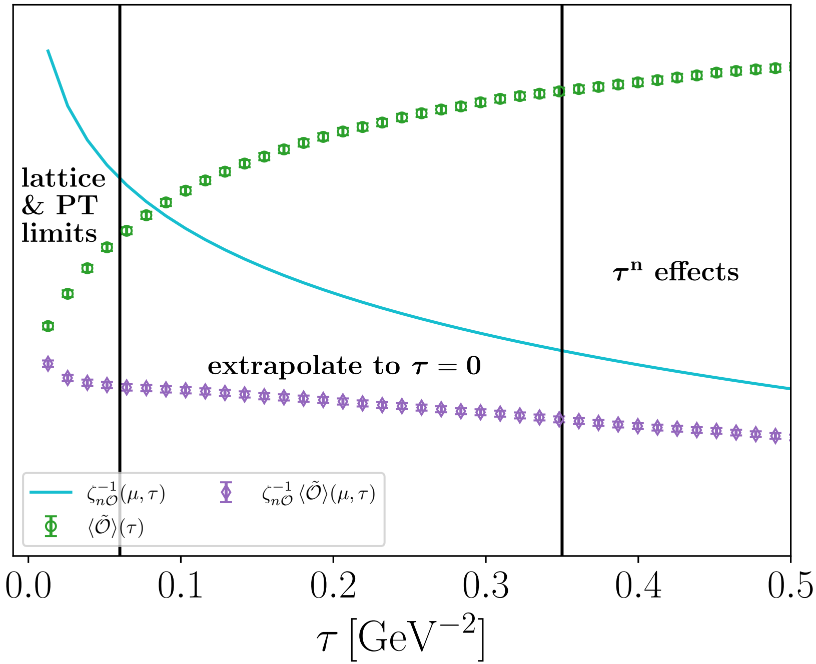
- Measure flowed matrix element $\langle\mathcal{O}\rangle(\tau)$ on the lattice



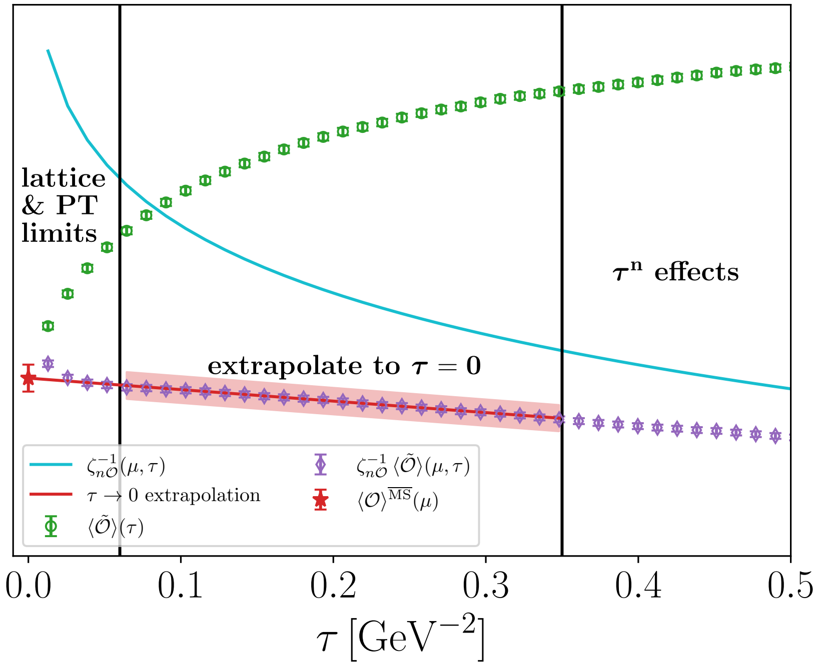
- Calculate perturbative matching coefficients $\zeta_{n\mathcal{O}}^{-1}(\mu,\tau)$



- Combine for 'matched' operator dependent on flow time τ and renormalisation scale μ



➤ Larger systematic effects at extremities



➤ Take $\tau \to 0$ result ➡ $\langle\mathcal{O}\rangle^{\overline{\mathrm{MS}}}(\mu)$

Lattice Details

- Exploratory setup using physical charm and strange quarks
  - ➥ $\Delta B = 0, 2$ ➡ $\Delta Q = 0, 2$, for generic heavy quark Q
- Exploratory study to test method ➡ neglect expensive+noisy eye diagrams

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- We use RBC/UKQCD's 2+1 flavour DWF + Iwasaki gauge action ensembles

	L	T	a^{-1}/GeV	am_l^{sea}	am_s^{sea}	M_π/MeV	srcs $\times$ N$_{\text{conf}}$
C1	24	64	1.7848	0.005	0.040	340	32×101
C2	24	64	1.7848	0.010	0.040	433	32×101
M1	32	64	2.3833	0.004	0.030	302	32×79
M2	32	64	2.3833	0.006	0.030	362	32×89
M3	32	64	2.3833	0.008	0.030	411	32×68
F1S	48	96	2.785	0.002144	0.02144	267	24×98

[Allton et al. '08]
[Aoki et al. '10]
[Blum et al. '14]
[Boyle et al. '17]

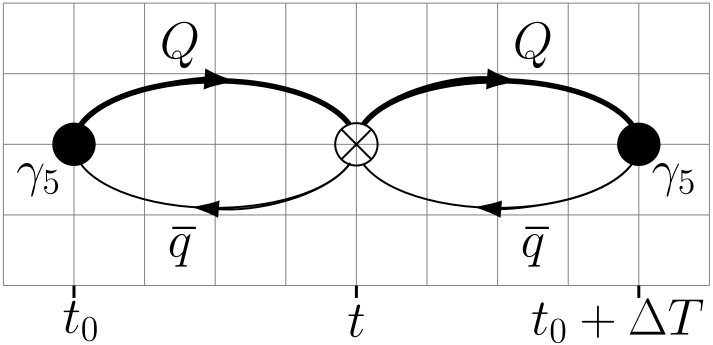
- For strange quarks tuned to physical value, $am_q \ll 1$ ✔
- For heavy b quarks, $am_q > 1$ ➡ large discretisation effects ✗
  - ➥ manageable for physical c quarks using stout-smeared Möbius DWF [Cho et. al '15]

Analysis and Results

- In the Standard Model, four operators contribute:

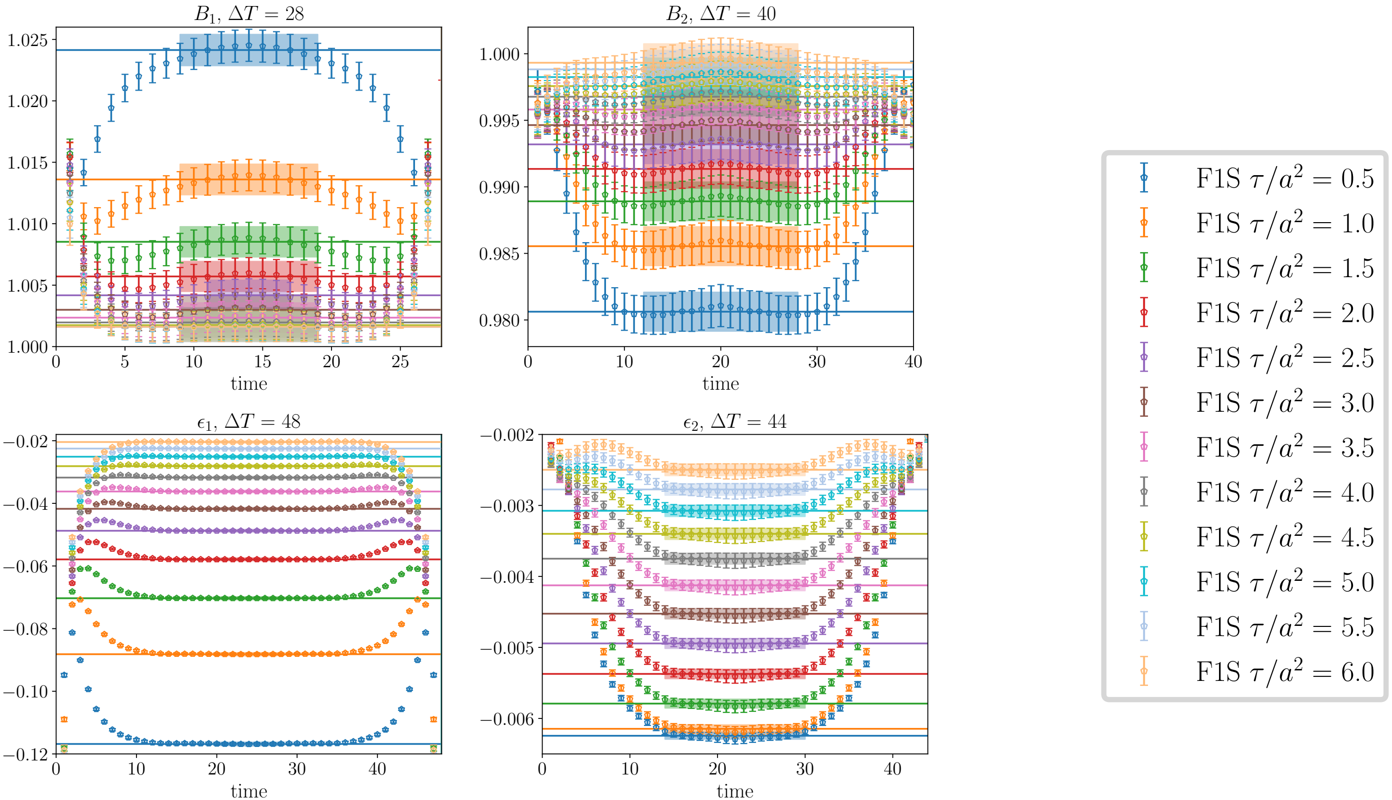
$$\mathcal{O}_1 \rightarrow B_1 \sim 1 \qquad \mathcal{O}_2 \rightarrow B_2 \sim 1$$
$$T_1 \rightarrow \epsilon_1 \sim 0 \qquad T_2 \rightarrow \epsilon_2 \sim 0$$

- Four-quark operators inserted in three-point correlation functions:

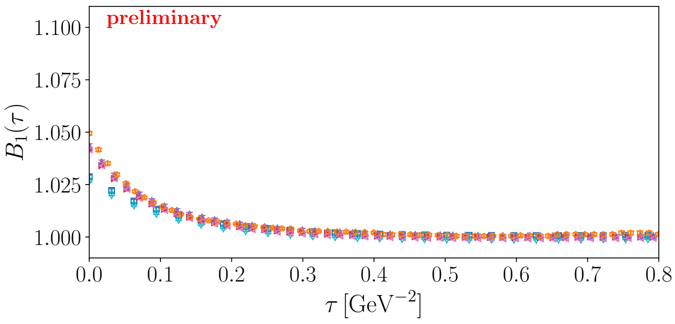


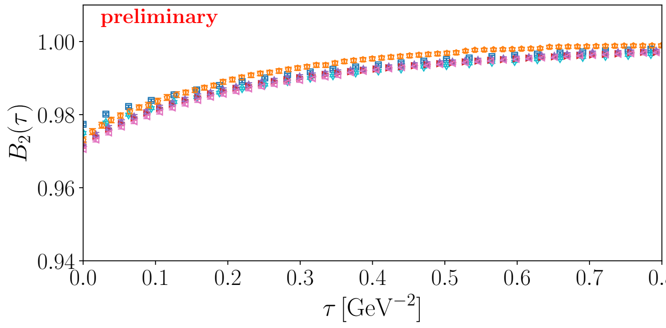
$$C^{\mathrm{3pt}}_{\mathcal{O}_i}(t, \Delta T, \tau) = \sum_{n,n'} \frac{\langle P_n|\mathcal{O}_i|P_{n'}\rangle(\tau)}{4M_n M_{n'}} e^{-(\Delta T - t)M_n} e^{-tM_{n'}} \underset{t_0 \ll t \ll t_0 + \Delta T}{\Longrightarrow} \frac{\langle P\rangle^2}{4M^2} \langle \mathcal{O}_i\rangle(\tau) e^{-\Delta T M}$$

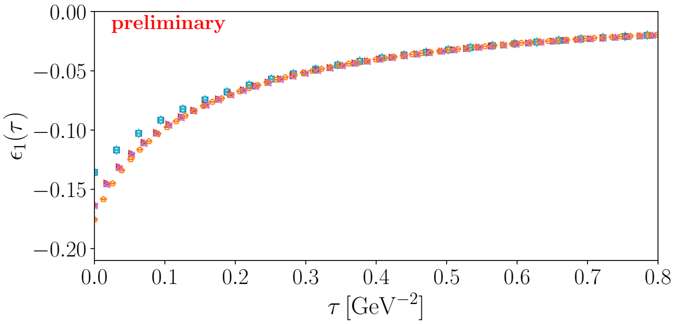
- To extract bag parameters, normalise with two-point correlation functions $\rightarrow B_i \propto \dfrac{\langle \mathcal{O}_i\rangle}{m^2 f_H^2}$

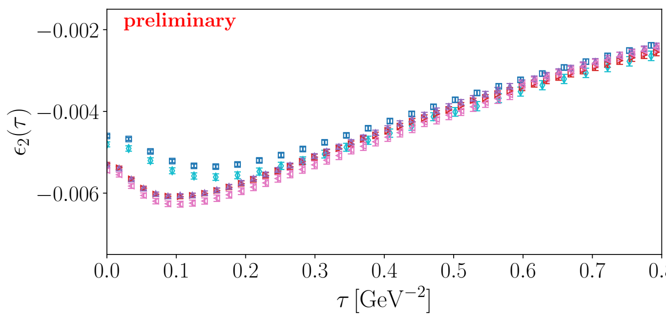


➤ Matrix elements extracted for each flow time ✔

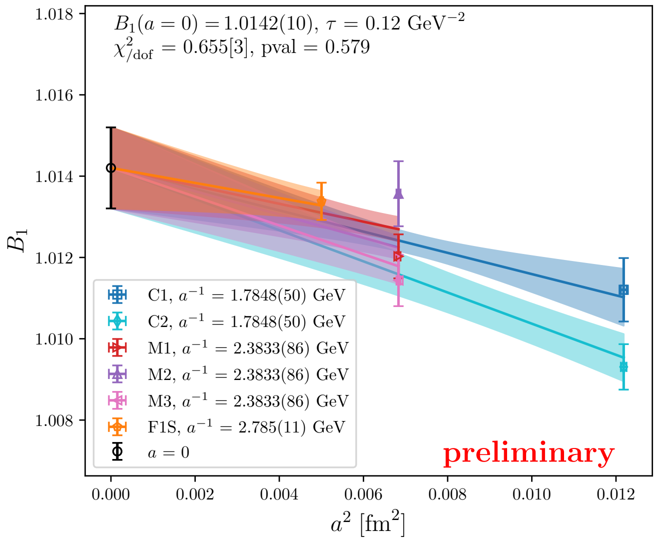


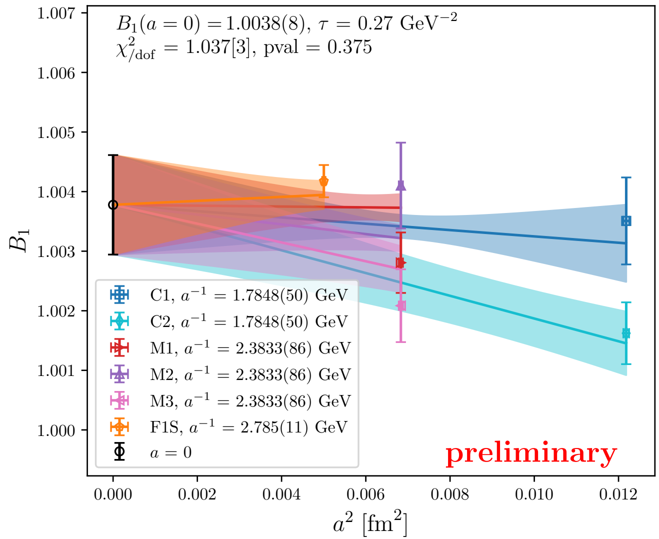






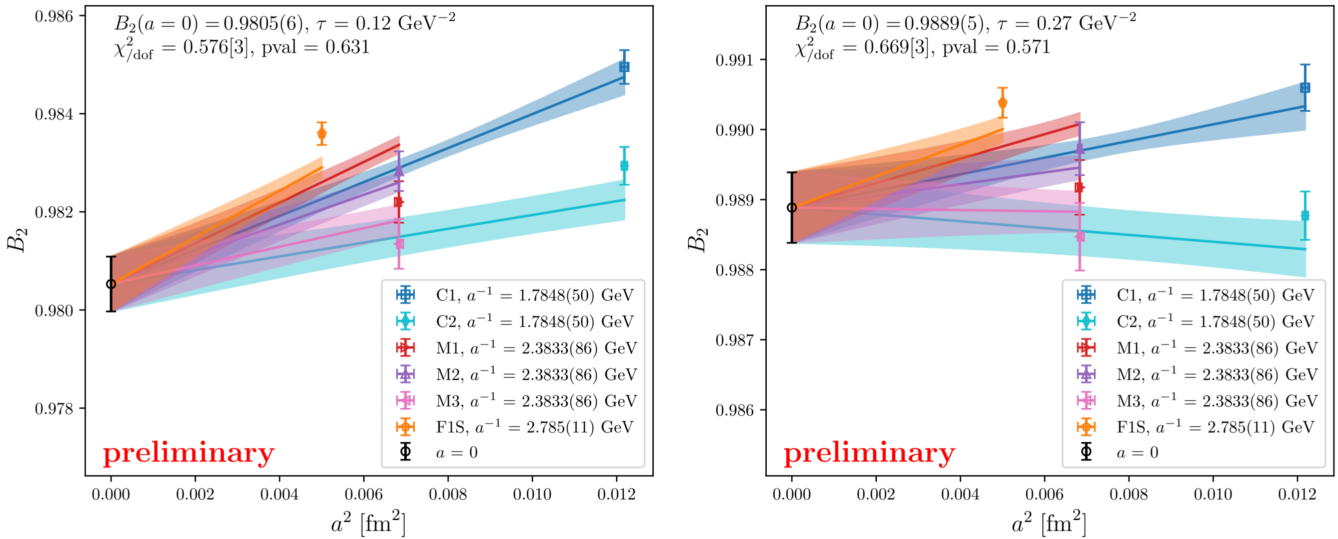
- Take continuum limit!





➤ Highly precise data resolves sea quark effects ➡ chiral-continuum limits

$$B_i^{\text{GF}}(a, M_\pi^{\text{latt}}, \tau) = B_i^{\text{GF,cont}}(\tau) + C\,a^2 + D\,a^2\left[(M_\pi^{\text{latt}})^2 - (M_\pi^{\text{phys}})^2\right]$$



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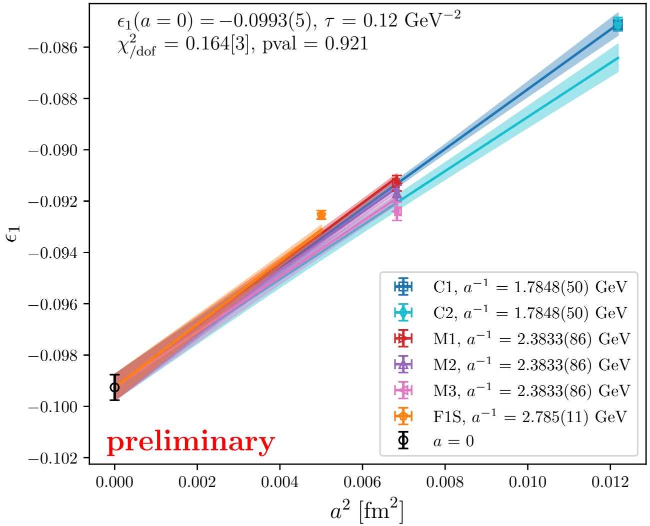

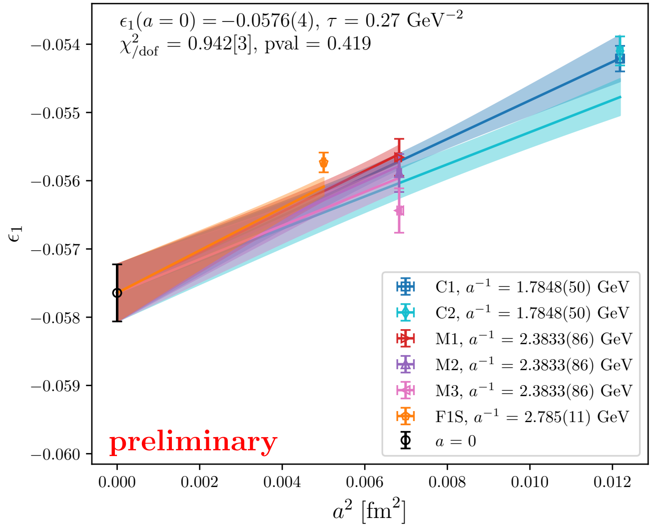

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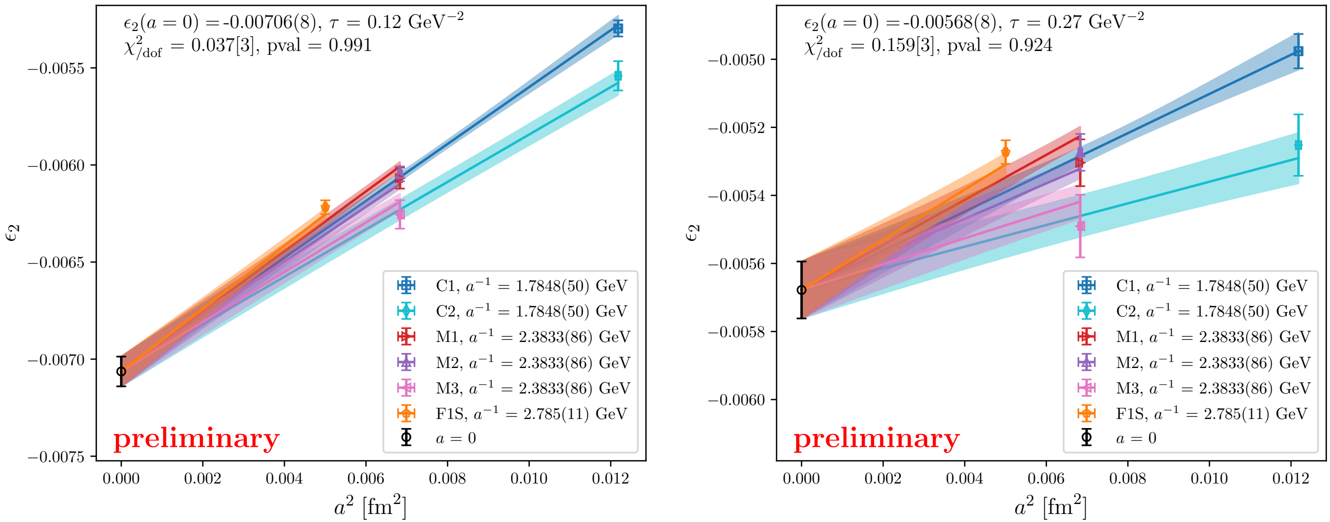

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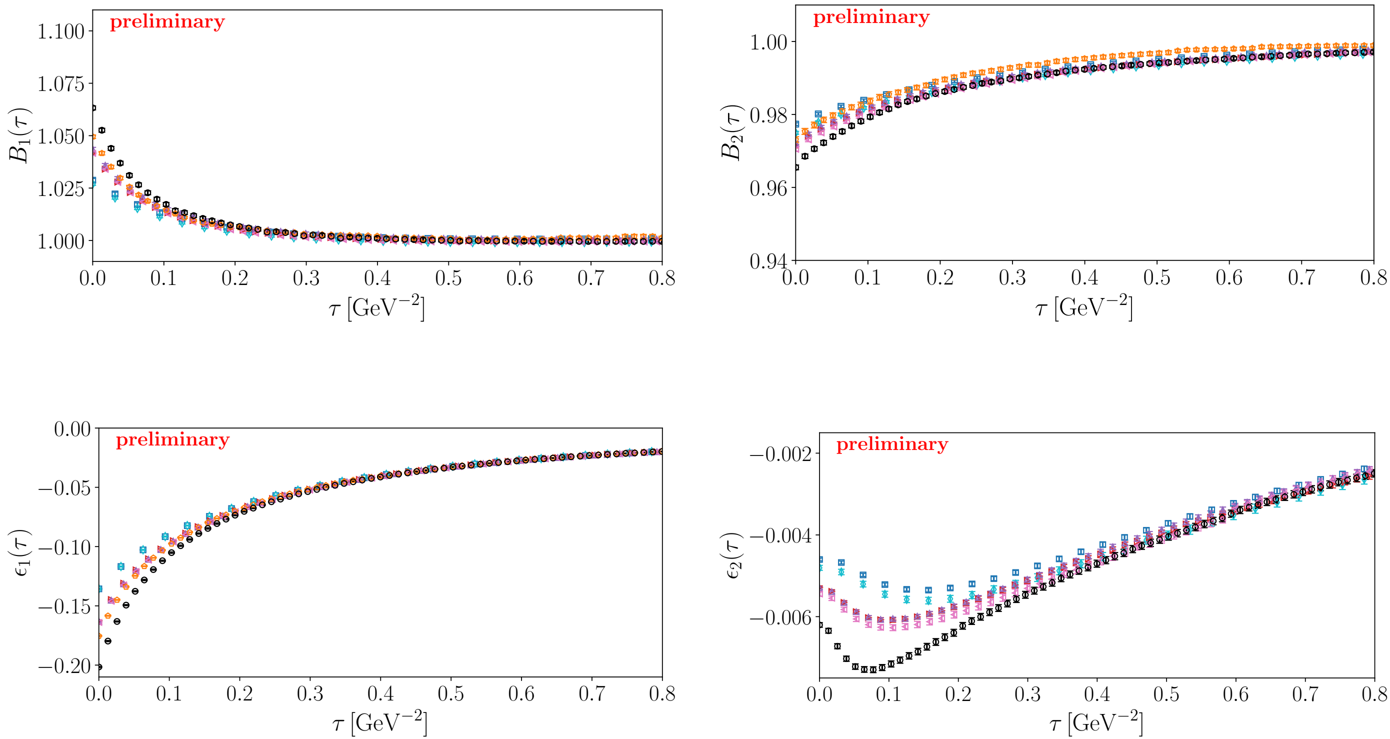
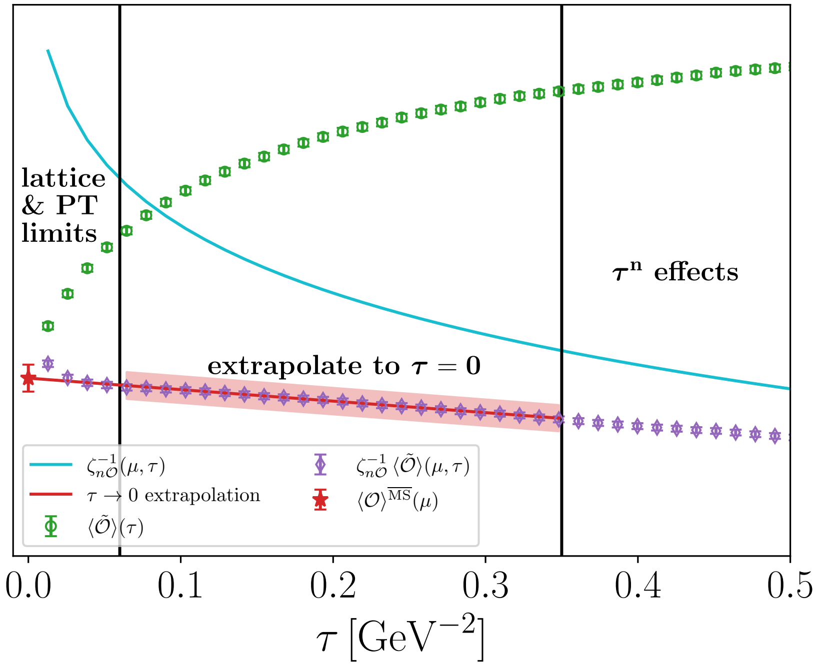
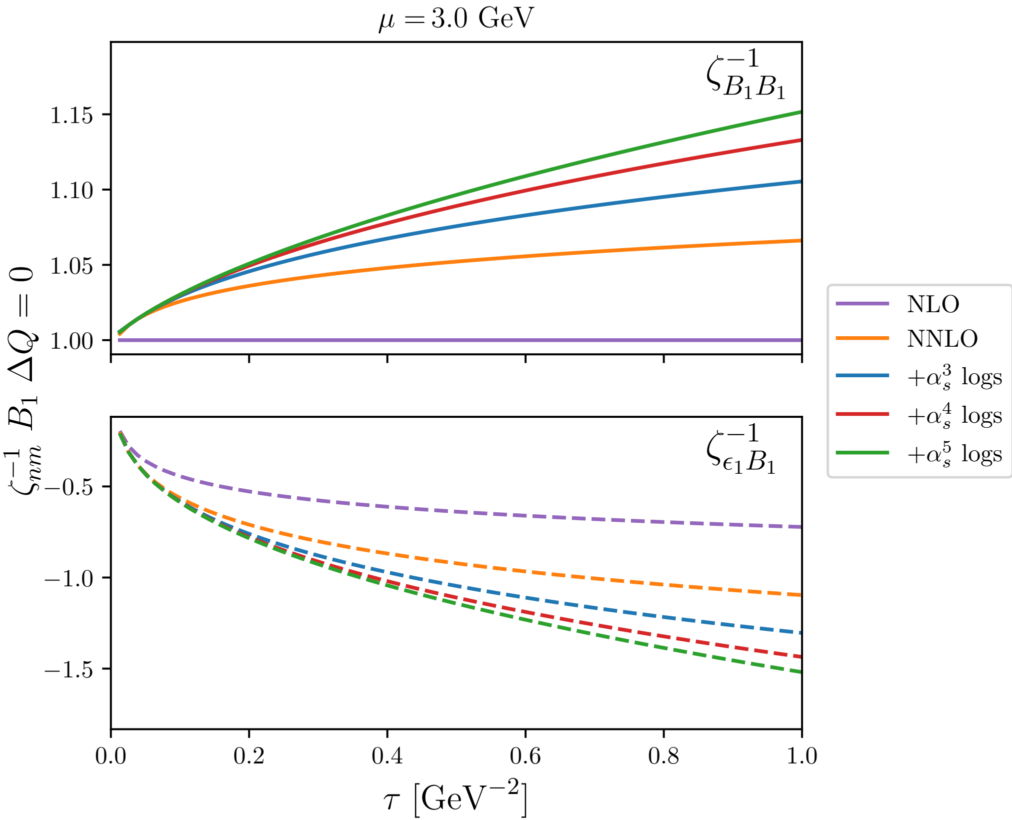


➤ Well-controlled chiral-continuum limits along flow time ✔


lattice & PT limits
τ^{n} effects
extrapolate to $\tau = 0$
$\zeta_{n\mathcal{O}}^{-1}(\mu,\tau)$
$\tau \to 0$ extrapolation
$\langle\tilde{\mathcal{O}}\rangle(\tau)$
$\zeta_{n\mathcal{O}}^{-1}\langle\tilde{\mathcal{O}}\rangle(\mu,\tau)$
$\langle\mathcal{O}\rangle^{\overline{\mathrm{MS}}}(\mu)$
0.0
0.1
0.2
0.3
0.4
0.5
$\tau\ [\mathrm{GeV}^{-2}]$

➤ Combine continuum limits of lattice data with perturbative matching coefficients $\zeta^{-1}_{nm}(\mu, \tau)$



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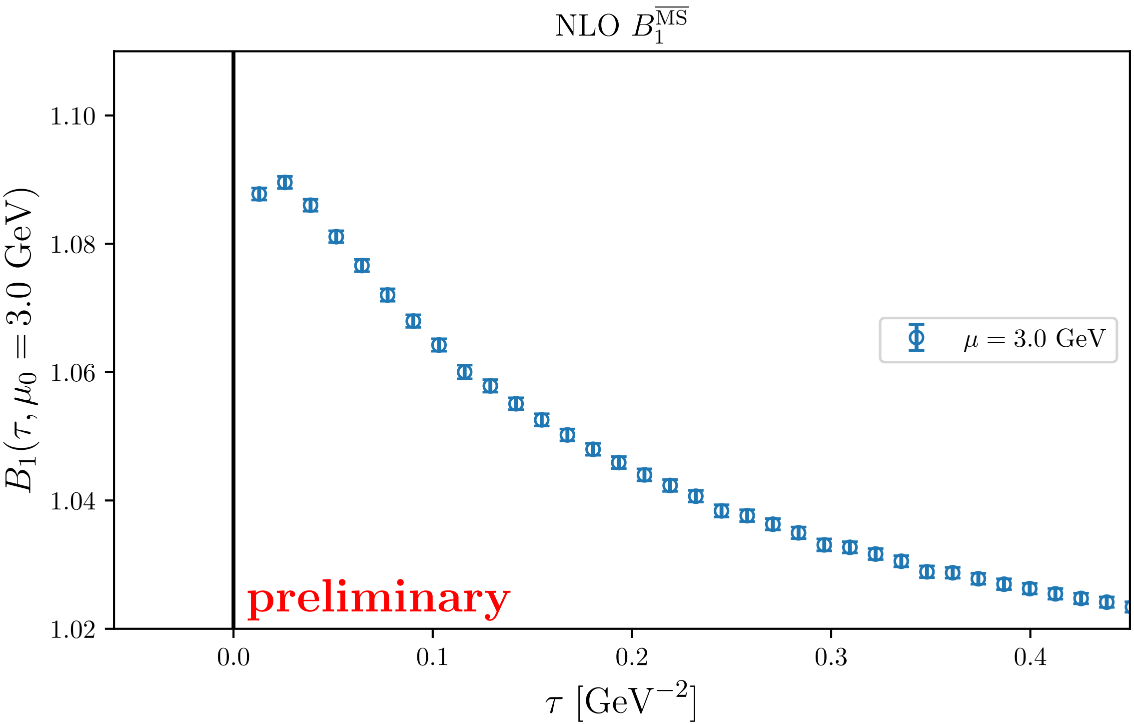
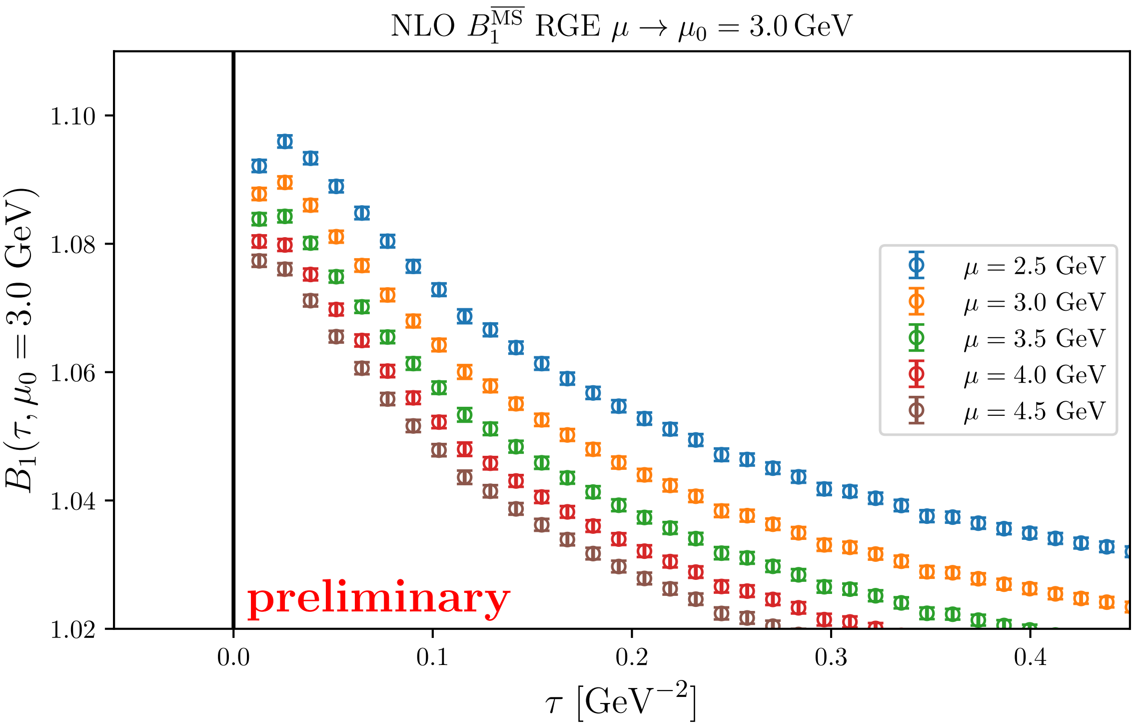
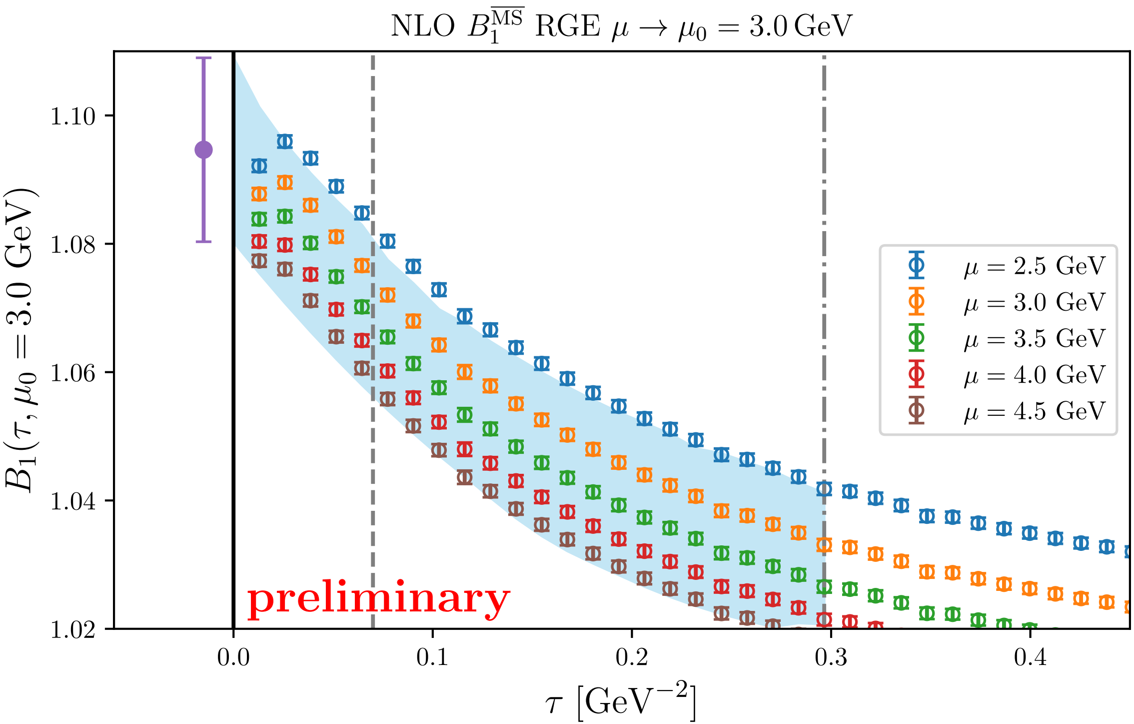


➤ Matching can be done for any scale μ ➡ run back to $\mu_0 = 3$ GeV for final results



Matched results: $\tau \to 0$ extrapolation

- $\tau \to 0$ limit should be RG-independent ➡ perform combined fits to better control extrapolation
  - ➥ Final result takes spread of all acceptable fits performed between $\tau_{\min}$ and $\tau_{\max}$



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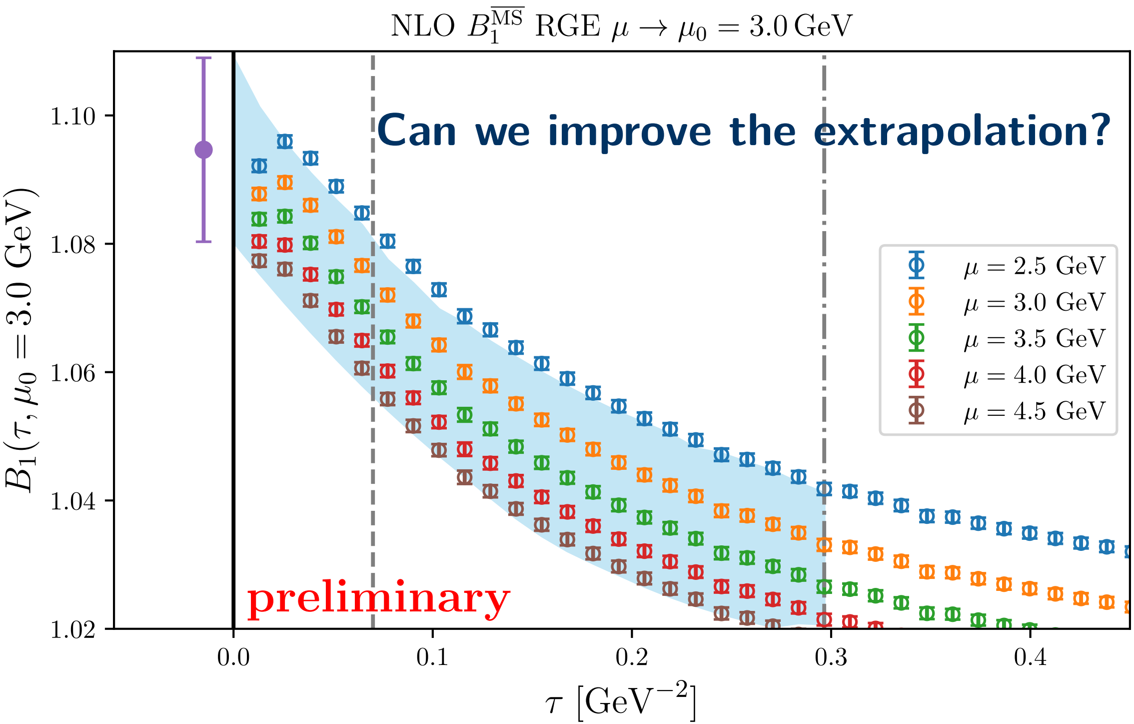


- We can RG-improve the matching procedure using the flow-time RGE:

$$\tau\partial_\tau\tilde{\mathcal{O}}(\tau) = \tilde{\gamma}(a_s(\mu), L_{\mu\tau})\tilde{\mathcal{O}}(\tau),$$

  for a flowed anomalous dimension

$$\tilde{\gamma}(a_s(\mu), L_{\mu\tau}) = (\tau\partial_\tau\zeta(\tau,\mu))\,\zeta^{-1}(\tau,\mu).$$

- Define a perturbative flow time coupled to renormalisation scale μ:

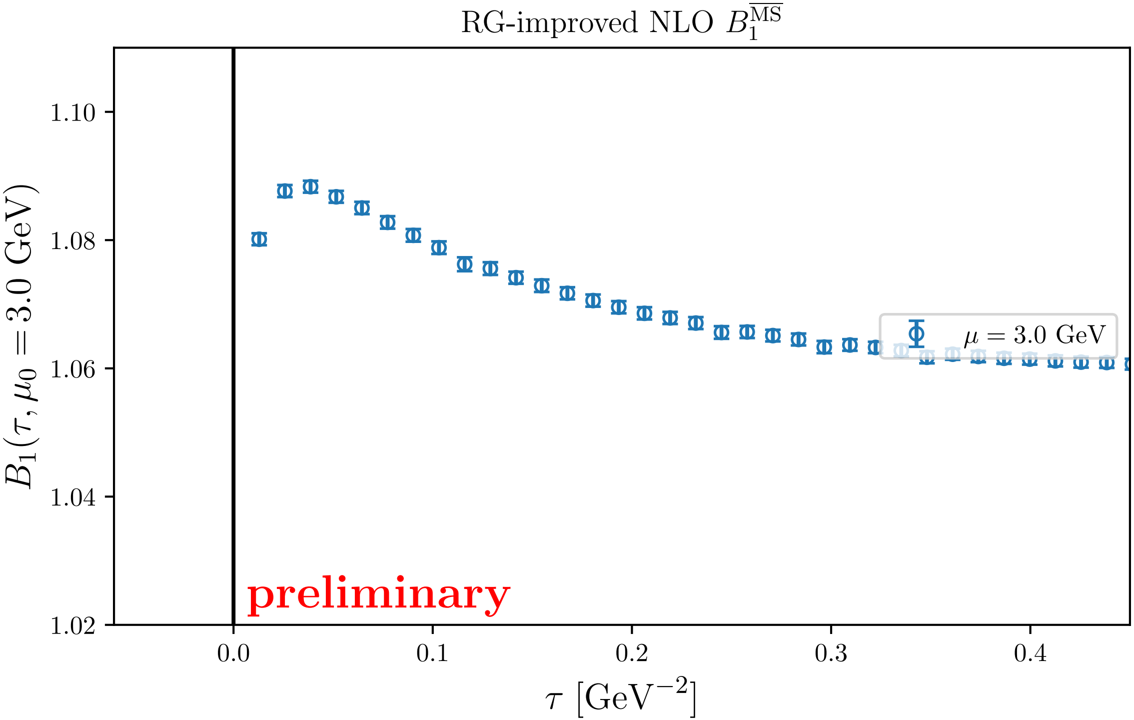
$$\tau_\mu = e^{-\gamma_{\mathrm{E}}}/2\mu^2$$

- Integrating the RGE from any lattice flow time τ to τ_μ will yield 'RG-improved' matched results

$$\mathcal{O}(\mu) = \zeta^{-1}(\tau_\mu,\mu)\,\exp\left[\int_\tau^{\tau_\mu} d\tau\,\tilde{\gamma}(a_s(\mu), L)\right]\tilde{\mathcal{O}}(\tau)$$

- Should decrease slope of fixed-order matching results and improve convergence of short-flow-time expansion
- Use perturbative flowed anomalous dimension at NLO and NNLO
  - ➥ could also calculate $\tilde{\gamma}$ non-perturbatively [Hasenfratz et. al, '22]

- We can RG-improve the matching procedure using the flow-time RGE:
  - now all same steps apply as before with “flatter data”



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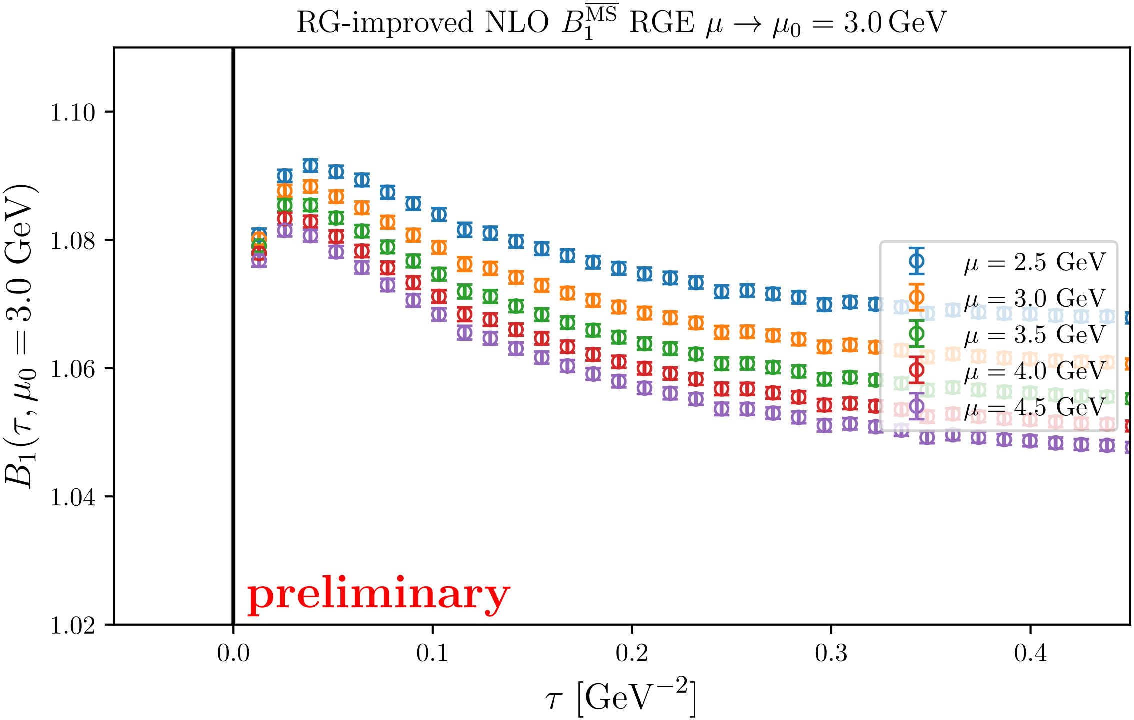

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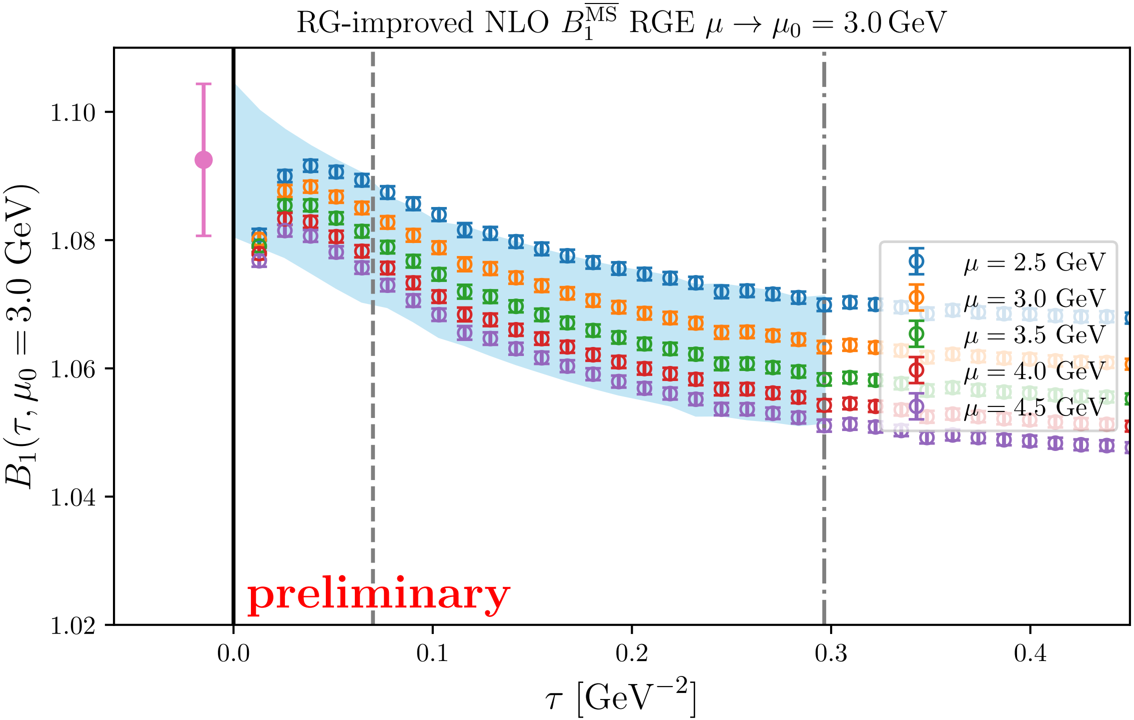
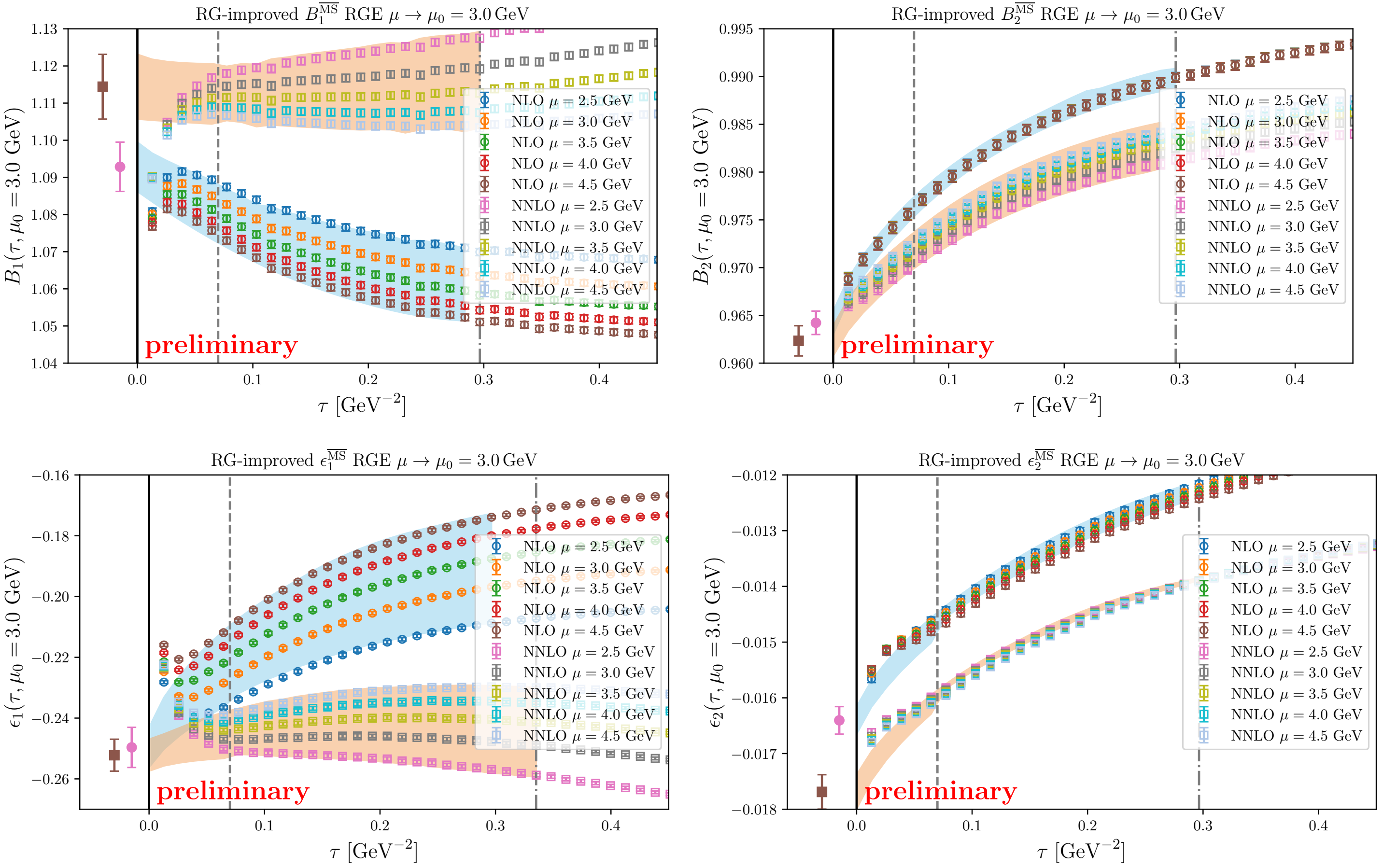



RG-improved $B_1^{\overline{\mathrm{MS}}}$ RGE $\mu \to \mu_0 = 3.0\,\mathrm{GeV}$
$B_1(\tau, \mu_0 = 3.0\ \mathrm{GeV})$
$\tau\ [\mathrm{GeV}^{-2}]$
preliminary
RG-improved $B_2^{\overline{\mathrm{MS}}}$ RGE $\mu \to \mu_0 = 3.0\,\mathrm{GeV}$
$B_2(\tau, \mu_0 = 3.0\ \mathrm{GeV})$
RG-improved $\epsilon_1^{\overline{\mathrm{MS}}}$ RGE $\mu \to \mu_0 = 3.0\,\mathrm{GeV}$
$\epsilon_1(\tau, \mu_0 = 3.0\ \mathrm{GeV})$
RG-improved $\epsilon_2^{\overline{\mathrm{MS}}}$ RGE $\mu \to \mu_0 = 3.0\,\mathrm{GeV}$
$\epsilon_2(\tau, \mu_0 = 3.0\ \mathrm{GeV})$
NLO $\mu = 2.5$ GeV
NLO $\mu = 3.0$ GeV
NLO $\mu = 3.5$ GeV
NLO $\mu = 4.0$ GeV
NLO $\mu = 4.5$ GeV
NNLO $\mu = 2.5$ GeV
NNLO $\mu = 3.0$ GeV
NNLO $\mu = 3.5$ GeV
NNLO $\mu = 4.0$ GeV
NNLO $\mu = 4.5$ GeV

Summary

- Finalising first lattice calculation of $\Delta Q = 0$ matrix elements for heavy meson lifetime ratios
- Gradient flow and short-flow-time expansion is an effective tool for renormalisation and matching
  - Proof of principle calculation of charm and strange quark masses [Black et al. '25]
- Scale dependence of short-flow-time expansion can be studied in detail

Outlook

- Perform large-scale simulations to extrapolate to B and B_s mesons
- 'Eye' diagrams need for absolute lifetime operators
  - to be included in both lattice simulations and perturbative matching

Thanks for the attention!

Backup Slides

- For lifetimes, the dimension-6 $\Delta Q = 0$ operators are:

$$\mathcal{O}_1^q = \bar{b}^\alpha \gamma^\mu (1-\gamma_5) q^\alpha \; \bar{q}^\beta \gamma_\mu (1-\gamma_5) b^\beta, \qquad \langle \mathcal{O}_1^q \rangle = \langle B_q | \mathcal{O}_1^q | B_q \rangle = f_{B_q}^2 M_{B_q}^2 B_1^q,$$

$$\mathcal{O}_2^q = \bar{b}^\alpha (1-\gamma_5) q^\alpha \; \bar{q}^\beta (1-\gamma_5) b^\beta, \qquad \langle \mathcal{O}_2^q \rangle = \langle B_q | \mathcal{O}_2^q | B_q \rangle = \frac{M_{B_q}^2}{(m_b+m_q)^2} f_{B_q}^2 M_{B_q}^2 B_2^q,$$

$$T_1^q = \bar{b}^\alpha \gamma^\mu (1-\gamma_5) (T^a)^{\alpha\beta} q^\beta \; \bar{q}^\gamma \gamma_\mu (1-\gamma_5) (T^a)^{\gamma\delta} b^\delta, \qquad \langle T_1^q \rangle = \langle B_q | T_1^q | B_q \rangle = f_{B_q}^2 M_{B_q}^2 \epsilon_1^q,$$

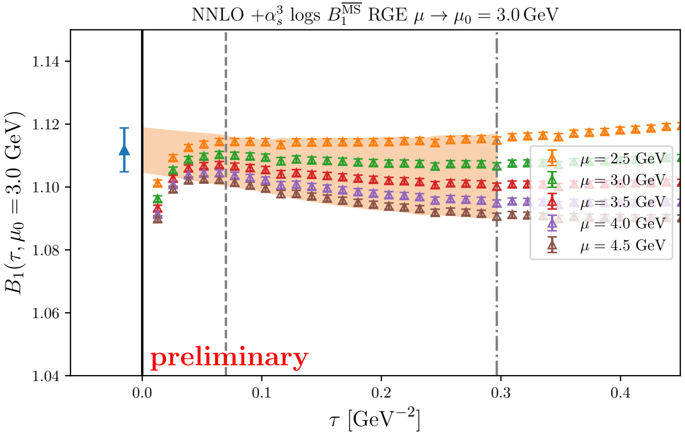
$$T_2^q = \bar{b}^\alpha (1-\gamma_5) (T^a)^{\alpha\beta} q^\beta \; \bar{q}^\gamma (1-\gamma_5) (T^a)^{\gamma\delta} b^\delta, \qquad \langle T_2^q \rangle = \langle B_q | T_2^q | B_q \rangle = \frac{M_{B_q}^2}{(m_b+m_q)^2} f_{B_q}^2 M_{B_q}^2 \epsilon_2^q.$$

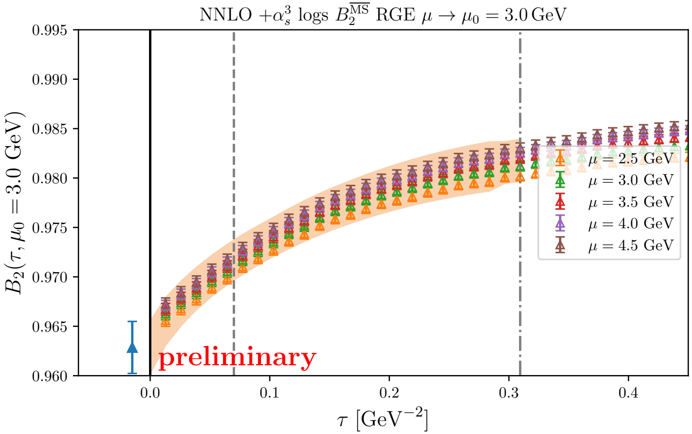
- For simplicity of computation, we rewrite these to be colour-singlet operators:

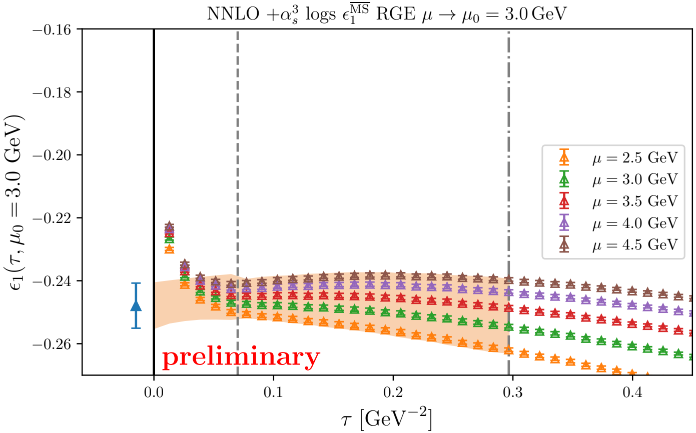
$$\begin{aligned} \mathcal{O}_1 &= \bar{b}^\alpha \gamma_\mu (1-\gamma_5) q^\alpha \; \bar{q}^\beta \gamma_\mu (1-\gamma_5) b^\beta \\ \mathcal{O}_2 &= \bar{b}^\alpha (1-\gamma_5) q^\alpha \; \bar{q}^\beta (1+\gamma_5) b^\beta) \\ \tau_1 &= \bar{b}^\alpha \gamma_\mu (1-\gamma_5) b^\alpha \; \bar{q}^\beta \gamma_\mu (1-\gamma_5) q^\beta \\ \tau_2 &= \bar{b}^\alpha \gamma_\mu (1+\gamma_5) b^\alpha \; \bar{q}^\beta \gamma_\mu (1-\gamma_5) q^\beta \end{aligned}$$

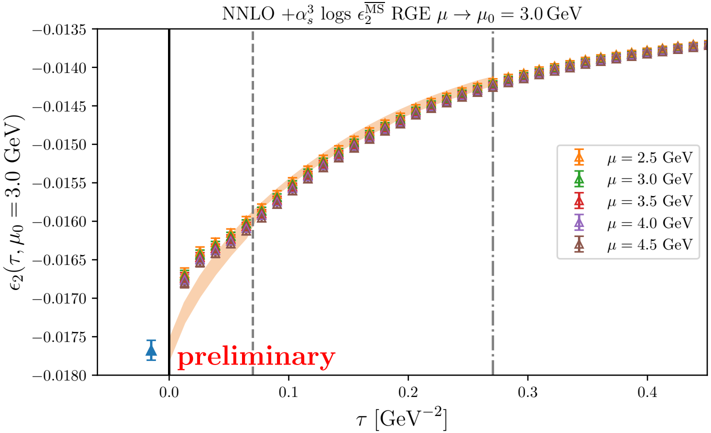
$$\begin{pmatrix} \mathcal{O}_1^+ \\ \mathcal{O}_2^+ \\ T_1^+ \\ T_2^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2N_c} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2N_c} & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \mathcal{O}_1^+ \\ \mathcal{O}_2^+ \\ \tau_1^+ \\ \tau_2^+ \end{pmatrix}$$

Consistency check — include α_s^3 logs at NNLO (not RG improved) A.2

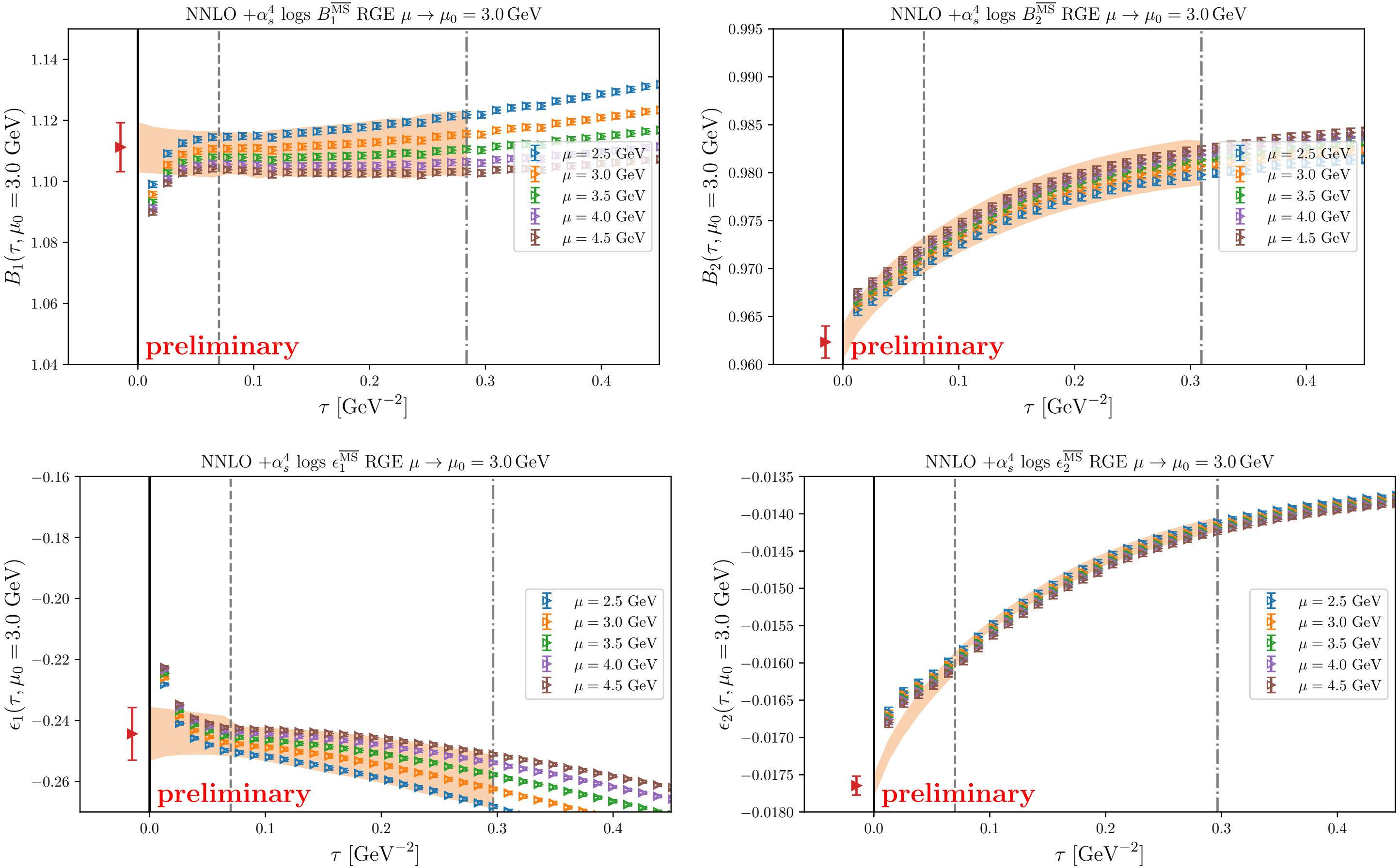








Consistency check — include α_s^4 logs at NNLO (not RG improved) A.3



Consistency check — include α_s^5 logs at NNLO (not RG improved)