# Perturbative Corrections to the subleading free-quark decay

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# Lifetimes

1. Heavy hadron **lifetimes are fundamental** properties, therefore important and interesting.
2. **Crucial for testing** our understanding of **QCD** in its interplay **with EW** interaction.

$B$ and $D$ meson lifetimes are known experimentally at 1‰ and 1% **[PDG Collaboration, Review of particle physics, PRD 110 (2024), 030001]**

$$\tau_{\rm exp}(B_d) = 1.517(4)\ \text{ps}, \quad \tau_{\rm exp}(B^+) = 1.638(4)\ \text{ps}, \quad \tau_{\rm exp}(B_s) = 1.520(5)\ \text{ps}$$
$$\tau_{\rm exp}(D^0) = 0.4101(15)\ \text{ps}, \quad \tau_{\rm exp}(D^+) = 1.040(7)\ \text{ps}, \quad \tau_{\rm exp}(D_s^+) = 0.504(4)\ \text{ps}$$

Until recently **theoretical precision strongly limited by** uncertainty due to $\mu$-dependence of the **free quark decay**, **only** available at **NLO**.

$$\Gamma(H_Q) = \frac{G_F^2 m_Q^5}{192\pi^3} \sum_{q_i} |V_{q_1 Q}|^2 |V_{q_2 q_3}|^2 \left( C_0^{q_1 q_2 q_3} + \mathcal{O}(\Lambda_{\rm QCD}/m_Q^2) \right)$$

**Precision studies** were **focused on lifetime ratios** (free quark decay cancels).

# Lifetimes

However, recent **determination of NNLO** corrections to free-quark decay **opened** the **road** for **precision tests** also **for lifetimes**.
**[Egner, Fael, Schönwald and Steinhauser, JHEP 02 (2025), 147]**

- Including the NNLO, $B$ meson lifetimes determined with uncertainty below $\sim 6\%$
  **[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald and Steinhauser, JHEP 04 (2025), 106]**
- For $D$ not yet included (currently large uncertainties)
  **[King, Lenz, Piscopo, Rauh, Rusov and Vlahos, JHEP 08 (2022), 241]**
  **[Gratrex, Melić and Nišandžić, JHEP 07 (2022), 058]**
- Further improvement possible by calculating higher orders in the HQE

Motivated by this **we explore PT corrections to the power suppressed terms** for $B$ and $D$ hadron lifetimes.

# Lifetimes

We consider the **CKM favoured decay channels** for the heavy hadron's **inclusive nonleptonic** decays

- $D$-decays
  - $c \to s\bar{d}u$ (massless)
- $B$-decays
  - $b \to c\bar{u}d$ (one mass)
  - $b \to c\bar{c}s$ (two masses)

we **keep** $m_c \sim m_b$ with $\rho = m_c^2/m_b^2$ and **light quarks** are **masless**.

# The weak effective Lagrangian

We consider **heavy** quark $Q$ **decays into three** quarks with **different flavors** $Q \to q_1 \bar{q}_2 q_3$ (no penguins)

$$\mathcal{L}_{\text{eff}} = -2\sqrt{2} G_F V_{q_2 q_3} V^*_{q_1 Q} (C_1(\mu)\mathcal{O}_1 + C_2(\mu)\mathcal{O}_2) + \text{h.c}\,,$$

with color singlet and color rearranged operators

$$\mathcal{O}_1 = (\bar{Q}^i \gamma^\mu P_L q_1^j)(\bar{q}_2^j \gamma_\mu P_L q_3^i)\,, \qquad \mathcal{O}_2 = (\bar{Q} \gamma^\mu P_L q_1)(\bar{q}_2 \gamma_\mu P_L q_3)\,,$$

- $C_{1,2}$ by **matching** to the SM at $\mu = M_W$ and **evolving down** to $\mu \sim m_Q \ll M_W$ via renormalization group at NLL.
- $\mathcal{O}_{1,2}$ **mix under renormalization**.
- Need to specify the **treatment of** $\gamma_5$ in dimensional regularization
  - The coefficients $C_{1,2}$ scheme-depedent.
  - The HQE correlators scheme-dependent.
  - The coefficients of the HQE (combination) scheme-independent.

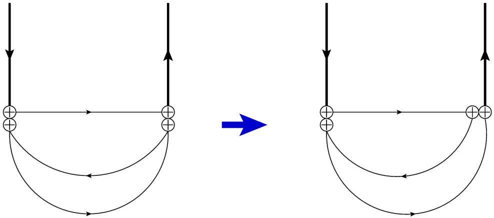
# The weak effective Lagrangian

We choose **NDR** with definition of evanescent operators that preserves **Fierz symmetry** in $D$-dimensions

[E. Bagan, P. Ball, V. M. Braun and P. Gosdzinsky, Nucl. Phys. B 432 (1994), 3-38]
[A.J. Buras and P.H. Weisz, Nucl. Phys. B 333 (1990) 6]

$$\begin{aligned} E_1 &= (\bar{Q}^i\gamma^\mu\gamma^\nu\gamma^\alpha P_L q_1^j)(\bar{q}_2^j\gamma_\mu\gamma_\nu\gamma_\alpha P_L q_3^i) - (16-4\epsilon)\mathcal{O}_1\,, \\ E_2 &= (\bar{Q}\gamma^\mu\gamma^\nu\gamma^\alpha P_L q_1)(\bar{q}_2\gamma_\mu\gamma_\nu\gamma_\alpha P_L q_3) - (16-4\epsilon)\mathcal{O}_2\,. \end{aligned}$$

Circumvents the algebraic inconsistencies arising when using anticommuting $\gamma_5$ in $D$ dimensions within **closed fermion loops** by **avoiding** them.



# HQE for inclusive nonleptonic decays

The $\Gamma_{H_Q\to X}(Q\to q_1\bar{q}_2 q_3)$ obtained from

$$\Gamma_{H_Q\to X} \quad \sim \quad \mathrm{Im}\,\langle H_Q| i\int dx\, T\left\{\mathcal{L}_{\rm eff}(x)\mathcal{L}_{\rm eff}(0)\right\}|H_Q\rangle$$

Since $m_Q \gg \Lambda_{\rm QCD}$ one can set up an expansion in $\Lambda_{\rm QCD}/m_Q$ (HQE)

$$\Gamma_{H_Q\to X} = \frac{G_F^2 m_Q^5}{192\pi^3}|V_{q_2 q_3}|^2|V_{q_1 Q}|^2\left[C_0\left(1-\frac{\mu_\pi^2}{2m_Q^2}\right)+C_{\mu_G}\frac{\mu_G^2}{2m_Q^2}\right]$$

perturbative and non-perturbative contributions are factorized in:

- **Wilson coefficients**: $C_i(\rho)$ have a perturbative expansion in $\alpha_s(\mu)$, obtained by matching to QCD.
  $C_0$**@NNLO [Egner, Fael, Schönwald and Steinhauser, JHEP 02 (2025), 147]**
  $C_{\mu_G}$**@LO [Bigi, Uraltsev and Vainshtein, Phys. Lett. B 293 (1992)] [Blok and Shifman, Nucl. Phys. B 399 (1993) & Nucl. Phys. B 399 (1993)]**

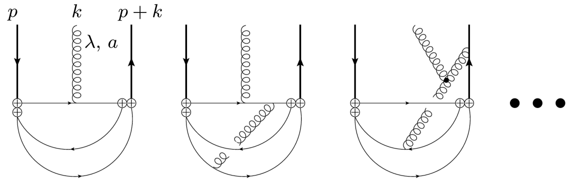
- **Forward ME of HQET operators**: called hadronic parameters

$$\begin{aligned}\mu_\pi^2 &= \langle H_Q|\bar{h}_v D_\perp^2 h_v|H_Q\rangle/2M_{H_Q}\\ \mu_G^2 &= g_s c_F(\mu)\langle H_Q|\bar{h}_v\sigma_{\alpha\beta}G_\perp^{\alpha\beta}h_v|H_Q\rangle/4M_{H_Q} = \frac{3}{4}(M_{H_Q^*}^2 - M_{H_Q}^2)\end{aligned}$$

At $\alpha_s/m_Q^2$ we only need to **determine the chromomagnetic** term

# Inclusive nonleptonic decay rate at $\mathcal{O}(\alpha_s/m_b^2)$

- **Take the amplitude** of the 3-point function ($Q \to Qg$) with kin. conf.




with $p^2 = m_Q^2$ and $k^\mu \sim \Lambda_{\rm QCD}$

- **Expand** to linear order in the small momentum $k$

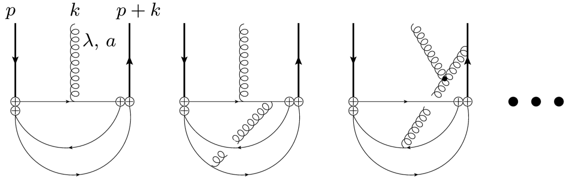
$$\mathcal{A}^\lambda_{Q\to Qg} = \mathcal{A}^\lambda_0 + \mathcal{A}_1^{\lambda\alpha} k_\alpha$$

- **Project** to the chromomagnetic operator

$$C_{\mu_G} \sim C_{A_0}\,\mathrm{Tr}(\mathcal{A}^\lambda_0 v_\lambda P_+) + C_{A_1}\frac{1}{c_F}\,\mathrm{Tr}(\mathcal{A}_1^{\lambda\alpha} P_+[\gamma_{\perp\,\alpha},\gamma_{\perp\,\lambda}]P_+)$$

At dim. 5 also operator $\bar h_v (v\cdot D)^2 h_v$, that contributes to higher orders in $1/m_Q$ after using the EOM (also the projector gets rid of it).

# Inclusive nonleptonic decay rate at $\mathcal{O}(\alpha_s/m_b^2)$



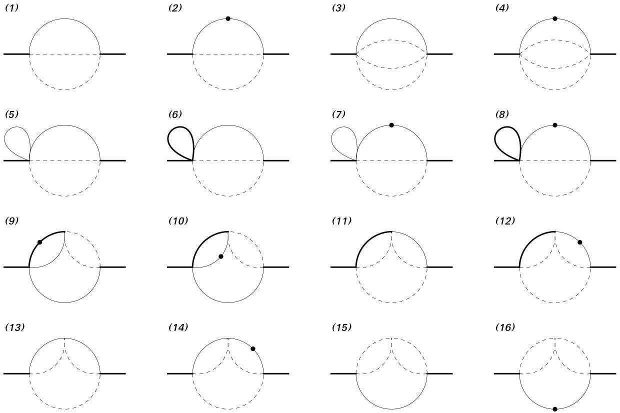

- **External legs** (definition of $h_v$ in HQET Lagrangian) expand and recipe

$$\frac{1}{p^2 - m_Q^2} \to -\frac{i}{2m_Q} P_-$$

- **Renormalization** is on shell for bottom and charm quarks and $\overline{\mathrm{MS}}$ for $\alpha_s$ and the weak effective and HQET Lagrangians

$$(\bar{h}_v \sigma_{\alpha\beta} G_\perp^{\alpha\beta} h_v)_B = Z_{\mathcal{O}_G} (\bar{h}_v \sigma_{\alpha\beta} G_\perp^{\alpha\beta} h_v)$$

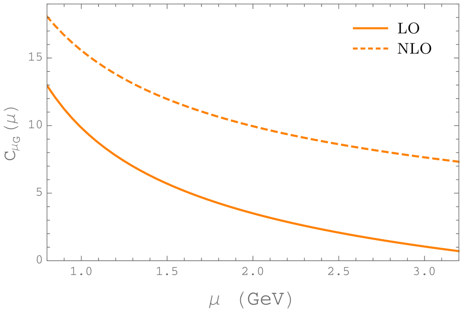
# Inclusive nonleptonic decay rate at $\mathcal{O}(\alpha_s/m_b^2)$




- $c \to s\bar{d}u$: cut 3 massless lines (numbers).
- $b \to c\bar{u}d$: cut 2 massless and 1 massive line (logs and polylogs).
- $b \to c\bar{c}s$: cut 1 massless and 2 equal mass lines (multiple polylogs).
  [Egner, Fael, Schönwald and Steinhauser, JHEP 02 (2025), 147]

# Results

## $c \to s\bar{d}u$ (massless)




| Parameter | Numerical value |
|---|---|
| $m_b$ | 4.7 GeV |
| $m_c$ | 1.6 GeV |
| $\rho = m_c^2/m_b^2$ | 0.116 |
| $\mu_\pi^2$ | 0.5 GeV$^2$ |
| $\mu_G^2$ | 0.35 GeV$^2$ |

- 12.4% **reduction in the $\mu$-dependence** of $C_{\mu_G}$ between $m_c/2 \le \mu \le 2m_c$
- $C_{\mu_G}$ LO numerically small due to accidental cancellations
  $\Rightarrow$ **large NLO corrections**.

# Results

$c \to s\bar{d}u$ (massless)

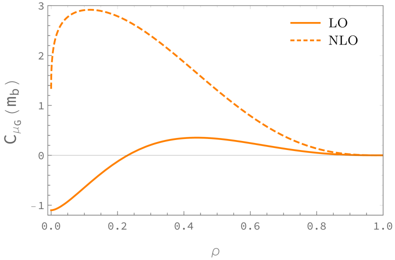
$$\frac{\Gamma(c \to s\bar{d}u)}{\Gamma_0 |V_{cs}|^2 |V_{ud}|^2} = C_0\left(1 - \frac{\mu_\pi^2}{2m_c^2}\right) + C_{\mu_G}\frac{\mu_G^2}{2m_c^2}$$
$$= (3.56_{\mathrm{LO}} + 0.49_{\mathrm{NLO}})_0 + (0.35_{\mathrm{LO}} + 0.43_{\mathrm{NLO}})_{\mu_G}$$

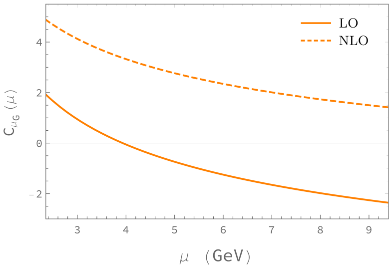
| **Contribution** | **% w.r.t LO** |
|---|---|
| NLO partonic | 13.7% |
| LO chromomagnetic | 9.9% |
| NLO chromomagnetic | 12.1% |

**NLO chromomagnetic** corrections are **significant** for $D$-hadrons
[Mannel, DM and Pivovarov, PRD 107 (2023), 114026]

# Results

$b \to c\bar{u}d$ (one mass)

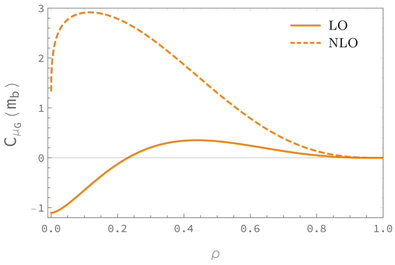


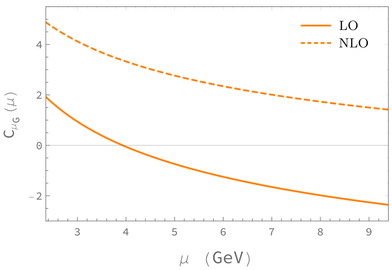




- $C_{\mu_G}$ **smaller for larger** $\rho$ due to smaller phase space for decay.
- $C_{\mu_G}$ LO numerically small due to accidental cancellations.
  $\Rightarrow$ **LO overwhelmed by NLO** corrections (specially at $\rho = 0.116$ and $\mu = m_b$)
- $C_{\mu_G}$ **changes sign** at NLO w.r.t LO.

# Results

$b \to c\bar{u}d$ (one mass)




- NLO $C_{\mu_G}$ remains positive for $m_b/2 \leq \mu \leq 2m_b \Rightarrow$ **NLO sets the sign**.
- $\mu$ **dependence** remains large but **reduced** by 20% $\Rightarrow$ need NNLO to reduce further.

# Results

$b \to c\bar{u}d$ (one mass)

$$
\begin{aligned}
\frac{\Gamma(b \to c\bar{u}d)}{\Gamma_0 |V_{ud}|^2 |V_{cb}|^2} &= C_0\left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) + C_{\mu_G}\frac{\mu_G^2}{2m_b^2} \\
&= (1.418_{\mathrm{LO}} + 0.104_{\mathrm{NLO}})_0 + (-0.004_{\mathrm{LO}} + 0.028_{\mathrm{NLO}})_{\mu_G}
\end{aligned}
$$

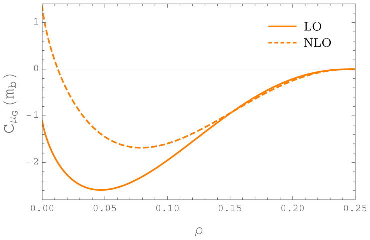
| **Contribution** | % **w.r.t LO** |
|---|---|
| NLO partonic | 7.3% |
| NNLO partonic | $\sim$ 0.3% |
| LO chromomagnetic | $-0.3$% |
| NLO chromomagnetic | 1.9% |

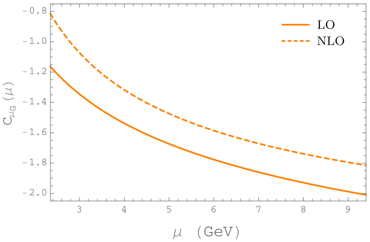
**NLO chromomagnetic** corrections **significant** for $b \to c\bar{u}d$

[Mannel, DM and Pivovarov, PRD 110 (2024), 094011]

# Results

## $b \to c\bar{c}s$ (two masses)

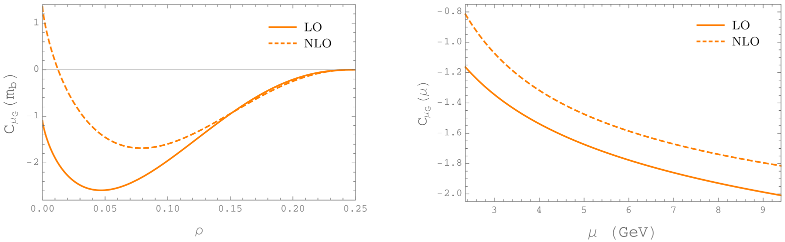



- $C_{\mu_G}$ **smaller for larger** $\rho$ due to smaller phase space for decay.
- **NLO** corrections **small for higher** $\rho$ and **very large for lower** $\rho$.
  $\Rightarrow$ numerical cancellations at low $\rho$.
- NLO corrections to $C_{\mu_G}$ are 12% (for $\rho = 0.116$ and $\mu = m_b$)

# Results

## $b \to c\bar{c}s$ (two masses)




- In contrast to $b \to c\bar{u}d$, **NLO corrections** to $C_{\mu_G}$ are **small**.
- $\mu$ **dependence of** $C_{\mu_G}$ **barely changes** from LO to NLO $\Rightarrow$ need NNLO?

# Results

## $b \to c\bar{c}s$ (two masses)

$$\frac{\Gamma(b \to c\bar{c}s)}{\Gamma_0 |V_{cb}|^2 |V_{cs}|^2} = C_0 \left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) + C_{\mu_G} \frac{\mu_G^2}{2m_b^2}$$

$$= (0.3434_{\mathrm{LO}} + 0.1155_{\mathrm{NLO}})_0 + (-0.0130_{\mathrm{LO}} + 0.0016_{\mathrm{NLO}})_{\mu_G}$$

| **Contribution** | **% w.r.t LO** |
|---|---|
| NLO partonic | 34% |
| NNLO partonic | $\sim$ 4% |
| LO chromomagnetic | $-4$% |
| NLO chromomagnetic | 0.5% |

**NLO chromomagnetic** corrections **small** for $b \to c\bar{c}s$
[Mannel, DM and Pivovarov, PRD 111 (2025), 094035]

# Summary

We have computed $\alpha_s$ **corrections** to the $(\Lambda_{\mathrm{QCD}}/m_Q)^2$ terms in the HQE of **inclusive nonleptonic** decays of heavy hadrons ($Q \to q_1 \bar{q}_2 q_3$).

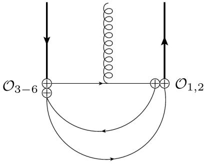
The **corrections** have a **significant impact** for $c \to s\bar{d}u$ (12%) and $b \to c\bar{u}d$ (2%), and are small for $b \to c\bar{c}s$ (0.5%).

For $c \to s\bar{d}u$ and $b \to c\bar{u}d$ the $\alpha_s(\Lambda_{\mathrm{QCD}}/m_Q)^2$ **essential to determine the size of power corrections** (comparable or bigger than LO).

Overall, **important for precision** studies of $B$ and $D$ hadron lifetimes.

# Outlook

**Penguin operators/diagrams** for $b \to c\bar{c}s$ decay.




[F. Krinner, A. Lenz and T. Rauh, Nucl. Phys. B 876 (2013), 31-54]

**Update of $D$-meson lifetimes** by including the recently obtained $\alpha_s^2$ and $\alpha_s(\Lambda_{\rm QCD}/m_Q)^2$

Computation of $\alpha_s$ corrections to **Darwin coefficient** might have impact on $\tau(B_s)/\tau(B_d)$.
[DM, PRD 109 (2024) no.7, 074030]