The $B^+ - B_d^0$ Lifetime Differences @ NNLO

Francesco Moretti

In collaboration with: U. Nierste, M. Steinhauser & P. Reeck

More than a Lifetime – Seigen, 25 Sep 2025

KIT 200 JAHRE

Institute for Theoretical Particle Physics

Collaborative Research Center TRR 257

P H

Particle Physics Phenomenology after the Higgs Discovery

Outline

- Introduction
- Heavy Quark Expansion
- Next-to-Next-to-Leading Order Calculation
- Summary & Conclusions

Introduction

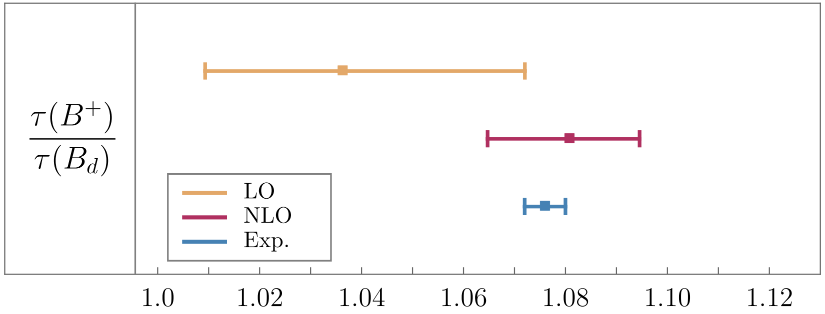
- **H**eavy **Q**uark **E**xpansion (HQE) provides a theoretical framework to systematically study decays of hadrons containing one heavy quark

See Maria Laura's talk

- HQE is also used for many other physical quantities, e.g. $\Delta\Gamma$ in the neutral B_s and B_d system

- Lifetimes ratios are theoretically clean, with more complicated contributions cancelling in the decay widths difference

- We can use precise theoretical predictions of lifetimes to test the HQE at higher orders in perturbation theory



- $\tau(B^+)/\tau(B_d)|_{\text{LO}} = 1.036^{+0.036}_{-0.027}$
- $\tau(B^+)/\tau(B_d)|_{\text{NLO}} = 1.081^{+0.014}_{-0.016}$
- $\tau(B^+)/\tau(B_d)|_{\text{exp.}} = 1.076 \pm 0.004$

[Black, Lang, Lenz, Wüthrich, 2024]

[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser, 2024]

Heavy Quark Expansion

- The first step to compute $\Gamma(H_b)$ is to perform and OPE integrating out the heavy W boson that mediates the weak decays

Effective $|\Delta B| = 1$ local Hamiltonian

μ_1 renormalisation scale

$$H = \frac{G_F}{\sqrt{2}} V_{cb}^* \sum_{\substack{u'=u,c \\ d'=d,s}} V_{u'd'} \left[C_1(\mu_1) Q_1^{u'd'}(\mu_1) + C_2(\mu_1) Q_2^{u'd'}(\mu_1) \right]$$

We focus on current-current operators only

- The Wilson coefficients $C_{1,2}$ encode the physics above the scale μ_1

- We work in the “historical” basis

Mixing under Fierz transformation

$$Q_1^{u'd'} = (\bar{b}_i c_j)_{V-A} (\bar{u}'_j d'_i)_{V-A} \qquad Q_2^{u'd'} = (\bar{b}_i c_i)_{V-A} (\bar{u}'_j d'_j)_{V-A}$$

Heavy Quark Expansion

- We relate the total decay rate $\Gamma(H_b)$ to the Hamiltonian *H* using the optical theorem

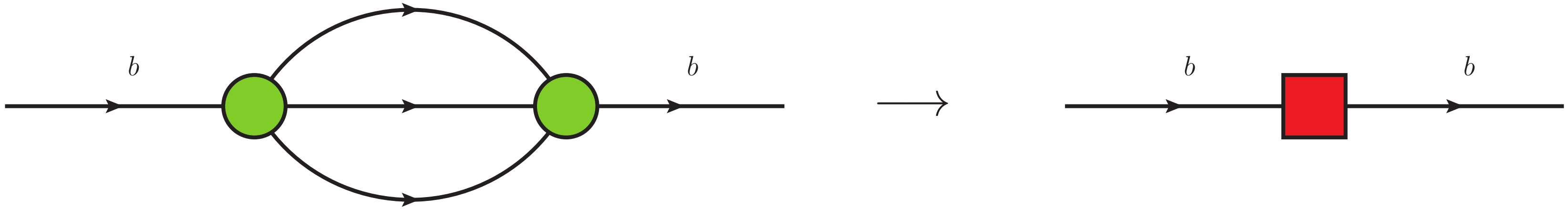
$$\Gamma(H_b) = \frac{1}{2M_{H_b}} \langle H_b | \mathcal{T} | H_b \rangle \qquad \mathcal{T} = \mathrm{Im}\, i \int d^4x \, \mathrm{T}\left\{ H(x)\, H(0) \right\}$$

Non-local correlator !

- HQE exploits the hierarchy of scales $m_b \gg \Lambda_{\mathrm{QCD}}$

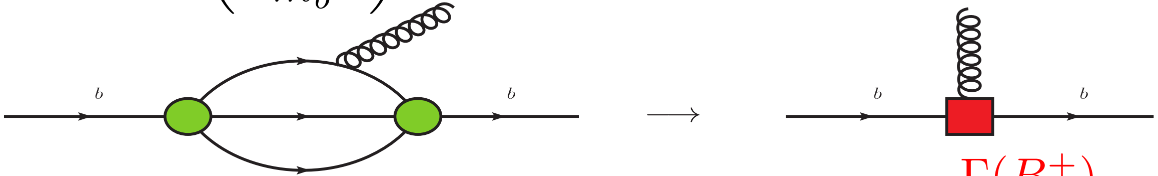
$$\mathcal{T} = \mathcal{T}^0 + \mathcal{T}^2 + \mathcal{T}^3 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right)^4 \quad \text{where} \quad \mathcal{T}^i \sim 1/m_b^i$$

- Leading order $\Longrightarrow$ Free b-quark decay $\longrightarrow$ All lifetimes are the same

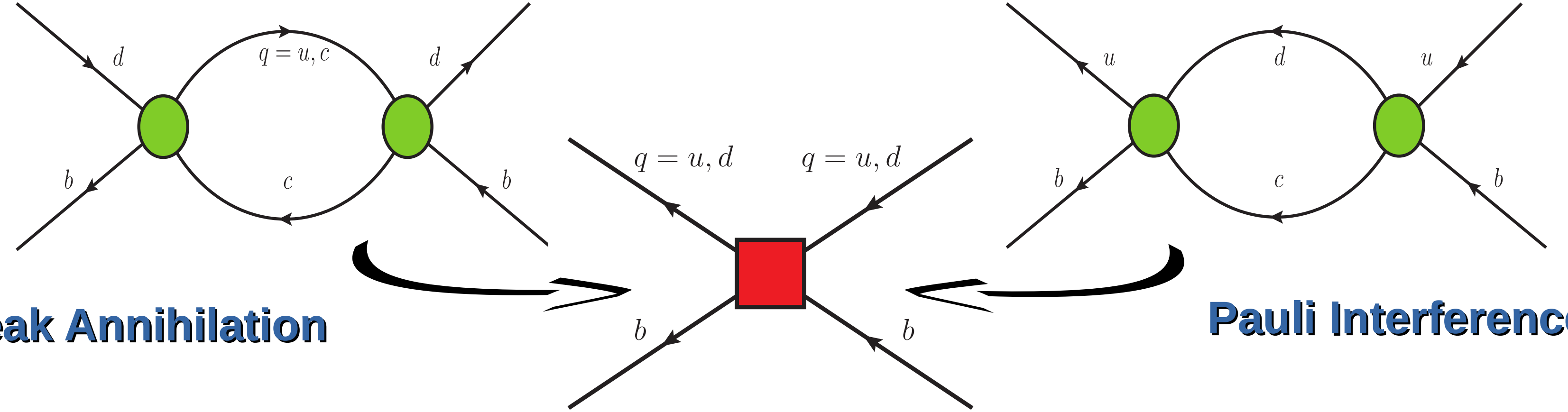


Heavy Quark Expansion

- First corrections arise at $\mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right)^2$ from chromomagnetic interactions



- Strong isospin $\Longrightarrow$ Negligible contributions to $\tau(B^+)/\tau(B_d)$ $\frac{\Gamma(B^+)}{\Gamma(B_d^0)} \approx 1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right)^3$
- At $\mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right)^3$ $\longrightarrow$ weak interaction of b-quarks with (light) valence quarks



Weak Annihilation

Pauli Interference

Heavy Quark Expansion

- We match the amplitudes onto a set of effective local $|\Delta B|=0$ operators

$$Q^q = (\bar{b}\, q)_{V-A}(\bar{q}\, b)_{V-A} \qquad Q_S^q = (\bar{b}\, q)_{S-P}(\bar{q}\, b)_{S+P}$$

$$T^q = (\bar{b}\, T^a q)_{V-A}(\bar{q}\, T^a b)_{V-A} \qquad T_S^q = (\bar{b}\, T^a q)_{S-P}(\bar{q}\, T^a b)_{S+P}$$

[Beneke, Buchalla, Greub, Lenz, Nierste, 2002]

- The decay rate is then decomposed as $\mathcal{T}^3 = \boxed{\mathcal{T}^u} + \boxed{\mathcal{T}^d}$

Weak Annihilation

Pauli Interference

- $$\mathcal{T}^u = \frac{G_F^2 m_b^2 |V_{cb}|^2}{6\pi} \left[|V_{ud}|^2 \left(F^u Q^d + F_S^u Q_S^d + G^u T^d + G_S^u T_S^d \right) + |V_{cd}|^2 \left(F^c Q^d + F_S^c Q_S^d + G^c T^d + G_S^c T_S^d \right) \right]$$

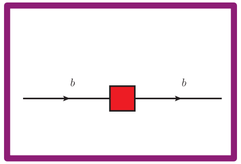
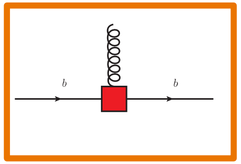
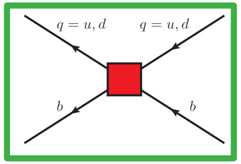
- $$\mathcal{T}^d = \frac{G_F^2 m_b^2 |V_{cb}|^2}{6\pi} \left[\left(F^d Q^u + F_S^d Q_S^u + G^d T^u + G_S^d T_S^u \right) \right]$$

Matching to $|\Delta B|=1$

Heavy Quark Expansion

- **Optical theorem:** $\Gamma(H_b) = \dfrac{1}{2M_{H_b}} \langle H_b | \mathcal{T} | H_b \rangle$ $\quad \mathcal{T} = \mathrm{Im}\, i \int d^4x \, \mathrm{T}\left\{ H(x)\, H(0) \right\}$

Non-local Correlator !

$$m_b \gg \Lambda_{\mathrm{QCD}}$$

$$\mathcal{T} = \boxed{\mathcal{T}^0} + \boxed{\mathcal{T}^2} + \boxed{\mathcal{T}^3} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right)^4$$

Effective $|\Delta B| = 1$ local Hamiltonian

$$H = \frac{G_F}{\sqrt{2}} V_{cb}^* \sum_{\substack{u'=u,c \\ d'=d,s}} V_{u'd'} \left[C_1(\mu_1) Q_1^{u'd'}(\mu_1) + C_2(\mu_1) Q_2^{u'd'}(\mu_1) \right]$$

μ_1 renormalisation scale

- We focus on contributions from dimension-6 operators $\Rightarrow \mathcal{T}^3 = \mathcal{T}^u + \mathcal{T}^d$

"Historical" basis:
$$Q_1^{u'd'} = (\bar{b}_i c_j)_{V-A} (\bar{u}_j' d_i')_{V-A}$$
$$Q_2^{u'd'} = (\bar{b}_i c_i)_{V-A} (\bar{u}_j' d_j')_{V-A}$$

- $\mathcal{T}^u = \dfrac{G_F^2 m_b^2 |V_{cb}|^2}{6\pi} \Big[|V_{ud}|^2 \left(F^u Q^d + F_S^u Q_S^d + G^u T^d + G_S^u T_S^d \right) + |V_{cd}|^2 \left(F^c Q^d + F_S^c Q_S^d + G^c T^d + G_S^c T_S^d \right) \Big]$

- $\mathcal{T}^d = \dfrac{G_F^2 m_b^2 |V_{cb}|^2}{6\pi} \Big[\left(F^d Q^u + F_S^d Q_S^u + G^d T^u + G_S^d T_S^u \right) \Big]$

Heavy Quark Expansion

- Using the isospin relation $\langle B_d^0 | Q^{u,d} | B_d^0 \rangle = \langle B^+ | Q^{d,u} | B^+ \rangle$

$$\Gamma(B_d) - \Gamma(B^+) = \frac{G_F^2 m_b^2 |V_{cb}|^2}{12\pi} f_B^2 M_B \left(|V_{ud}|^2 \vec{F}^u + |V_{cd}|^2 \vec{F}^c - \vec{F}^d \right) \cdot \vec{B}$$

where

$$\vec{F}^q(z, \mu_0) = \begin{pmatrix} F^q(z,\mu_0) \\ F_S^q(z,\mu_0) \\ G^q(z,\mu_0) \\ G_S^q(z,\mu_0) \end{pmatrix} \quad \text{and} \quad \vec{B}(\mu_0) = \begin{pmatrix} B_1(\mu_0) \\ B_2(\mu_0) \\ \epsilon_1(\mu_0) \\ \epsilon_2(\mu_0) \end{pmatrix} \quad \text{for } q = u,d,c$$

$$z = \frac{m_c^2}{m_b^2}$$

Bag parameters

- We conveniently decompose the Wilson coefficients as

$$F^u(z,\mu_0) = C_1^2(\mu_1) F_{11}^u(z,\mu_1,\mu_0) + C_1(\mu_1) C_2(\mu_1) F_{12}^u(z,\mu_1,\mu_0) + C_2^2(\mu_1) F_{22}^u(z,\mu_1,\mu_0)$$

μ_1 dependence cancels

Heavy Quark Expansion

- Hadronic matrix elements

$$\langle B_d^0|Q^{u,d}|B_d^0\rangle = \langle B^+|Q^{d,u}|B^+\rangle$$

$$\langle B^+|(Q^u - Q^d)(\mu_0)|B^+\rangle = f_B^2 M_B^2 B_1(\mu_0) \qquad \langle B^+|(Q_S^u - Q_S^d)(\mu_0)|B^+\rangle = f_B^2 M_B^2 B_2(\mu_0)$$

$$\langle B^+|(T^u - T^d)(\mu_0)|B^+\rangle = f_B^2 M_B^2 \epsilon_1(\mu_0) \qquad \langle B^+|(T_S^u - T_S^d)(\mu_0)|B^+\rangle = f_B^2 M_B^2 \epsilon_2(\mu_0)$$

$|\Delta B| = 0$ renormalisation scale

Isospin-breaking combinations

- Non-perturbative evaluation of the bag parameters

Lattice QCD

Preliminary studies

[Di Pierro, Sachrajda, 1998]
[Di Pierro, Sachrajda ,Michael, 1999]
[Becirevic, 2001]
[Lin, Detmold, Meinel, 2022]
[Black, Harlander, Lange, Rago, Shindler, Witzel, 2023]
[Black, Harlander, Lange, Rago, Shindler, Witzel, 2024]

HQET Sum Rules

[Kirk, Lenz and Rauh, 2017]
[Black, Lang, Lenz, Wuthrich, 2024]
[King, Lenz, Rauh, 2021]

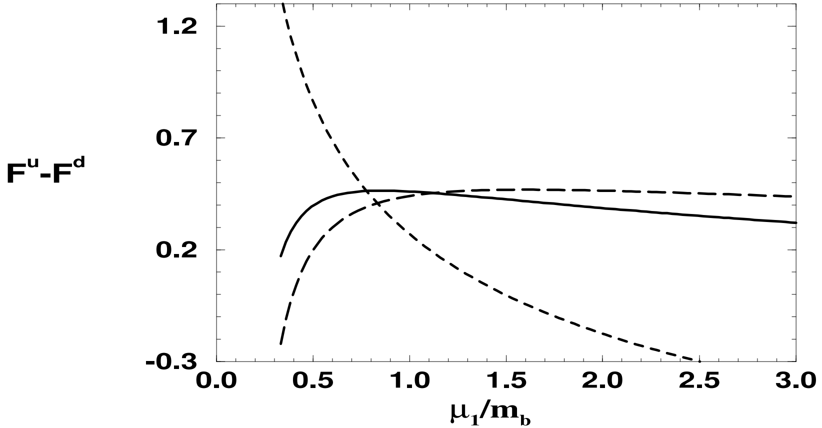
Matching @ NLO

- Next-to-Leading order calculation of the Wilson coefficients

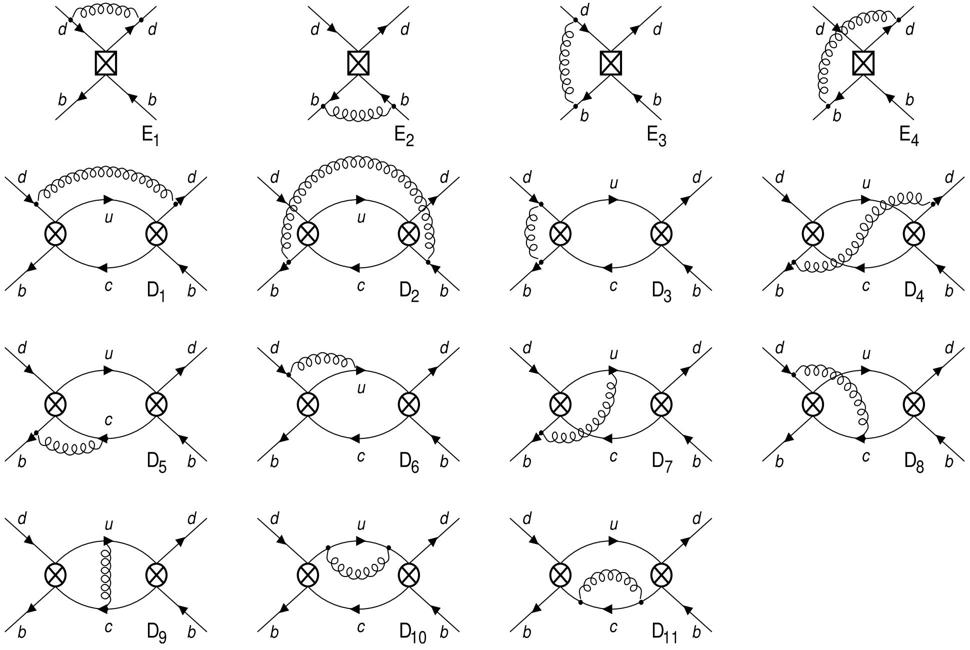
$m_c = 0$

- *[Ciuchini, Franco, Lubicz, Mescia, 2001]*
- *[Franco, Lubicz, Mescia, Tarantino, 2002]*
- *[Beneke, Buchalla, Greub, Lenz, Nierste, 2002]*

No contribution from $c\bar{c}$ intermediate states at NLO



Feynman diagrams



NNLO QCD Calculations

Workflow

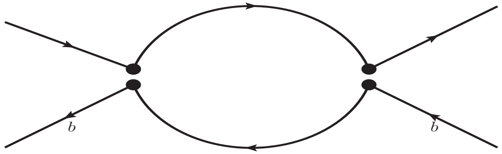
[Nogueira, 1991]	QGRAF	Diagram generation based on model files
[Gerlach, Herren, Lang, 2022] [Seidensticker, 1999]	TAPIR + EXP	Implementation of Feynman rules & mapping over topologies
[Vermaseren, 2000] [Reeck, Shtabovenko, Steinhauser, 2024]	FORM	Reduction of Feynman amplitudes in terms of scalar integrals
[Klappert, Lange, Maierhöfer, Usovitsch, 2023]	KIRA	Integration-by-parts reduction to master integrals
[Wolfram Research, 2024]	FORM + MATHEMATICA	Implementation of master integrals & renormalisation

$|\Delta B| = 1$ Renormalisation

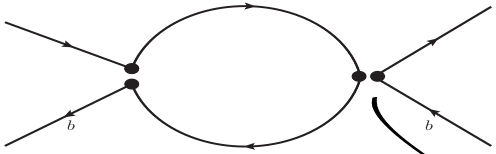
- Leading CKM contribution @ NNLO $\rightarrow |V_{cd}| = 0$

- "Full theory" given by $|\Delta B| = 1$ contributions $\Longrightarrow$ "Historical" basis operators $Q_{1,2}$

WA

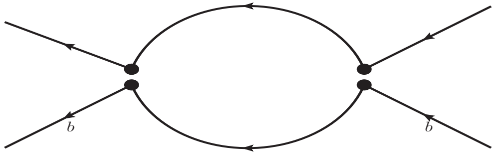


$\Longrightarrow$

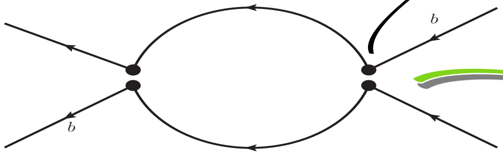


Fierzed operators

PI



$\Longrightarrow$

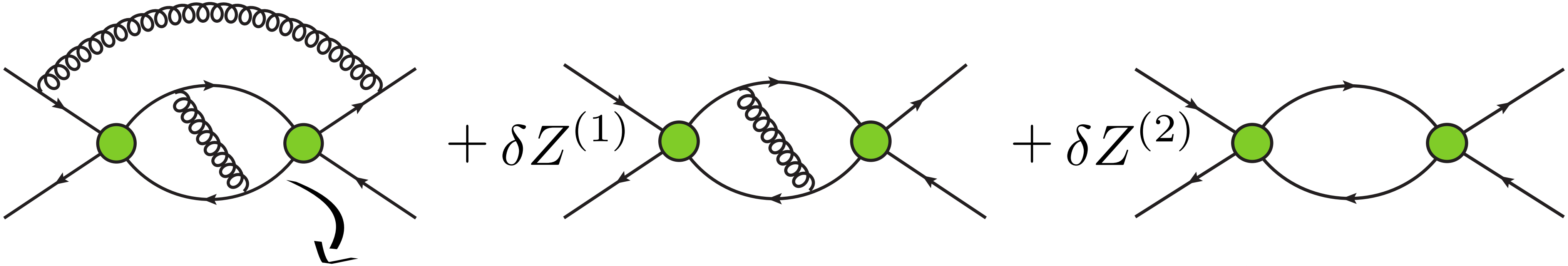


More complicated to evaluate

Choice of evanescent operators is crucial !

- Wilson coefficients $C_{1,2}$ mix upon renormalisation according to
  - $C_j^b = Z_{ij} C_i$
  - $Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_{ij}^{(k)}$
  - $Z_{ij}^{(k)} = \sum_{l=0}^{k} \frac{1}{\epsilon^l} Z_{ij}^{(k,l)}$

$|\Delta B| = 1$ Renormalisation



- $\mathcal{O}(2k)$ diagrams

- MIs from recent projects:
  [Reeck, Shtabovenko, Steinhauser, 2024]
  ↘ Log-linear series in $x = m_c/m_b$

$$\boxed{\begin{aligned} E[Q_1] =& (\bar{q}_1^i \gamma^{\mu_1\mu_2\mu_3} P_L b^j)(\bar{q}_2^j \gamma_{\mu_1\mu_2\mu_3} P_L q_3^i) \\ & - (16 - 4\epsilon - 4\epsilon^2) Q_1 \end{aligned}}$$

- Lower order counter-terms
- Mixing matrix up to $\mathcal{O}(\alpha_s^2)$ and appropriate choice of Evanescent operators:
  [Egner, Fael, Schönwald, Steinhauser, 2024]
- $\overline{\textbf{MS}}$ renormalisation of charm quark mass
- on-shell renormalisation of bottom quark mass

$|\Delta B|=0$ Renormalisation

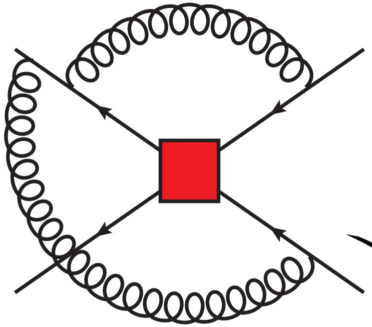
- We do not fix the scheme of evanescent operators @ NNLO, e.g.
$$E[Q^q]^{(3)} = \bar{b}\gamma^{\mu_1}\cdots\gamma^{\mu_7}(1-\gamma_5)q\,\bar{q}\gamma_{\mu_7}\cdots\gamma_{\mu_1}(1-\gamma_5)b - (64+e\epsilon+f\epsilon^2)Q^q$$

- Wilson coefficients $F,\cdots,G_S$ mix upon renormalisation (see previous slides)

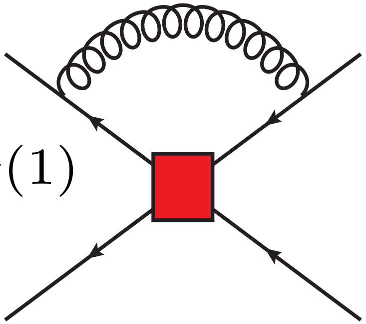
- We derive the two loop renormalisation matrix for $|\Delta B|=0$ basis and extract the two-loop anomalous dimension
$$\frac{\mathrm{d}}{\mathrm{d}\,\ln\mu}\vec{F} = \hat{\gamma}^{\mathrm{T}}\cdot\vec{F}$$

- Recent results presented in [Aebischer, Morell, Pesut, Virto, 2025] with fixed scheme of $E[Q]$

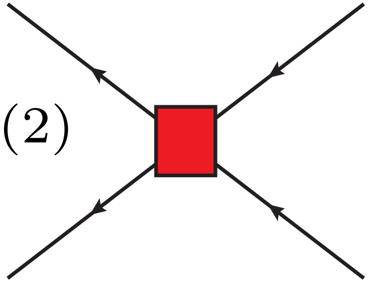
$|\Delta B|=0$ Renormalisation



$+ \quad \delta Z^{(1)}$



$+ \quad \delta Z^{(2)}$



- $\mathcal{O}(1k)$ diagrams

- MIs from recent projects:
  [Reeck, Shtabovenko, Steinhauser, 2024]

$$E[Q] = (\bar{b}\gamma^{\mu_1\mu_2\mu_3} P_L d)(\bar{d}\gamma_{\mu_3\mu_2\mu_1} P_L b) - (8 - 4\epsilon - a\epsilon^2) Q$$

[Beneke, Buchalla, Greub, Lenz, Nierste, 2002]

- Lower order counter-terms
- Mixing matrix know up to $\mathcal{O}(\alpha_s)$. We derived the renormalisation of the $|\Delta B|=0$ basis at $\mathcal{O}(\alpha_s^2)$.
- $\overline{\textbf{MS}}$ renormalisation of charm quark mass
- on-shell renormalisation of bottom quark mass

Matching & Wilson Coefficients

- We perform the matching between the “full” $|\Delta B|=1$ theory and the effective $|\Delta B|=0$ theory in **$d=4-2\epsilon$**

- We set $p_{d,u}=0$ and $p_b^2=m_b^2 \implies$ We generate **IR spurious divergences**

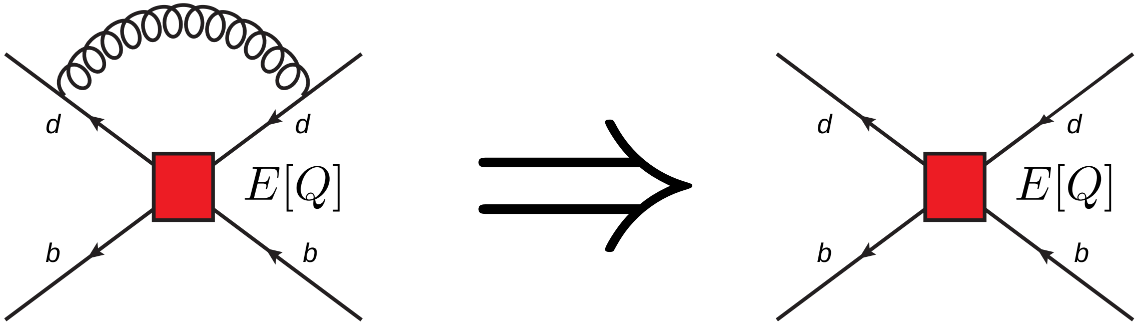
- The IR behaviour of the two theories is the same $\implies$ cancellation of $1/\epsilon_{\rm IR}$ poles

- Yet, spurious poles give finite contributions when they multiply $\mathcal{O}(\epsilon)$ parts of evanescent structures $\implies$ **Evanescent operators must be included in the matching**

Scale-less integral

$\downarrow$

Vanishing in dimreg



Uncancelled counter-terms

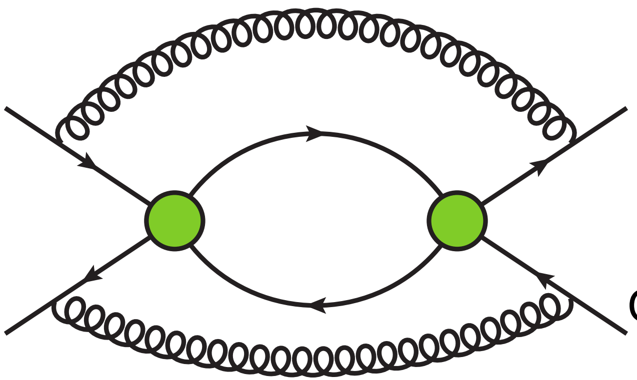
Matching & Wilson Coefficients

- Our calculation is still ongoing $\Rightarrow$ Internal cross checks to be starting soon

- Meanwhile, we performed many checks on the preliminary results such as gauge independence and cancellation of IR divergences

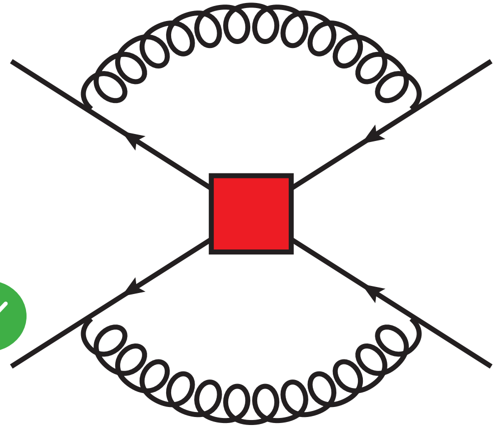
- We also checked the independence of the Wilson coefficients $F, \cdots, G_S$ on μ_1:
  - We perform the matching at $\mu_1 \rightarrow F_{ij}(x, \mu_1)$
  - We use Renormalisation Group Equation $F_{ij}(x, \mu_1) \longrightarrow F_{ij}(x, \mu_1, \boldsymbol{\mu_0})$

    $\mu_0 \sim m_b$



IR cancellation

Gauge independence



Summary & Conclusions

Summary & Conclusions

- Lifetimes ratios provide a clean theoretical ground to test HQE and compare with precise experimental results

- We focus on the dimension-6 operators contribution coming from spectator effects

- In the framework of HQE, we need to match the non-local correlators in the $|\Delta B|=1$ side onto a set of local $|\Delta B|=0$ operators

- So far, this has been done only at $\mathcal{O}(\alpha_s)$. Computation of the matching at $\mathcal{O}(\alpha_s^2)$ is still ongoing, with independent checks to be done soon.

- We checked the correct IR cancellation in the matching of some particular diagrams and the gauge independence of the final Wilson coefficient

- Future outlooks: Exact cancellation of the evanescent scheme $\Rightarrow$ additional complete independent check **See Fabian's talk !**

Summary & Conclusions

- Lifetimes ratios provide a clean theoretical ground to test HQE and compare with precise experimental results
- We focus on the dimension-6 operators contribution coming from spectator effects
- In the framework of HQE, we need to match the non-local correlators in the $|\Delta B|=1$ side onto a set of local $|\Delta B|=0$ operators
- So far, this has been done only at $\mathcal{O}(\alpha_s)$. Computation of the matching at $\mathcal{O}(\alpha_s^2)$ is still ongoing, with independent checks to be done soon.
- We checked the correct IR cancellation in the matching of some particular diagrams and the gauge independence of the final Wilson coefficient
- Future outlooks: Exact cancellation of the evanescent scheme $\Rightarrow$ additional complete independent check **See Fabian’s talk !**

Thank You!