Perturbative corrections to the free quark decay

More than a lifetime, Siegen 2025

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Introduction

Motivation

- The lifetimes of mesons are fundamental properties of particles.
- The decays are due to an interplay of the weak and strong interactions.

$\Rightarrow$ Flavor physics is important to test the Standard Model and search for new physics.

- We consider hadrons with heavy bottom quarks $\boxed{m_b \gg \Lambda_{QCD}}$:
  - Systematic theoretical framework of the Heavy Quark Expansion.
  - Well defined non-perturbative effects.
- Many and highly precise experimental measurements, which will improve in the future (LHCb, Atlas, CMS, BESIII, BelleII).

The Heavy Quark Expansion

Heavy Quark Expansion (HQE): Decay width of a B-meson:

$$\Gamma(B) = \Gamma_3 + \underbrace{\Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \ldots}_{\text{talk by D. Moreno}} + 16\pi^2 \underbrace{\left(\tilde{\Gamma}_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \ldots \right)}_{\text{talk by F. Moretti}} .$$

- Γ_i, $\tilde{\Gamma}_i$: perturbative short distance coefficient functions
- $\mathcal{O}_i$, $\tilde{\mathcal{O}}_i$: local operators bilinear in the heavy quark fields
- $\Gamma_3 = \Gamma(b)$: total decay width of a free b quark

$$\boxed{\Gamma_i = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^k \Gamma_i^{(k)}}$$

The Heavy Quark Expansion – Status

$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{4\pi}\Gamma_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^3 \Gamma_i^{(3)} + \dots$$

	semileptonic decays
$\Gamma_3^{(3)}$	[Fael, Schönwald, Steinhauser '20]
$\Gamma_5^{(1)}$	[Alberti, Gambino, Nandi '13] [Mannel Pivovarov, Rosenthal '15]
$\Gamma_6^{(1)}$	[Mannel, Moreno, Pivovarov '19]
$\Gamma_7^{(0)}$	[Dassinger, Mannel, Turczyk '06]
$\Gamma_8^{(0)}$	[Mannel, Turczyk, Uraltsev '10]
$\tilde{\Gamma}_6^{(1)}$	[Lenz, Rauh '13]

	non-leptonic decays
$\Gamma_3^{(1)}$	[Ho-Kim, Pham '83; Altarelli, Petrarca '91] [Bagan et al. '94; Krinner, Lenz Rauth '13] [Lenz, Nierste, Ostermaier '97]
$\Gamma_3^{(2)}$	[Czarnecki, Slusarczyk, Tkachov '05] [Egner, Fael, Schönwald, Steinhauser '24]
$\Gamma_5^{(1)}$	[Mannel Moreno, Pivovarov '23,'24]
$\Gamma_6^{(0)}$	[Lenz, Piscopo, Rusov '20] [Mannel, Moreno, Pivovarov '20]
$\tilde{\Gamma}_6^{(1)}$	[Beneke, Buchalla, Greub, Lenz, Nierste '02] [Franco, Lubicz, Mescia, Tarantino '02]
$\tilde{\Gamma}_7^{(0)}$	[Gabbiani, Onishchenko, Petrov '03]

The Heavy Quark Expansion – Status

Semileptonic decays

- Inclusive determination of $|V_{cb}|$:
  - $\boxed{|V_{cb}| = (41.69 \pm 0.63) \times 10^{-3}}$ (new q^2 moments from Belle(II)) [Bernlochner et al '22]
  - $\boxed{|V_{cb}| = (41.97 \pm 0.48) \times 10^{-3}}$ [Finauri, Gambino '23]

Non-leptonic decays

[PDG; Lenz, Piscopo Rusov '23]

$$\Gamma_{exp}(B^+)[ps^{-1}] = 0.611 \pm 0.002, \qquad \Gamma_{SM}(B^+)[ps^{-1}] = 0.58^{+0.11}_{-0.07}$$

$$\Gamma_{exp}(B_d)[ps^{-1}] = 0.658 \pm 0.002, \qquad \Gamma_{SM}(B_d)[ps^{-1}] = 0.63^{+0.11}_{-0.07}$$

$$\Gamma_{exp}(B_s)[ps^{-1}] = 0.657 \pm 0.002, \qquad \Gamma_{SM}(B_s)[ps^{-1}] = 0.63^{+0.11}_{-0.07}$$

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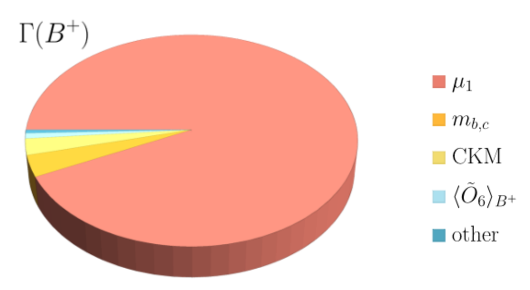
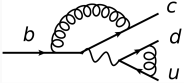


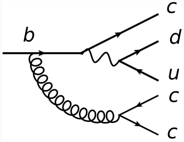
[Abrecht, Bernlochner, Lenz, Rosov '24]

Decay channels at NNLO

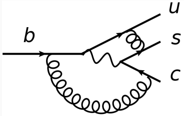
Four decay channels with charm quarks in the final state at $\mathcal{O}(\alpha_s^2)$:

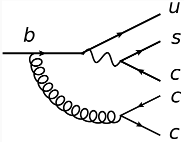
- $\boxed{b \to c\bar{u}d}$ $(\propto \boxed{V_{cb}}\,)$





- $\boxed{b \to u\bar{c}s}$ $(\propto \boxed{V_{ub}}\,)$





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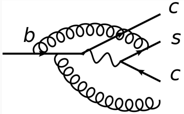

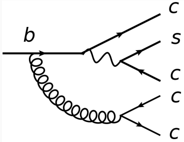

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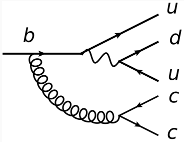

What is known?

Previous calculations including finite charm quark mass for Γ_3:

- $\boxed{b \to c\bar{u}d}$: $\mathcal{O}\left(\alpha_s^1\right)$ [Bagan, Ball, Braun, Gosdzinsky '94]
- $\boxed{b \to c\bar{c}s}$: $\mathcal{O}\left(\alpha_s^1\right)$ [Bagan, Ball, Fiol, Gosdzinsky '95; Krinner, Lenz, Rauh '13]

Calculations at $\mathcal{O}\left(\alpha_s^2\right)$: [Czarnecki, Slusarczyk, Tkachov '06]

- Only massless quarks in the final state $b \to u\bar{u}d$.
- Only one operator of the effective theory was considered.

This talk: NNLO calculation of all non-leptonic decay channels.

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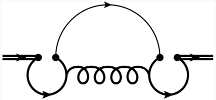
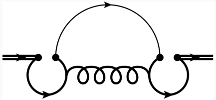
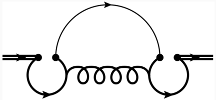
Small caveat: penguin topologies for $b \to c\bar{c}s$ not (yet) included.



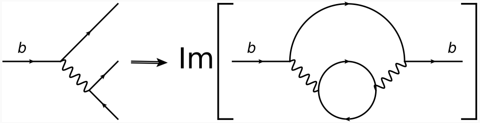




Calculation setup

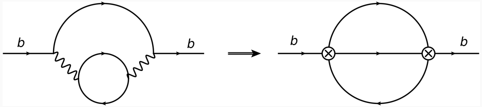
- Optical theorem:

$$\Gamma(b \to c\bar{q}_1 q_2) = \frac{1}{m_b}\mathrm{Im}\left[\mathcal{M}(b \to b)\right]$$



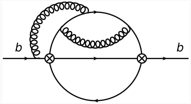
- Integrate out W-boson

$$\frac{1}{(m_W^2 - p^2)} \to \frac{1}{m_W^2}$$



and arrive at the weak effective theory.

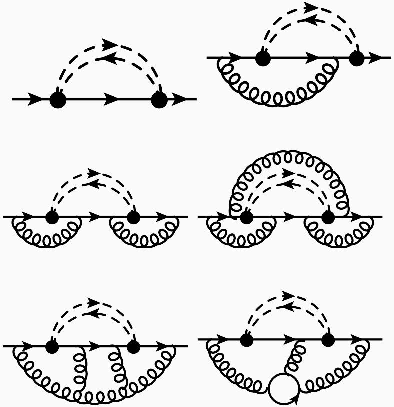
At $\mathcal{O}\left(\alpha_s^2\right)$ calculate imaginary part of 4-loop diagrams.



Semileptonic Decays

Simplicity of semileptonic decays

$$\boxed{\Gamma(b \to c\ell\nu) = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 \left[X_0(\rho) + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n(\rho)\right]}$$



- $\mathcal{O}(\alpha_s^3)$ corrections require the imaginary part of 5-loop forward scattering diagrams.
- The leptonic loop can be integrated trivially.
- After considering the expansion in the mass difference $\boxed{\delta = 1 - m_c/m_b}$ the leptonic system factorizes completely.

$\Rightarrow$ $\boxed{\text{We can reduce a 5-loop calculation to } 3 \times 1 \times 1\text{-loop.}}$

Results - Semileptonic Decays

$$\boxed{\Gamma(b \to c\ell\nu) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right]}$$

$$\begin{aligned} X_3 = \delta^5 \bigg(& \frac{266929}{810} - \frac{5248 a_4}{27} + \frac{2186\pi^2\zeta_3}{45} - \frac{4094\zeta_3}{45} - \frac{1544\zeta_5}{9} - \frac{656 l_2^4}{81} + \frac{1336}{405}\pi^2 l_2^2 \\ & + \frac{44888\pi^2 l_2}{135} - \frac{9944\pi^4}{2025} - \frac{608201\pi^2}{2430} \bigg) + \delta^6 \bigg(-\frac{284701}{540} + \frac{2624 a_4}{9} - \frac{1093\pi^2\zeta_3}{15} \\ & + \frac{391\zeta_3}{3} + \frac{772\zeta_5}{3} + \frac{328 l_2^4}{27} - \frac{668}{135}\pi^2 l_2^2 - \frac{1484\pi^2 l_2}{3} + \frac{4972\pi^4}{675} + \frac{591641\pi^2}{1620} \bigg) + \mathcal{O}(\delta^7 \ln^2(\delta)) \,, \end{aligned}$$

with $l_2 = \ln(2)$, $a_4 = \mathrm{Li}_4(1/2)$, $\zeta_i = \sum_{j=1}^{\infty} j^{-i}$ and $\mu_s = m_b$.

- We have calculated the expansion up to δ^{12} (for general color factors).
- A subset of color factors has been independently been computed up to δ^9.
  [Czakon, Czarnecki, Dowling (2021)]

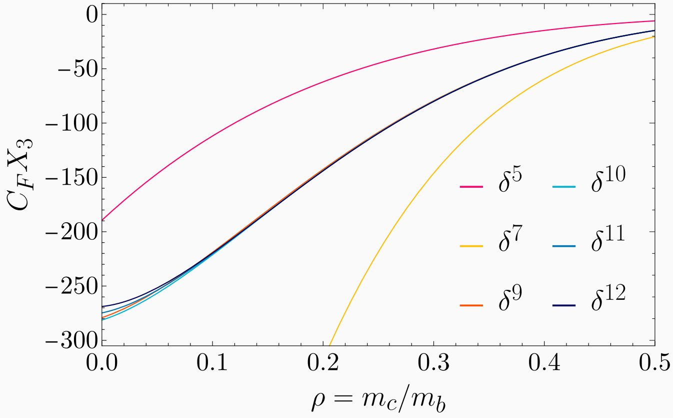
Results - Semileptonic Decays

- We see a good convergence at the physical point of $\rho = m_c/m_b \approx 0.28$.
- We find:

$$\boxed{X_3(\rho = 0.28) = -68.4 \pm 0.3}$$

- We use the difference of the last two expansion terms to estimate the uncertainty.
- For $\rho \to 0$ we can extract values for $b \to u\ell\nu$:

$$\boxed{X_3^u = -202 \pm 20}$$



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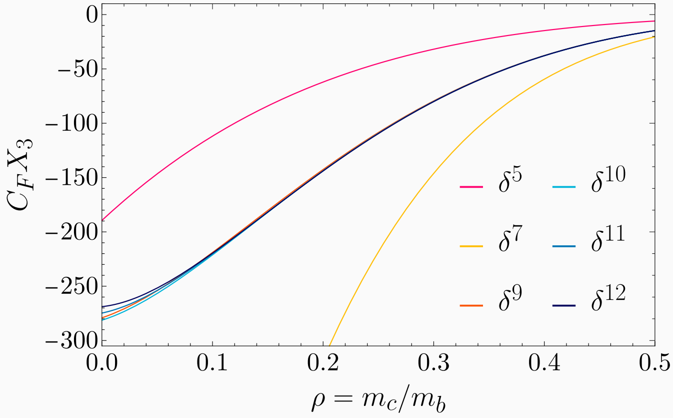
- New calculations: [Fael Usovitsch '23; Chen, Li[2], Wang[2], Fu '23]

$$\boxed{X_3^u = -200.9 \pm 2.0}$$



The Kinetic Heavy Quark Mass

- The kinetic mass scheme is tailored for $b \to c$ transitions.
- It is based on the heavy quark – hadron mass relation in HQET:

Definition

$$\boxed{m^{\text{kin}} = m^{\text{OS}} - \overline{\Lambda}(\mu)\big|_{\text{pert}} - \frac{\mu_\pi^2(\mu)\big|_{\text{pert}}}{2m^{\text{kin}}} + \ldots}$$

- Λ is the B-meson binding energy.
- μ_π is the kinetic energy induced by the residual motion of the heavy quark.

[Bigi et al., PRD 56 (1997); Czarnecki, Melnikov, Uraltsev, PRL 80 (1998)]

- Key mass scale is m_b, not M_B:

$$\Gamma_{\rm sl} \simeq \frac{G_F^2 |V_{cb}|^2}{192\pi^3} (M_B - \overline{\Lambda})^5$$

Results – Kinetic Mass

$$
\begin{aligned}
\frac{m^{\text{kin}}}{m^{\text{OS}}} = 1 &- \frac{\alpha_s^{(n_l)}}{\pi} C_F \left(\frac{4}{3}\frac{\mu}{m^{\text{OS}}} + \frac{1}{2}\frac{\mu^2}{(m^{\text{OS}})^2} \right) + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^2 C_F \Bigg\{ \frac{\mu}{m^{\text{OS}}} \left[C_A \left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9} l_\mu \right) + n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \\
&+ \frac{\mu^2}{(m^{\text{OS}})^2} \left[C_A \left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12} l_\mu \right) + n_l T_F \left(\frac{13}{18} - \frac{1}{3} l_\mu \right) \right] \Bigg\} + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^3 C_F \Bigg\{ \frac{\mu}{m^{\text{OS}}} \Bigg[C_A^2 \Bigg(-\frac{130867}{1944} \\
&+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27} \right) l_\mu - \frac{121}{27} l_\mu^2 \Bigg) + C_A n_l T_F \Bigg(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3 \\
&+ \left(-\frac{1654}{81} + \frac{8\pi^2}{27} \right) l_\mu + \frac{88}{27} l_\mu^2 \Bigg) + C_F n_l T_F \left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81} l_\mu - \frac{16}{27} l_\mu^2 \right) \Bigg] \\
&+ \frac{\mu^2}{(m^{\text{OS}})^2} \Bigg[C_A^2 \Bigg(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36} \right) l_\mu - \frac{121}{72} l_\mu^2 \Bigg) + C_A n_l T_F \Bigg(\frac{13699}{1296} - \frac{23\pi^2}{54} \\
&- \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9} \right) l_\mu + \frac{11}{9} l_\mu^2 \Bigg) + C_F n_l T_F \left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27} l_\mu - \frac{2}{9} l_\mu^2 \right) \Bigg] \Bigg\}, (4)
\end{aligned}
$$

Non-leptonic Decays

Effective operators

$$\mathcal{H}_{\mathrm{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{q_{1,3}=u,c} \sum_{q_2=d,s} V_{q_1 b} V^*_{q_2 q_3} \Big(C_1(\mu_b) O_1^{q_1 q_2 q_3} + C_2(\mu_b) O_2^{q_1 q_2 q_3} \Big) + \mathrm{h.c.}$$

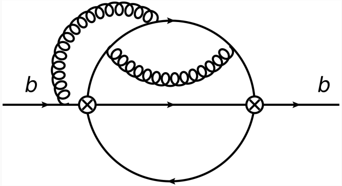
with physical operators

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\beta)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\alpha),$$
$$O_2^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\alpha)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\beta).$$

- Consider all possible combinations of operators:

$$\boxed{\{O_1 \times O_1, O_1 \times O_2, O_2 \times O_1, \dots\}}$$

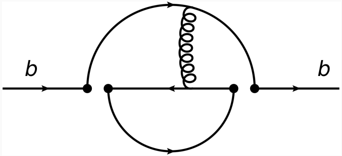
- How to treat γ_5 in dimensional regularization?



Treatment of γ_5

- How do we have to treat traces with one insertion of γ_5 in $d = 4 - 2\epsilon$?

$$\boxed{\mathrm{Tr}[\gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma \gamma_5] \cdot \mathrm{Tr}[\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma \gamma_5]}$$



- Use anticommuting γ_5 to be consistent with calculation of matching coefficients and anomalous dimensions [Buras, Weisz '90; Gorbahn, Haisch '05] .

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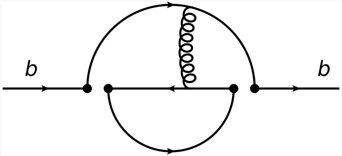


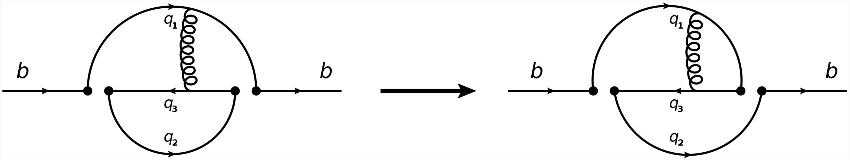
- Use anticommuting γ_5 to be consistent with calculation of matching coefficients and anomalous dimensions [Buras, Weisz '90; Gorbahn, Haisch '05] .
- Use Fierz identities [Bagan, Ball, Braun, Gosdzinsky '94] :

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\beta)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\alpha) \xrightarrow{\text{Fierz}} (\bar{q}_2^\beta \gamma^\mu P_L b^\beta)(\bar{q}_1^\alpha \gamma_\mu P_L q_3^\alpha) = O_2^{q_2 q_1 q_3}$$

Treatment of γ_5

- Fierzed operators correspond to unfierzed operators with switched color indices

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\beta)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\alpha) \xrightarrow{\text{Fierz}} (\bar{q}_2^\beta \gamma^\mu P_L b^\beta)(\bar{q}_1^\alpha \gamma_\mu P_L q_3^\alpha) = O_2^{q_2 q_1 q_3}$$



- Applying Fierz identities to one vertex leads to one instead of two spin lines:

$$\boxed{\mathrm{Tr}\left[\ldots\gamma_5\ldots\right]\mathrm{Tr}\left[\ldots\gamma_5\ldots\right] \to \mathrm{Tr}\left[\ldots\gamma_5\ldots\gamma_5\ldots\right]}$$

- Caveat: Fierz identities are valid in $d = 4$ dimensions!

Evanescent operators

- Fierz symmetry can be restored order by order in perturbation theory by choosing the correct evanescent operators [Buras, Weisz '90; Herrlich, Nierste '94] .

$$E_1^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\alpha) - (16 - 4\epsilon + A_1\epsilon^2) O_1^{q_1q_2q_3},$$
$$E_2^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\beta) - (16 - 4\epsilon + A_2\epsilon^2) O_2^{q_1q_2q_3},$$
$$E_1^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\alpha) - (256 - 224\epsilon + B_1\epsilon^2) O_1^{q_1q_2q_3},$$
$$E_2^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\beta) - (256 - 224\epsilon + B_2\epsilon^2) O_2^{q_1q_2q_3}$$

- Evanescent operators and physical operators mix under renormalization:

$$\begin{pmatrix} O_P \\ O_E \end{pmatrix} = Z \begin{pmatrix} O_{P,B} \\ O_{E,B} \end{pmatrix}$$

$\rightarrow$ Evanescent operators contribute to physical result at higher orders.

Evanescent operators

- Fix $\{A_1, A_2, B_1, B_2\}$ by imposing a symmetric anomalous dimension matrix:

[Buras, Weisz '90]

$$\mu \frac{\mathrm{d}C_i}{\mathrm{d}\mu} = \gamma_{ij} C_j, \qquad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \quad \text{with } \gamma_{11} = \gamma_{22}, \quad \gamma_{12} = \gamma_{21}$$

- This condition yields, e.g.

$$A_1 = A_2 = -4, \qquad B_1 = -\frac{45936}{125}, \qquad B_2 = -\frac{115056}{115}$$

$$\begin{aligned}
E_1^{(1),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\alpha) - (16 - 4\epsilon - 4\epsilon^2) O_1^{q_1q_2q_3}, \\
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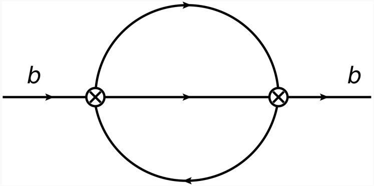
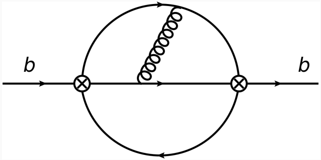
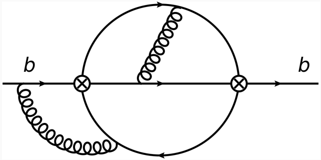
E_1^{(2),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\alpha) - (256 - 224\epsilon - \frac{45936}{125}\epsilon^2) O_1^{q_1q_2q_3}, \\
E_2^{(2),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\beta) - (256 - 224\epsilon - \frac{115056}{115}\epsilon^2) O_2^{q_1q_2q_3}
\end{aligned}$$

Evanescent operators







$$\{O_1, O_2, E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)}\}$$
$$\times \{O_1, O_2, E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)}\}$$

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$$\times \{O_1, O_2, E_1^{(1)}, E_2^{(1)}\}$$

$$\{O_1, O_2\} \times \{O_1, O_2\}$$

Calculation setup

- Generate diagrams with `QGRAF` [Nogueira '93]
- Mapping on integral families with `TAPIR` [Gerlach, Herren, Lang '22] , `EXP` [Harlander, Seidensticker, Steinhauser '98] .
- Reduction to master integrals with `Kira` [Klappert, Lange, Maierhöfer, Usovitsch '21] .
    - Choose good basis of master integrals, where ϵ and $\rho = m_c/m_b$ factorize, with `ImproveMasters.m` [Smirnov, Smirnov '20] .

$\boxed{b \to c\bar{u}d}$	$\boxed{b \to c\bar{c}s}$
1308 diagrams	1308 diagrams
42 families	49 families
385 (18) masters	644 (26) masters

Calculation of the master integrals

1. Set up differential equation for master integrals using IBP relations [Chetyrkin, Tkachov '81] :

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\rho}\vec{I} = A(\epsilon, \rho) \cdot \vec{I}}$$

2. At $\mathcal{O}(\alpha_s)$ we can solve the master integrals analytically.
3. Master integrals for $b \to c\bar{c}s$ have also been used for subleading power corrections. [Mannel, Moreno, Pivovarov '25]

See talk by Daniel Moreno.

Calculation of the master integrals

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$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\rho}\vec{I} = A(\epsilon, \rho) \cdot \vec{I}}$$

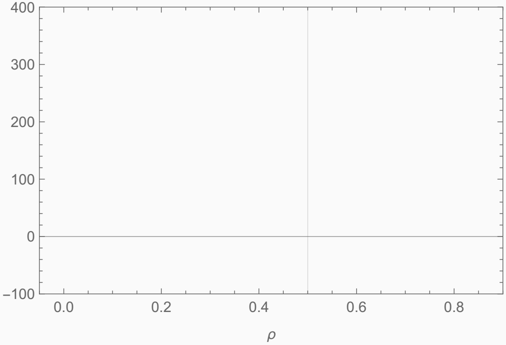
2. Make a general expansion ansatz in $\rho = m_c/m_b$ around certain point ρ_0:

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i, j, m, n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

3. Insert the ansatz into the differential equation and get linear equations between the c.
4. Solve the resulting linear system in terms of a few boundary conditions.
5. Determine the remaining boundaries by matching to numerical results obtained with `AMFlow` [Liu, Ma '22] .
6. Expand around several expansion points and match expansions in between.
   $\rightarrow$ Cover $\rho \in [0, 1]$.

Master integrals: Example

1. Taylor expand around $\rho_0 = 0.5$.
2. Obtain numerical values at $\rho = 0.5$ with `AMFlow` and use as matching condition.



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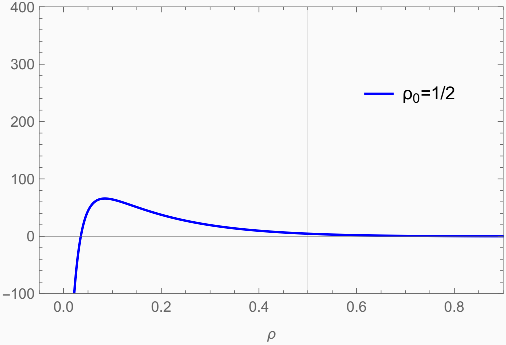

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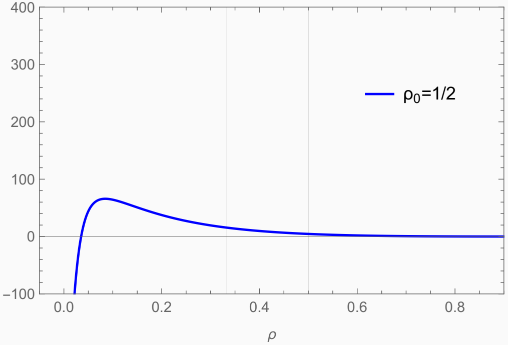
3. Expand around $\rho_0 = 1/3$.
4. Evaluate expansions of step 2 at $\rho = 0.4$:

$\Rightarrow$ Determine expansion coefficients of expansion of step 3 by matching to numerical results of step 4.



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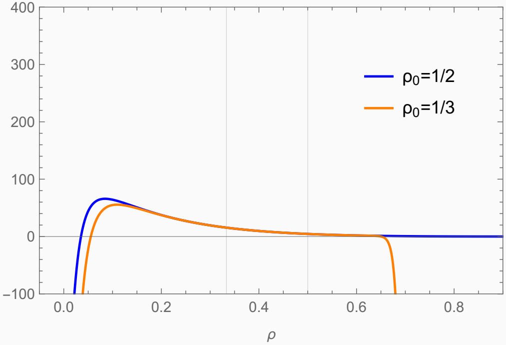

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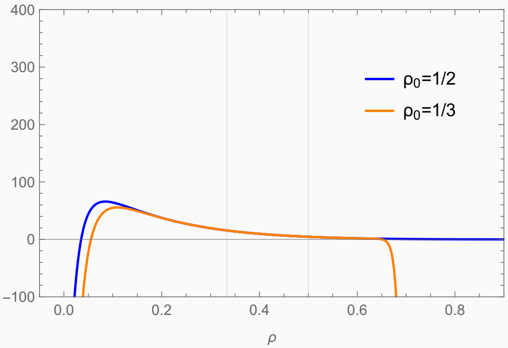

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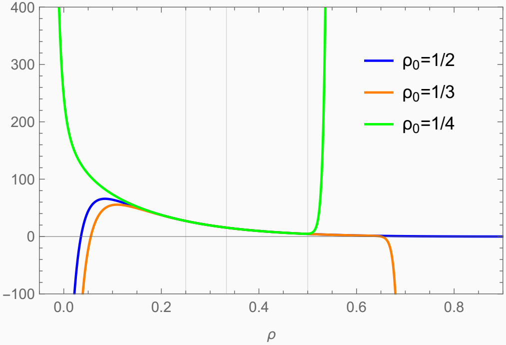

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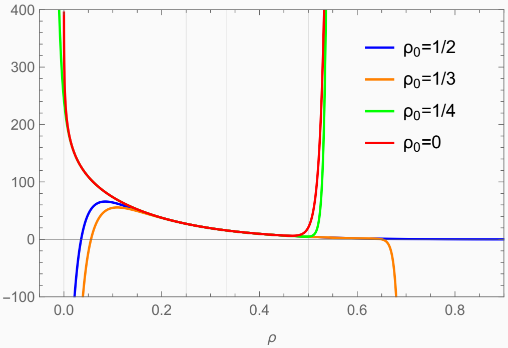


Results

Results

$$\boxed{\Gamma\left(b \to c\bar{q}_1 q_2\right) = \frac{G_f^2 m_b^5 \left|V_{cb}\right|^2}{192\pi^3}\left[C_1^2(\mu)G_{11} + C_1(\mu)C_2(\mu)G_{12} + C_2^2(\mu)G_{22}\right]}$$

with

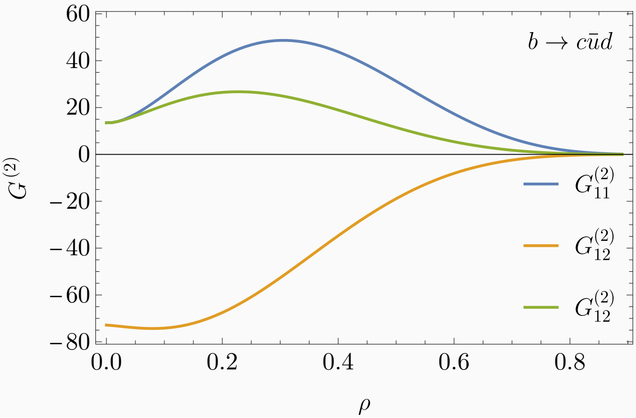
$$\boxed{G_{ij} = G_{ij}^{(0)} + \frac{\alpha_s}{\pi}G_{ij}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 G_{ij}^{(2)} + \dots}$$

- We obtain $G_{ij}^{(1)}$ fully analytically.
- We obtain $G_{ij}^{(2)}$ as semi-analytic expansions around various values of ρ.
- Both $G_{ij}^{(n)}$ and $C_i(\mu)$ depend on the choice of evanescent operators and the γ_5 scheme.
- Fierz identites: $\boxed{(\bar{q}_1^\alpha \gamma^\mu P_L b^\beta)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\alpha) \xrightarrow{\text{Fierz}} (\bar{q}_2^\beta \gamma^\mu P_L b^\beta)(\bar{q}_1^\alpha \gamma_\mu P_L q_3^\alpha)}$

$$\Rightarrow \quad G_{11}(\rho = 0) = G_{22}(\rho = 0)$$

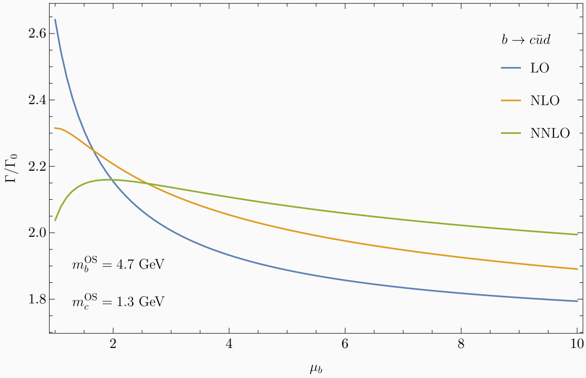
Results: $b \to c\bar{u}d$

$$\Gamma\left(b \to c\bar{u}d\right) = \frac{G_f^2 m_b^5 \left|V_{cb}\right|^2}{192\pi^3}\left[C_1^2(\mu)G_{11} + C_1(\mu)C_2(\mu)G_{12} + C_2^2(\mu)G_{22}\right]$$



Results $b \to c\bar{u}d$

- Dependence on γ_5 scheme and evanescent operators drops out for Γ.
- Renormalization scale dependence is reduced.



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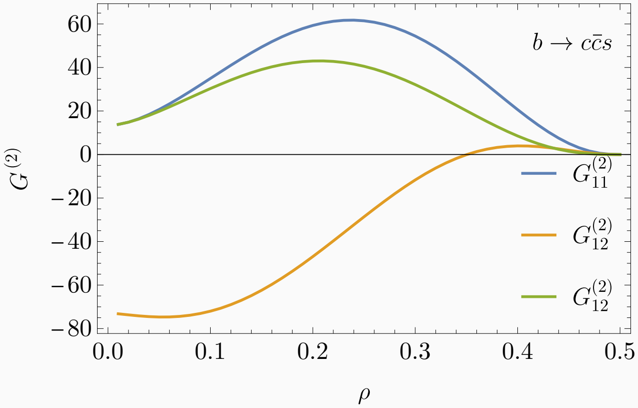
$$= \Gamma_0 \left[1.89907 + 1.77538\left(\frac{\alpha_s}{\pi}\right) + 14.1081\left(\frac{\alpha_s}{\pi}\right)^2\right]\Bigg|_{\mu=m_b}$$

[Egner, Fael, Schönwald, Steinhauser]

Results: $b \to c\bar{c}s$

$$\Gamma\left(b \to c\bar{c}s\right) = \frac{G_f^2 m_b^5 V_{cb} V_{cs}^*}{192\pi^3}\left[C_1^2(\mu)G_{11} + C_1(\mu)C_2(\mu)G_{12} + C_2^2(\mu)G_{22}\right]$$



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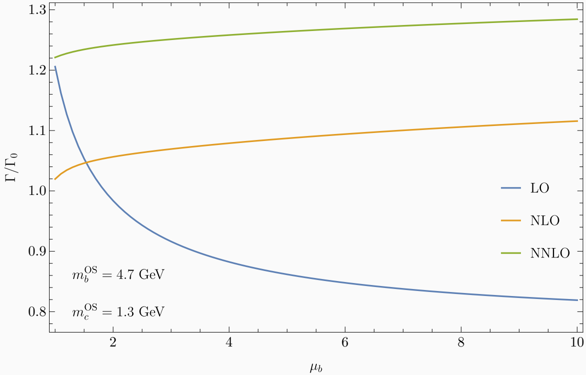


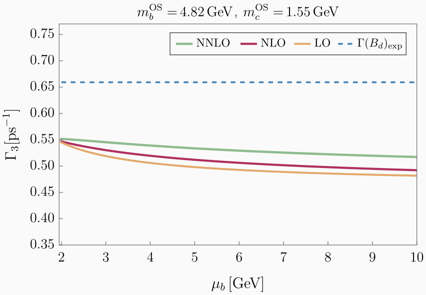
$$\Gamma\left(b \to c\bar{c}s\right) = \frac{G_f^2 m_b^5 V_{cb} V_{cs}^*}{192\pi^3}\left[C_1^2(\mu)G_{11} + C_1(\mu)C_2(\mu)G_{12} + C_2^2(\mu)G_{22}\right]$$

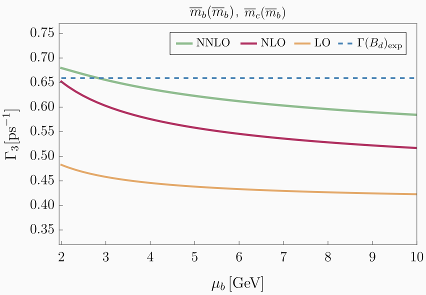
$$= \Gamma_0\left[0.86706 + 3.15768\left(\frac{\alpha_s}{\pi}\right) + 37.3426\left(\frac{\alpha_s}{\pi}\right)^2\right]\Bigg|_{\mu=m_b}$$

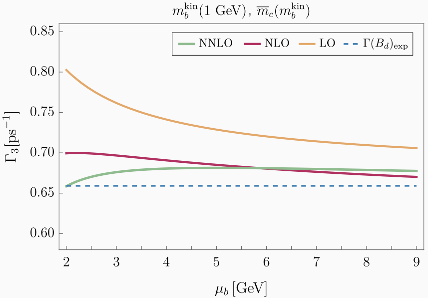
[Egner, Fael, Schönwald, Steinhauser]

Update of B meson lifetimes

[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser '24]







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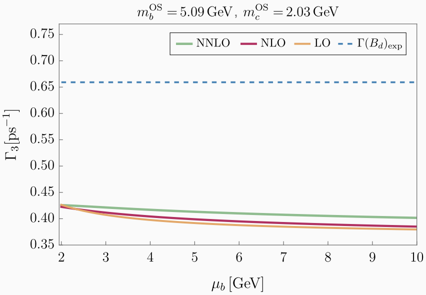

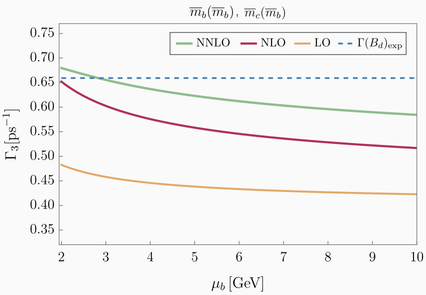

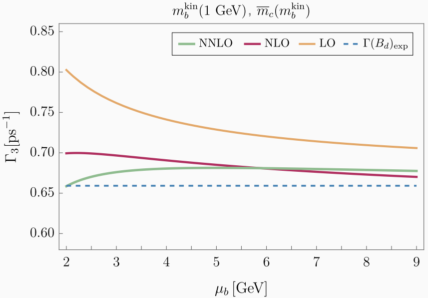

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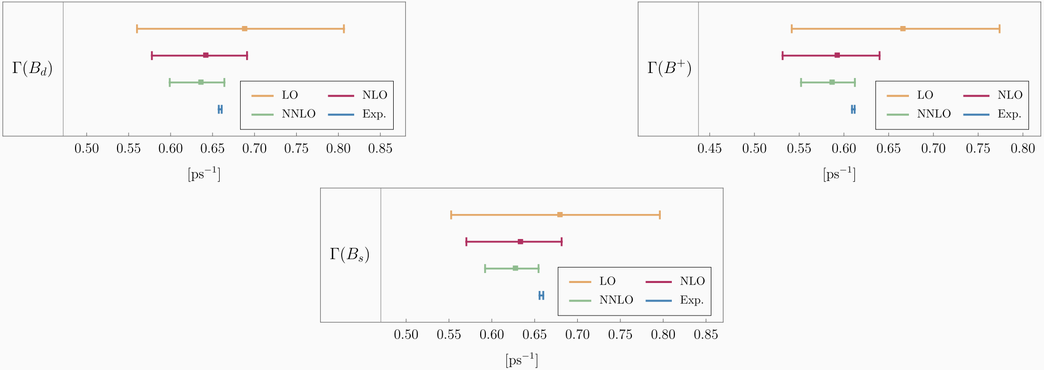
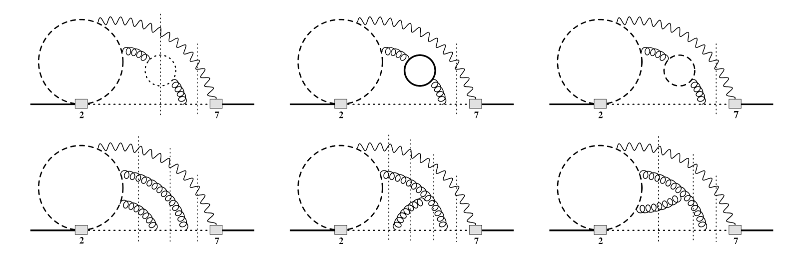


- We see significant improvement in scale and mass scheme dependence.
- Also subleading contributions to $b \to c\bar{u}d$ are important.

$b \to s\gamma$ at NNLO

[Steinhauser CKM conference '25]

- Experimental: $\boxed{\mathcal{B}_{s\gamma}^{\mathrm{exp}} = (3.49 \pm 0.19) \times 10^{-4}}$ for $E_\gamma > 1.6\,\mathrm{GeV}$ [HFLAV '25]
- Theory: $\boxed{\mathcal{B}_{s\gamma}^{\mathrm{theory}} = (3.40 \pm 0.17) \times 10^{-4}}$ [Misiak, Rehman, Steinhauser '20]
- 3% uncertainty from interpolation of charm quark mass effects at NNLO!

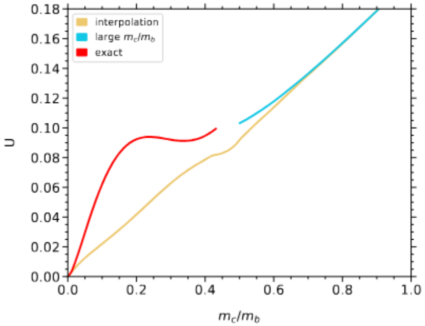


[Steinhauser CKM conference '25]

Impact of exact $Q_{1,2} - Q_7$

$$\frac{\Delta\mathcal{B}_{s\gamma}}{\mathcal{B}_{s\gamma}} \simeq U(z,\delta) \equiv \frac{\alpha_s^2(\mu_b)}{8\pi^2}\,\frac{C_1^{(0)}(\mu_b)F_1(z,\delta) + \left(C_2^{(0)}(\mu_b) - \frac{1}{6}C_1^{(0)}(\mu_b)\right)F_2(z,\delta)}{C_7^{(0)\mathrm{eff}}(\mu_b)}.$$

$G_{17}^{(2)}(z,\delta) = F_1(z,\delta) +$ terms known since $<$2015 $\quad$ $G_{27}^{(2)}(z,\delta) = F_2(z,\delta) +$ terms known since $<$2015



$b \to s\gamma$ – Charm mass dependence at NNLO

[Steinhauser CKM conference '25]

- Experimental: $\boxed{B_{s\gamma}^{\text{exp}} = (3.49 \pm 0.19) \times 10^{-4}}$ for $E_\gamma > 1.6\,\text{GeV}$ [HFLAV '25]
- Theory: $\boxed{B_{s\gamma}^{\text{theory}} = (3.54 \pm 0.14) \times 10^{-4}}$ [Czaja, Czakon, Huber, Misiak, Niggetiedt, Rehman, Schönwald, Steinhause ?soon?]

Other changes:

- Inclusion of 4- and 5-particle contributions at NLO.
- No normalization to $b \to ue\bar{\nu}$ anymore.

Summary and Outlook

Summary

Summary:

- We have calculated NNLO QCD corrections to non-leptonic b decays.
- We calculated the contributions of the operators O_1 and O_2.
- We included also CKM suppressed decay channels.
- We see that the NNLO corrections can be significant and that the renormalization scale dependence is reduced.
- We updated the phenomenological analysis in a well-suited quark mass definition.
  $\Rightarrow$ Needs information beyond the free quark decay!

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- We have calculated NNLO QCD corrections to non-leptonic b decays.
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  $\Rightarrow$ Needs information beyond the free quark decay!

Outlook:

- Include penguin contributions to $b \to c\bar{c}s$.
- Is the application to D meson decays possible?

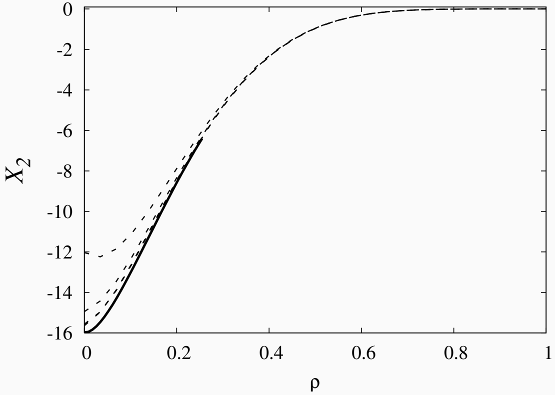
Thank you for your attention!

The Heavy-Daughter Expansion

- Perform the expansion in the limit $m_c \sim m_b$: $\delta = 1 - \rho = 1 - \frac{m_c}{m_b} \ll 1$
- Limit has been shown to converge well down to $m_c/m_b \to 0$ at 2-loop order.
  [Czarnecki, Dowling, Piclum (Phys. Rev. D 78 (2008))]

$$\Gamma(b \to c\ell\nu) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right]$$

with $\Gamma_0 = G_F^2 m_b^2 |V_{cb}|^2 / 192\pi^3$



Asymptotic Expansion

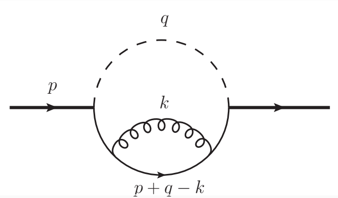
The expansion in $\delta = 1 - m_c/m_b$ is similar to the threshold expansion for the kinetic mass:

- We use the method of regions to perform the expansion.
  [Beneke, Smirnov (Nucl. Phys. B (1998))]
- Loop momenta can either scale hard $k_i \sim m_b$ or ultra-soft $k_i \sim \delta m_b$
  (regions have been cross-checked with `Asy`). [Pak, Smirnov (Eur. Phys. J. C (2011))]
- The momentum of the electron-neutrino loop can be integrated trivially.
- The properties of the asymptotic expansion allow to factorize the leptonic system completely.

$\Rightarrow$ We can reduce our 5-loop to 3-loop integrals with integer powers without any integration-by-parts.

Asymptotic Expansion – Example

Look at the 1-loop integral (we already integrated out the electron-neutrino loop):



$$\sim \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[(p+q)^2 - m_c^2]^2[k^2][(p+q+k)^2 - m_c^2]}$$

- Case 1: q has to be ultra-soft, k is hard ($q \sim \delta m_b$, $k \sim m_b$);

$$\to \int \frac{\mathrm{d}q}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^2} \times \int \frac{\mathrm{d}k}{[k^2][(k+p)^2 - m_b^2]}$$

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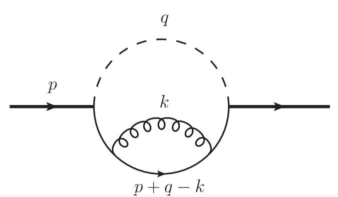


$$\sim \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[(p+q)^2 - m_c^2]^2[k^2][(p+q+k)^2 - m_c^2]}$$

- Case 2: q and k are ultra-soft ($q, k \sim \delta m_b$);

$$\to \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^2[k^2][2p\cdot k + \underbrace{2p\cdot q + 2m_b^2\delta}_{\text{fixed combination}}]}$$

$$= \int \frac{\mathrm{d}q}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^\beta} \times \int \frac{\mathrm{d}k}{[k^2][2p\cdot k + 1]}$$

Details of the Calculation

- We can always perform the integrations over the electron-neutrino loop and lepton momenta analytically via 1-loop tensor reduction. The remaining loop integration have the following scalings:

	scaling	n. regions
$\mathcal{O}(\alpha_s)$	h, u	2
$\mathcal{O}(\alpha_s^2)$	hh, hu, uu	4
$\mathcal{O}(\alpha_s^3)$	hhh, huu, hhu, uuu	8

- In case a single region with either hard or ultra-soft scaling remains we can also integrate it out analytically.
- The remaining two- or three-loop integrals have integer powers of the propagators and can be reduced to master integrals via IBP reduction.

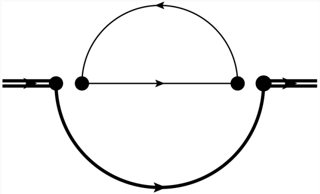
Details of the calculation

- Now also the hard region with up to three loop on-shell master integrals contributes.
  [Melnikov, van Ritbergen (Nucl. Phys. B (2000) ; Lee, Smirnov (JHEP (2011))]
- Several subtleties with `FORM`:
  - Major obstacle is to keep the size of `FORM` expressions as low as possible.
  - Efficient expansion of denominators and memory management.
  - Automated partial fractioning and sector/families mapping.
    [LIMIT, F. Herren, PhD Thesis, KIT, 2020]
- Intermediate `FORM` expressions of $\mathcal{O}(100)$ GB.
- About 25M three-loop integrals with positive and negative indices up to 12 had to be reduce. We used (a private version of) `FIRE` together with `LiteRed` (also standalone).

Master integrals: Expand and Match

Which expansion for which expansion point?

- Depends on singular points of the $A(\epsilon, \rho)$:
  - $\boxed{b \to c\bar{u}d}$ & $\boxed{b \to u\bar{c}s}$: $\{0, 1/3, 1\}$
  - $\boxed{b \to c\bar{c}s}$: $\{0, 1/4, 1/2, 1\}$
  - $\boxed{b \to u\bar{u}d}$: $\{0, 1/2, 1\}$



$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

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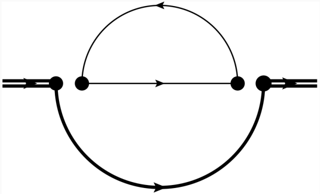

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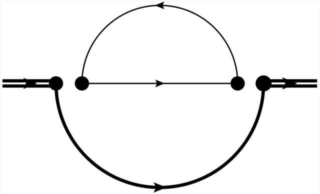
- Expansion around $\{2, 4\}$-particle threshold

$$\to \quad I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i,j,m,n]\,\epsilon^j (\rho-\rho_0)^{n/2} \log^m(\rho-\rho_0)$$

Master integrals: Expand and Match

Which expansion for which expansion point?



- Depends on singular points of the $A(\epsilon, \rho)$:
  - $\boxed{b \to c\bar{u}d}$ & $\boxed{b \to u\bar{c}s}$: $\{0, 1/3, 1\}$
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- Expansion around $\rho_0 \in \{0, 1\}$ or 3-particle threshold

$$\to \quad I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^{n} \log^m(\rho - \rho_0)$$

Master integrals: Expand and Match

Which expansion for which expansion point?

- Depends on singular points of the $A(\epsilon, \rho)$:
  - $\boxed{b \to c\bar{u}d}$ & $\boxed{b \to u\bar{c}s}$: $\{0, 1/3, 1\}$
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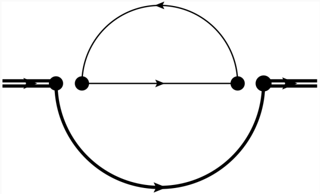


$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

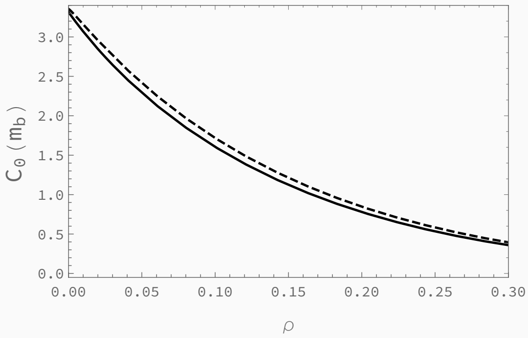
- Expansion around regular point

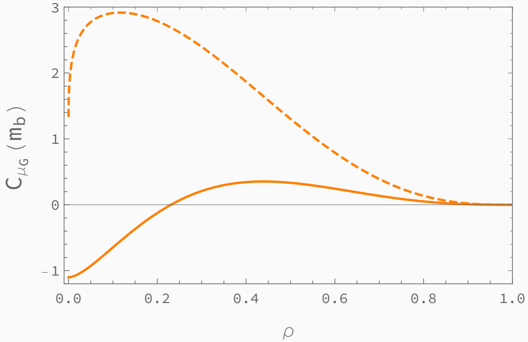
$$\to \quad I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{n=0}^{n_{\max}} c[i,j,0,n]\, \epsilon^j (\rho - \rho_0)^n$$

Subleading power corrections to $b \to c\bar{u}d$

[Mannel, Moreno, Pivovarov '24]

$$\Gamma(B \to X_c) = \frac{G_F^2 m_b^5}{192\pi^3}|V_{ud}|^2|V_{cb}|^2 \left(C_0 - C_{\mu_\pi}\frac{\mu_\pi^2}{2m_b} + C_{\mu_G}\frac{\mu_G^2}{2m_b^2} + \ldots \right)$$





$\mathcal{O}(\alpha_s)$ corrections to C_{μ_G} are important and of the same size as free $\mathcal{O}(\alpha_s^2)$ corrections.

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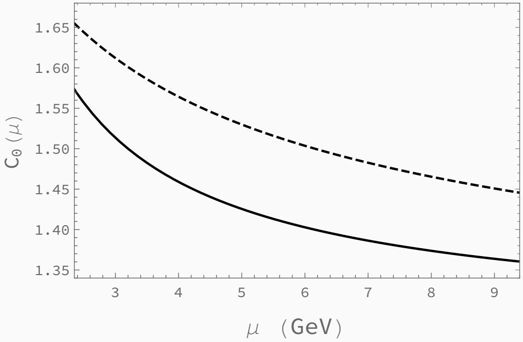

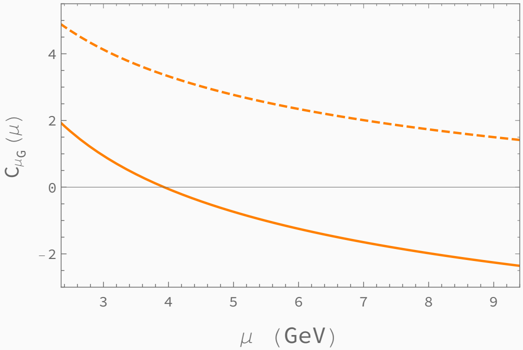

$\boxed{\mathcal{O}(\alpha_s) \text{ corrections fix the sign of } C_{\mu_G}.}$