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Factorization, Resummation and Endpoint Singularities at Subleading Power

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CERN

Factorization at Subleading Power

- A simple scale separation is violated by endpoint divergences at subleading power

- Failure of $\overline{\mathrm{MS}}$ subtraction scheme

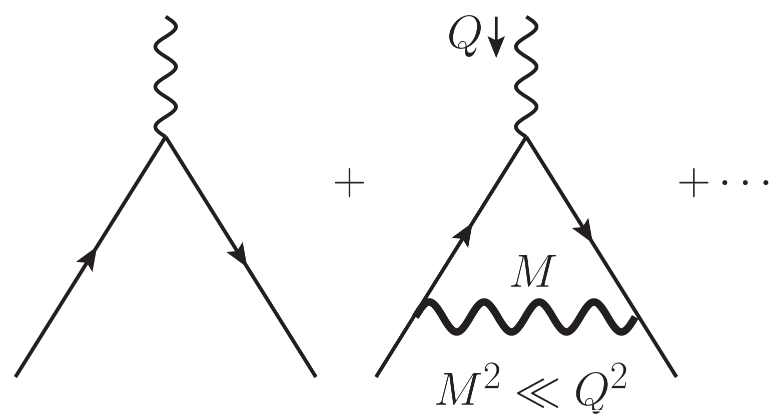
  $h \to \gamma\gamma$ mediated by b-quark loop is a simple example:
  no non-perturbative effect

- Evaluating the scattering amplitudes in the high energy limit is a fundamental problem of QFT

- Resummation of large logarithms is necessary for a reliable theoretical prediction

Leading vs. Subleading

Leading Power



- Factorization is local

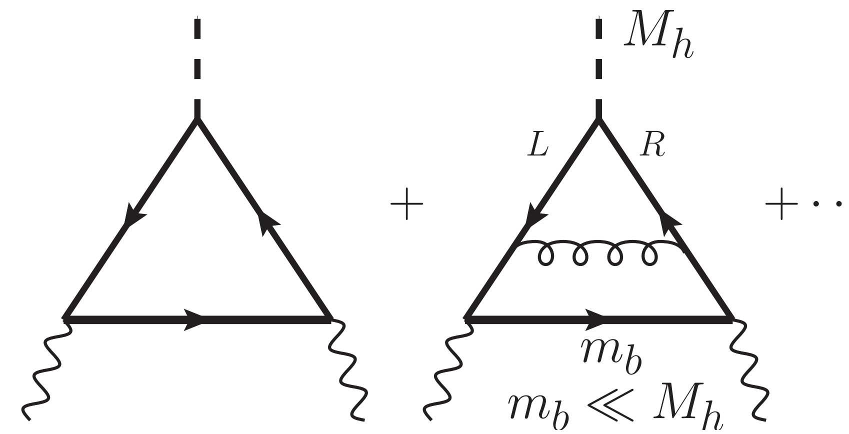
$$H(Q^2,\mu)J_n(M,\mu,\nu/Q)J_{\bar{n}}(M,\mu,\nu/Q)S(M,\mu,\nu/M)$$

$$\sim 1+\frac{\alpha}{4\pi}\left(-L^2+4L+c_0\right)+\mathcal{O}(\alpha^2) \qquad L=\ln\frac{-Q^2}{M^2}$$

- No convolution, no endpoint

- rapidity divergence

Subleading Power



$$\lambda \sim \frac{m_b}{M_h}$$

$$L=\ln\frac{-M_h^2}{m_b^2}$$

- Factorization is non-local

$$H_1 O_1 + H_2 \otimes O_2 + H_3 \otimes J \otimes J \otimes S$$

$$\sim y_b m_b\left\{\left(\frac{L^2}{2}-2\right)+\frac{C_F\alpha_s}{4\pi}\left[-\frac{L^4}{12}-L^3+\cdots\right]\right\}+\mathcal{O}(\lambda^2)$$

- Endpoint-divergent convolutions

- rapidity divergence

Rapidity divergences at LO

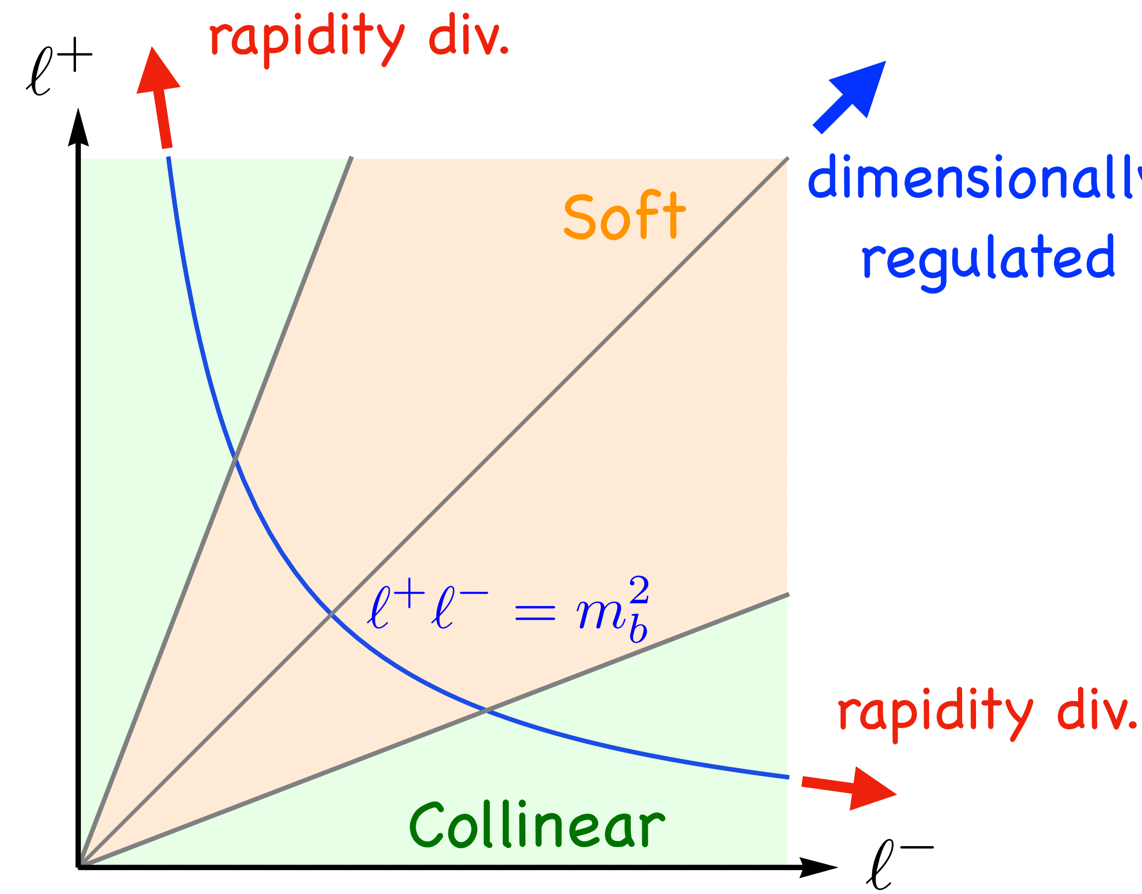
$$\int d^d\ell \frac{1}{(\ell^2 - m^2)(n\cdot\ell + i\varepsilon)(-\bar{n}\cdot\ell + i\varepsilon)}$$

$$\sim \int_{m_b^2}^{\infty} \frac{d(\ell^+\ell^-)}{(\ell^+\ell^-)} (\ell^+\ell^- - m_b^2)^{-\epsilon} \int_0^{\infty} \frac{d\ell^+}{\ell^+}$$

divergence at $\ell^+\ell^- \to +\infty$ is dimensionally regulated

rapidity divergence

Additional regulator is needed !!!



analytical regulator: $\left(\frac{\nu}{\ell^+}\right)^\eta$ Becher, Bell, '11

soft mode is scaleless

rapidity div. cancel between col. & anti-col.

rapidity regulator: $\left|\frac{\ell^+ - \ell^-}{\nu}\right|^\eta$ Chiu, Jain, Neill, Rothstein, '12

exponential regulator: $e^{-\tau(\ell^+ + \ell^-)}$ Li, Neill, Zhu, '16

symmetric with $\ell^+ \leftrightarrow \ell^-$

non-trivial soft mode

Rapidity divergences at LO

We prefer a regulator:

- gives non-trivial soft function
- symmetric with $\ell^+ \leftrightarrow \ell^-$
- easy for higher order calculation

ZLL, Neubert, 1912.08818(JHEP)

$$\Theta(\ell^+ - \ell^-)\left(-\frac{M_h \ell^+}{\nu^2}\right)^\eta + \Theta(\ell^- - \ell^+)\left(-\frac{M_h \ell^-}{\nu^2}\right)^\eta$$

hard $$\frac{1}{\epsilon^2} - \frac{L_h}{\epsilon} + \frac{L_h^2}{2} - 2 - \frac{\pi^2}{12}$$ $$L_h = \ln\frac{-M_h^2}{\mu^2}$$

col.+ anti-col. $$\left(\frac{2}{\eta} + 2\ln\frac{-M_h^2}{\nu^2}\right)\left(\frac{1}{\epsilon} - L_m\right)$$ $$L_m = \ln\frac{m_b^2}{\mu^2}$$

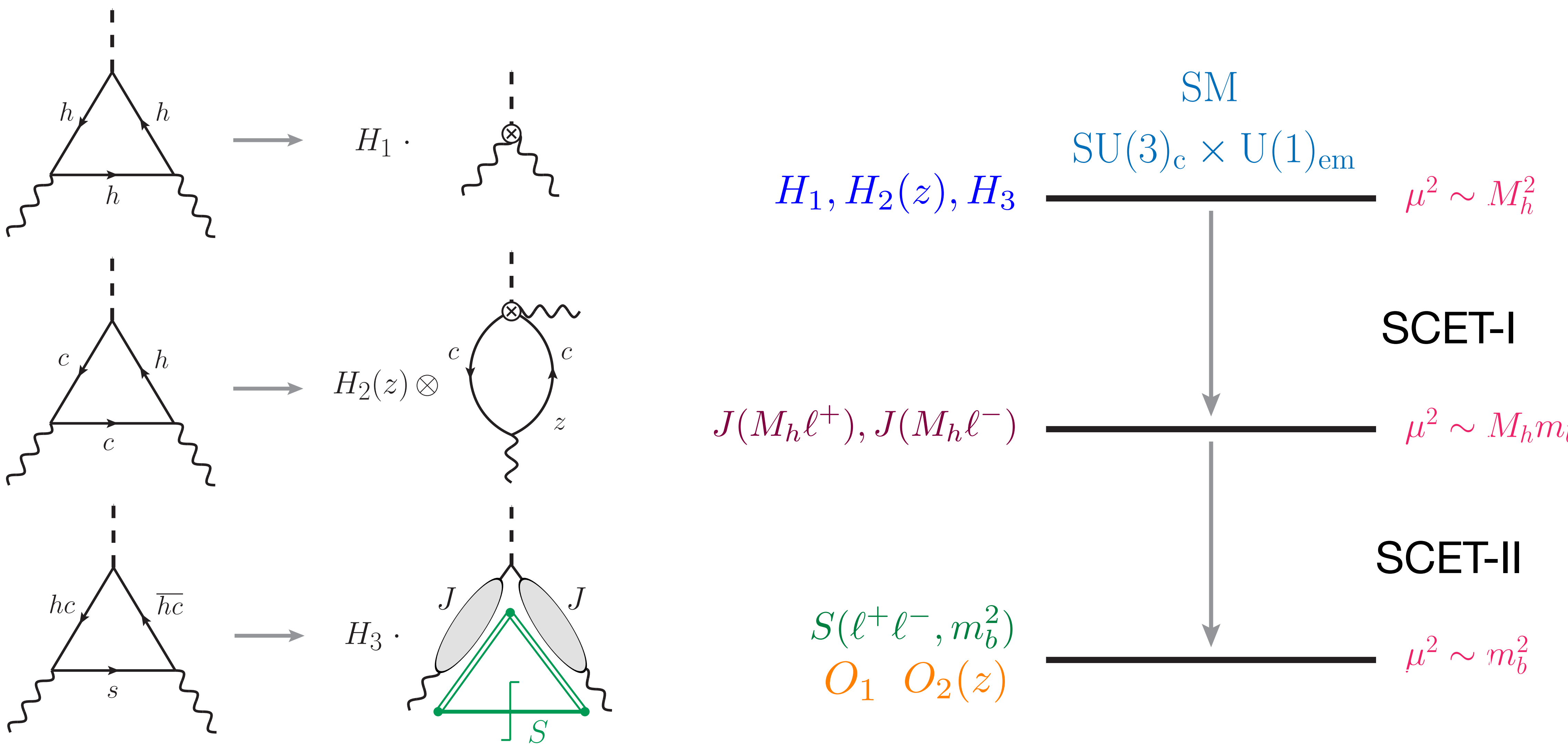
soft $$-\left(\frac{2}{\eta} + \ln\frac{-M_h^2}{\nu^2} + \ln\frac{m_b^2}{\nu^2}\right)\left(\frac{1}{\epsilon} - L_m\right) - \frac{1}{\epsilon^2} + \frac{L_m}{\epsilon} - \frac{L_m^2}{2} + \frac{\pi^2}{12}$$

$$H + C + \bar{C} + S = \frac{L^2}{2} - 2$$ $$L = \ln\frac{-M_h^2}{m_b^2}$$

Factorization Theorem at Subleading Power and Endpoint Divergences

ZLL, M. Neubert, 1912.08818 (JHEP)

Factorization at Bare Level



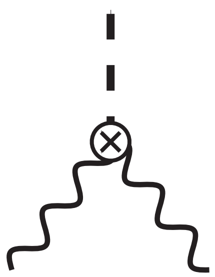
- perfect scale separation with $z \sim \mathcal{O}(1)$ and $\ell^+, \ell^- \sim \mathcal{O}(m_b)$
- factorization violated at $z \ll 1\,,\quad \ell^-/\ell^+ \gg 1 \quad \ell^+, \ell^- \sim M_h$

Factorization at Bare Level

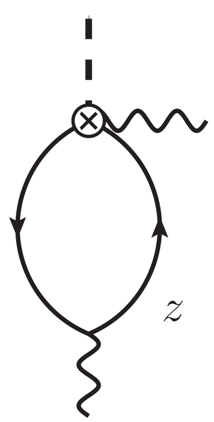
$$\mathcal{M}(h \to \gamma\gamma) = H_1 O_1 + 4\int_0^1 dz \left(-\frac{zM_h^2}{\nu^2}\right)^\eta H_2(z)\, O_2(z)$$
$$+ 2H_3 \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^{\ell_-} \frac{d\ell_+}{\ell_+} \left(-\frac{M_h \ell_-}{\nu^2}\right)^\eta J(M_h\ell_+) J(-M_h\ell_-) S(\ell_+\ell_-, m_b^2)$$
$$+ (\ell_+ \leftrightarrow \ell_-)$$

- each ingredient is well defined with dimensional regulator
- regulator is friendly for higher order calculations
- all the divergences cancel, but highly non-trivial !!!
- divergent convolution gives $\frac{1}{\eta}$ and $\frac{1}{\epsilon}$ poles

Matrix Elements



$$\langle\gamma\gamma|O_1|h\rangle = \frac{m_b}{e_b^2}\langle\gamma\gamma|\, h(0)\,\mathcal{A}^{\perp\mu}_{n_1}(0)\,\mathcal{A}^{\perp}_{n_2,\mu}(0)|h\rangle$$

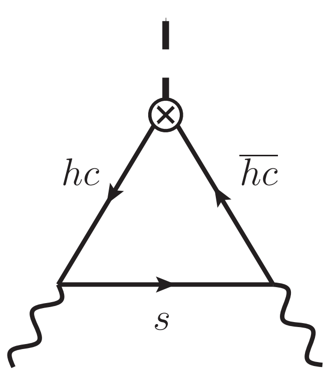


$$\langle\gamma\gamma|O_2(z)|h\rangle = \langle\gamma\gamma|h(0)\left[\bar{\mathcal{X}}_{n_1}(0)\,\gamma_\perp^\mu\,\frac{\bar{\not{n}}_1}{2}\,\delta(zM_h + i\bar{n}_1\cdot\partial)\,\mathcal{X}_{n_1}(0)\right]\mathcal{A}^{\perp}_{n_2,\mu}(0)|h\rangle$$

$$\lim_{z\to 0} O_2(z) = \frac{1}{\epsilon} - L_m + \frac{C_F\alpha_s}{4\pi}\left(\frac{m_b^2}{\mu^2}\right)^{-\epsilon}$$

non-vanish at z=0, factorization violated

$$\times\left\{\frac{1}{\epsilon^2}(\ln z + 3) + \frac{1}{\epsilon}\left(\frac{\ln^2 z}{2} - \frac{1}{2} - \frac{\pi^2}{6}\right) + \frac{\ln^3 z}{6} + \left(\frac{\pi^2}{6} - \frac{1}{2}\right)\ln z + \frac{\pi^2}{6} - 2\zeta_3 + 3\right\}$$



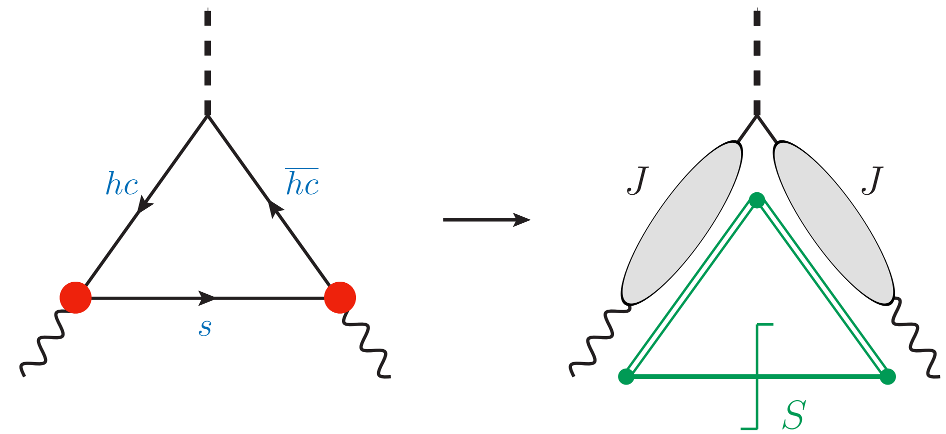
$$\langle\gamma\gamma|O_3|h\rangle = \langle\gamma\gamma|T\left\{h(0)\,\bar{\mathcal{X}}_{n_1}(0)\,\mathcal{X}_{n_2}(0), i\int d^Dx\,\mathcal{L}^{(1/2)}_{q\,\xi_{n_1}}(x), i\int d^Dy\,\mathcal{L}^{(1/2)}_{\xi_{n_2}q}(y)\right\}|h\rangle$$

$$\mathcal{L}^{(1/2)}_{q\,\xi_{n_1}}(x) = \bar{q}_s(x_-)\left[\not{\mathcal{A}}^{\perp}_{n_1}(x) + \not{\mathcal{G}}^{\perp}_{n_1}(x)\right]\mathcal{X}_{n_1}(x)$$

coupling of soft quark to hard-collinear fields

Factorization of O_3 at Operator Level

$$\mathcal{L}^{(1/2)}_{q\,\xi_{n_1}}(x) = \bar{q}_s(x_-)\left[\not{\mathcal{A}}^{\perp}_{n_1}(x) + \not{\mathcal{G}}^{\perp}_{n_1}(x)\right]\mathfrak{X}_{n_1}(x)$$



$$\langle\gamma\gamma|O_3|h\rangle = \langle\gamma\gamma|T\left\{h(0)\,\bar{\mathfrak{X}}_{n_1}(0)\,\mathfrak{X}_{n_2}(0), i\int d^Dx\,\mathcal{L}^{(1/2)}_{q\,\xi_{n_1}}(x), i\int d^Dy\,\mathcal{L}^{(1/2)}_{\xi_{n_2}q}(y)\right\}|h\rangle$$

hard-col. quark field $\quad \mathfrak{X}_{n_i}(x) = \dfrac{\not{n}_i\not{\bar{n}}_i}{4}\,W^{\dagger}_{n_i}(x)\,b(x)$

decouple with soft mode

$$\mathfrak{X}_{n_1}(x) \to S_{n_1}(x_-)\,\mathfrak{X}_{n_1}(x)\,,$$
$$\mathcal{G}^{\perp\mu}_{n_1}(x) \to S_{n_1}(x_-)\,\mathcal{G}^{\perp\mu}_{n_1}(x)\,S^{\dagger}_{n_1}(x_-)$$

collinear & soft Wilson line

$$W_{n_i}(x) = P\exp\left[ig_s\int_{-\infty}^{0} ds\,\bar{n}_i\cdot G_{n_i}(x+s\bar{n}_i)\right]$$
$$S_{n_i}(x) = P\exp\left[ig_s\int_{-\infty}^{0} dt\,n_i\cdot G_s(x+tn_i)\right]$$

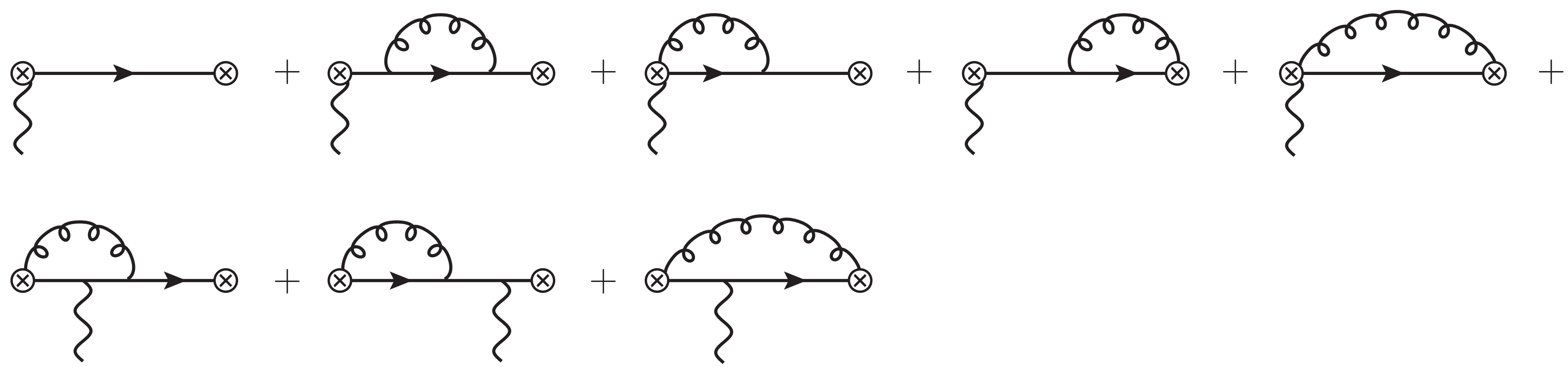
hard-collinear sector $\qquad$ hard-anti-collinear sector

$$O_3 = h(0)\int d^Dx\int d^Dy\,T\left\{\left[\left(\not{\mathcal{A}}^{\perp}_{n_1}(x) + \not{\mathcal{G}}^{\perp}_{n_1}(x)\right)\mathfrak{X}_{n_1}(x)\right]^{\alpha i}\bar{\mathfrak{X}}^{\beta j}_{n_1}(0)\right\}\cdot T\left\{\mathfrak{X}^{\beta k}_{n_2}(0)\left[\bar{\mathfrak{X}}_{n_2}(y)\left(\not{\mathcal{A}}^{\perp}_{n_2}(y) + \not{\mathcal{G}}^{\perp}_{n_2}(y)\right)\right]^{\gamma l}\right\}$$

$$\times\,T\left\{\left[S^{\dagger}_{n_2}(y_+)\,q_s(y_+)\right]^{\gamma l}\left[\bar{q}_s(x_-)\,S_{n_1}(x_-)\right]^{\alpha i}\left[S^{\dagger}_{n_1}(0)\,S_{n_2}(0)\right]^{jk}\right\}$$

soft sector

Radiative Jet Function



$$\langle\gamma(k_1)|\, T\left[\left(\mathcal{A}_{n_1}^{\perp}(x) + \mathcal{G}_{n_1}^{\perp}(x)\right)\mathcal{X}_{n_1}(x)\right]\bar{\mathcal{X}}_{n_1}(r)\,|0\rangle \sim \int \frac{d^D p}{(2\pi)^D}\,\frac{i\bar{n}_1\cdot p}{p^2+i0}\,J\big(p^2,(p-k_1)^2\big)\,e^{-ip\cdot(x-r)+ik_1\cdot x}$$

light-like

Renormalization: $J(p^2,\mu) = \int_0^{\infty} dx\, Z_J(p^2, xp^2;\mu)\, J^{(0)}(xp^2)$ reach endpoint region

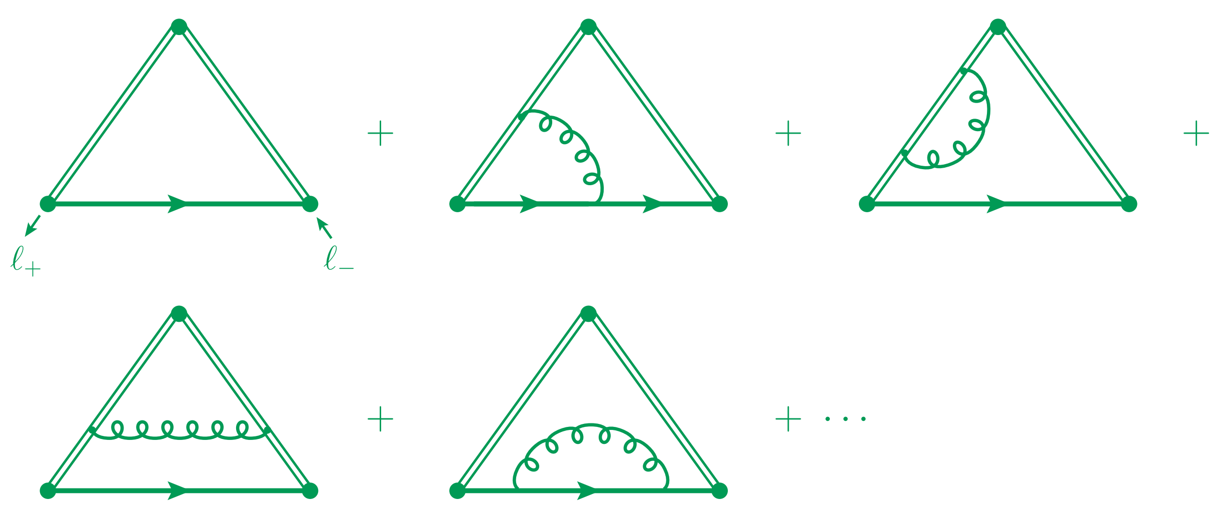
First obtained in $B^- \to \gamma\ell^-\bar{\nu}$ Bosch, Hill, Lange, Neubert, '03

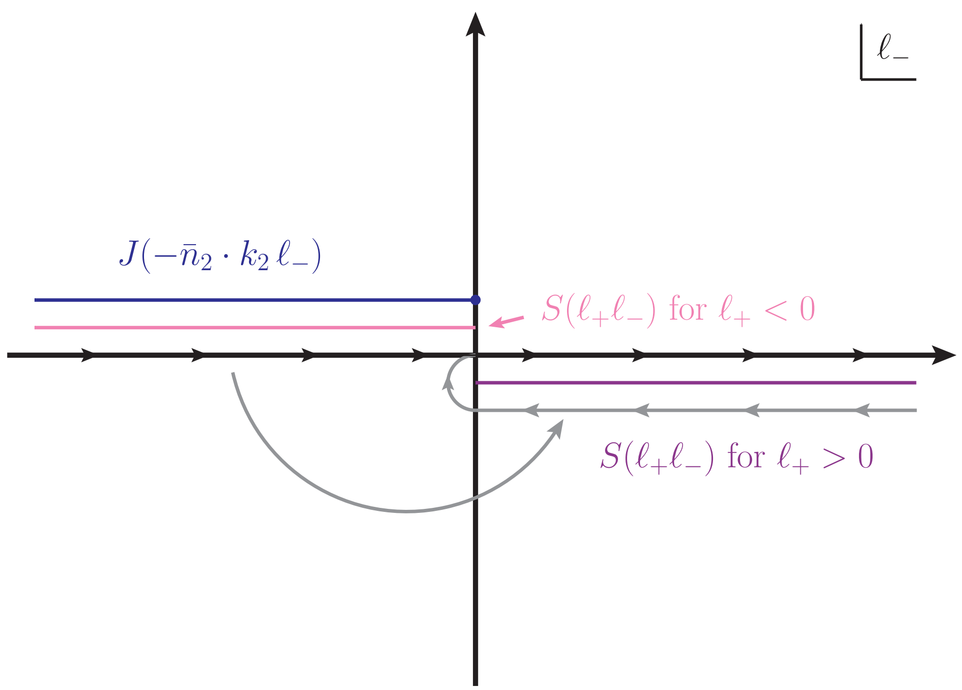
Local / Non-local

$$Z_J(p^2, xp^2;\mu) = \left[1 + \frac{C_F\alpha_s}{4\pi}\left(-\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{-p^2}{\mu^2}\right)\right]\delta(1-x) + \frac{C_F\alpha_s}{2\pi\epsilon}\Gamma(1,x) + \mathcal{O}(\alpha_s^2)$$

with $\Gamma(y,x) = \left[\frac{\theta(y-x)}{y(y-x)} + \frac{\theta(x-y)}{x(x-y)}\right]_+$ Lange-Neubert Kernel

Soft-Quark Soft Function





$$\frac{i}{2}\, \mathcal{S}(\ell_+\ell_-)\, P_n = -\frac{(4\pi)^{1-\epsilon}}{N_c}\, e^{\epsilon\gamma_E} \int dr_1\, e^{ir_1\ell_-} \int dr_2\, e^{-ir_2\ell_+}$$
$$\times\, \mu^{2\epsilon}\, \langle 0|\, T\, \mathrm{Tr}\, \bar{P}_{\bar{n}}\, S_{\bar{n}}(0, r_1\bar{n})\, q_s(r_1\bar{n})\, \bar{q}_s(r_2 n)\, S_n(r_2 n, 0)\, P_n\, |0\rangle$$

$$S(\ell_+\ell_-) = \frac{1}{2\pi i}\left[\mathcal{S}(\ell_+\ell_- + i0) - \mathcal{S}(\ell_+\ell_- - i0)\right] \qquad \mathcal{S}(\ell_+\ell_-) \equiv \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}}\, \mathcal{S}_1(\ell)$$

$$S(\ell_+\ell_-) \sim m_b\left[S_a(\ell_+\ell_-)\,\theta(\ell_+\ell_- - m_b^2) + S_b(\ell_+\ell_-)\,\theta(m_b^2 - \ell_+\ell_-)\right]$$

Reproduce NLO amplitude

$$\mathcal{M}(h\to\gamma\gamma) = H_1\, O_1 \quad \cdots\cdots\cdots\cdots \quad T_1$$

$$+\,4\int_0^1 dz \left(-\frac{zM_h^2}{\nu^2}\right)^{\eta} H_2(z) O_2(z) \quad \cdots\cdots\cdots\cdots \quad T_2$$

$$+\,2H_3\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^{\ell_-}\frac{d\ell_+}{\ell_+}\left(-\frac{M_h\ell_-}{\nu^2}\right)^{\eta} J(M_h\ell_+)J(-M_h\ell_-)S(\ell_-\ell_+,m_b^2) \quad \cdots\cdots \quad T_3$$

$$T_1 = \frac{1}{\epsilon^2} - \frac{L_h}{\epsilon} + \frac{L_h^2}{2} - 2 - \frac{\pi^2}{12} + \frac{C_F\alpha_s}{4\pi}\left(-\frac{1}{2\epsilon^4} + \cdots\right),$$

$$T_2 = \left[\frac{2}{\eta} + 2\ln\frac{-M_h^2 - i0}{\nu^2}\right]\left[\frac{1}{\epsilon} - L_m + \frac{C_F\alpha_s}{4\pi}\left(-\frac{3}{\epsilon^3} + \cdots\right)\right] + \frac{C_F\alpha_s}{4\pi}\left(-\frac{2}{\epsilon^4} + \cdots\right),$$

$$T_3 = -\left[\frac{2}{\eta} + \ln\frac{-M_h^2 - i0}{\nu^2} + \ln\frac{m_b^2}{\nu^2}\right]\left[\frac{1}{\epsilon} - L_m + \frac{C_F\alpha_s}{4\pi}\left(-\frac{3}{\epsilon^3} + \cdots\right)\right]$$
$$-\frac{1}{\epsilon^2} + \frac{L_m}{\epsilon} - \frac{L_m^2}{2} + \frac{\pi^2}{12} + \frac{C_F\alpha_s}{4\pi}\left(\frac{5}{2\epsilon^4} + \cdots\right)$$

NLO amplitude is perfectly reproduced up to constant terms

because of endpoint divergences, the poles in T_2 and T_3 are higher than the poles in $H_2(z)\,, O_2(z)\,, J(p^2)\,, S(\ell^+\ell^-)$

Endpoint Divergences

$\mathcal{O}(\alpha_s)$	$\lim_{z\to 0} H_2(z)$	$\lim_{z\to 0} O_2(z)$
LO	$\frac{1}{z}$	$\frac{1}{\epsilon} - L_m$
NLO	$\frac{c_0}{z} + \frac{c_1}{z^{1+\epsilon}}$	$a_0 + a_1 z^{\epsilon}$
NNLO	$\frac{c_0'}{z} + \frac{c_1'}{z^{1+\epsilon}} + \frac{c_2'}{z^{1+2\epsilon}}$	$a_0' + a_1' z^{\epsilon} + a_2' z^{2\epsilon}$

Can we use plus expansion to resolve endpoint divergences?

$$z^{-1+\eta} = \frac{1}{\eta}\delta(z) + \left(\frac{1}{z}\right)_+ + \eta\left(\frac{\ln z}{z}\right)_+ + \cdots$$

undefined convolution arises

$$\int_0^1 dz \left(\frac{1}{z}\right)_+ \ln z$$

difficult to renormalize endpoint divergences before performing convolution !!!

Refactorization

PhD thesis of Philipp

At the endpoint, the refactorization conditions read

$$[\![H_2(z)]\!] \equiv \lim_{z\to 0} H_2(z) = H_3 \frac{J(zM_h^2)}{z}$$

$$[\![O_2(z)]\!] \equiv \lim_{z\to 0} \langle\gamma\gamma|\, O_2(z)\,|h\rangle = \int_0^\infty \frac{d\ell_+}{\ell_+} J(M_h\,\ell_+)\, S(\ell_+\ell_-)\,,$$

Then $[\![H_2(z)]\!] \otimes [\![O_2(z)]\!]$ has the same integrand with $H_3\, O_3$,
but has different integral region

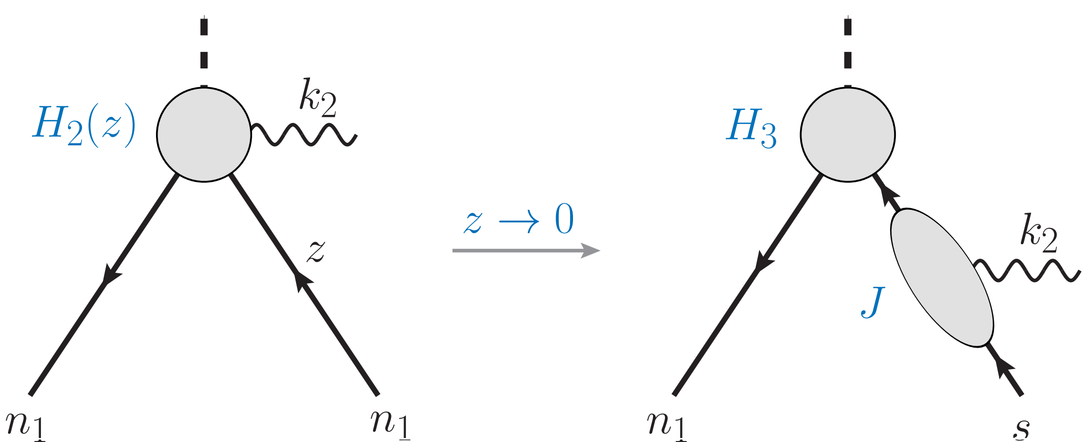
$$\int_0^1 dz [\![H_2(z)]\!][\![O_2(z)]\!] = H_3 \int_0^{M_h} \frac{d\ell^-}{\ell^-} J(M_h\ell^-) \int_0^\infty \frac{d\ell^+}{\ell^+} J(M_h\ell^+) S(\ell^+\ell^-)$$

useful to do subtraction

$$\int_0^1 dz H_2(z) O_2(z) \to \int_0^1 dz\,(H_2(z)O_2(z) - [\![H_2(z)]\!][\![O_2(z)]\!]) + \int_0^1 dz [\![H_2(z)]\!][\![O_2(z)]\!]$$

Refactorization at Operator Level

ZLL, Mecaj, Neubert, Wang, '20



$$H_3^{(0)}\,\langle b(k_1)\,\bar b(zk_1)\,\gamma(k_2)|\,T\left\{h\,\bar{\mathcal X}_{n_1}\mathcal X_{n_2}(0),\, i\int d^Dy\,\mathcal L^{(1/2)}_{\xi_{n_2}q}(y)\right\}|h\rangle$$

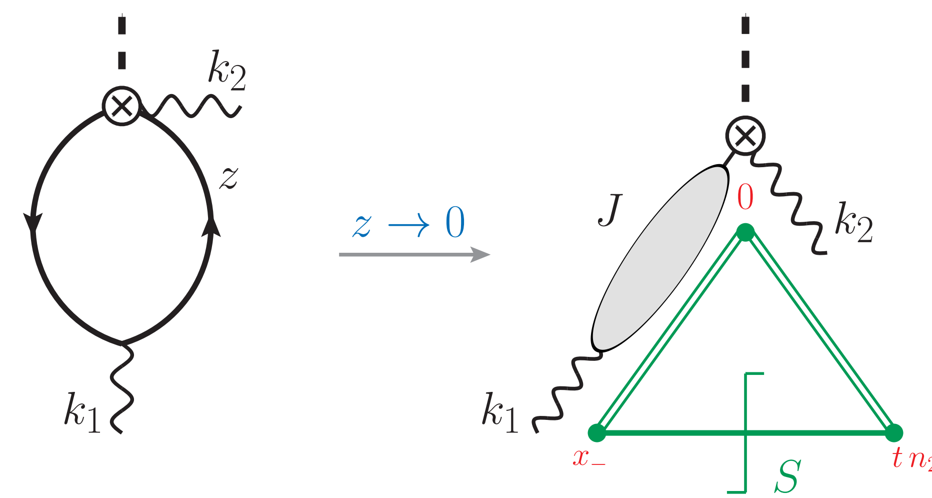
↓

$$H_3^{(0)}\,\langle b(k_1)\,\bar b(zk_1)\,\gamma(k_2)|\,T\Big\{\bar{\mathcal X}_{n_1}(0)\,S^\dagger_{n_1}(0)\,S_{n_2}(0)\,\mathcal X_{n_2}(0)$$

$$\times\, i\int d^Dy\,\bar{\mathcal X}_{n_2}(y)\,\big(\mathcal A\!\!\!/^{\perp}_{n_2}(y)+\mathcal G\!\!\!/^{\perp}_{n_2}(y)\big)\,S^\dagger_{n_2}(y_+)\,q_s(y_+)\Big\}\,|0\rangle$$

↓

$$[\![H_2^{(0)}(z)]\!] = \frac{[\![\bar H_2^{(0)}(z)]\!]}{z} = -\frac{H_3^{(0)}}{z}\,J(zM_h^2)$$



$$O_2^{(0)}(t) = h(0)\,\bar{\mathcal X}_{n_1}(0)\,\gamma_\perp^\mu\,\frac{\bar{n}\!\!\!/_1}{2}\,\mathcal X_{n_1}(t\bar n_1)\,\mathcal A^\perp_{n_2,\mu}(0)$$

↓

$$h(0)\,\bar{\mathcal X}_{n_1}(0)\,S^\dagger_{n_1}(0)\,S_{n_2}(0)\,\gamma^\mu_\perp\,\frac{n\!\!\!/_2}{2}\,S^\dagger_{n_2}(tn_2)\,q_s(tn_2)\,\mathcal A^\perp_{n_2,\mu}(0)$$

↓

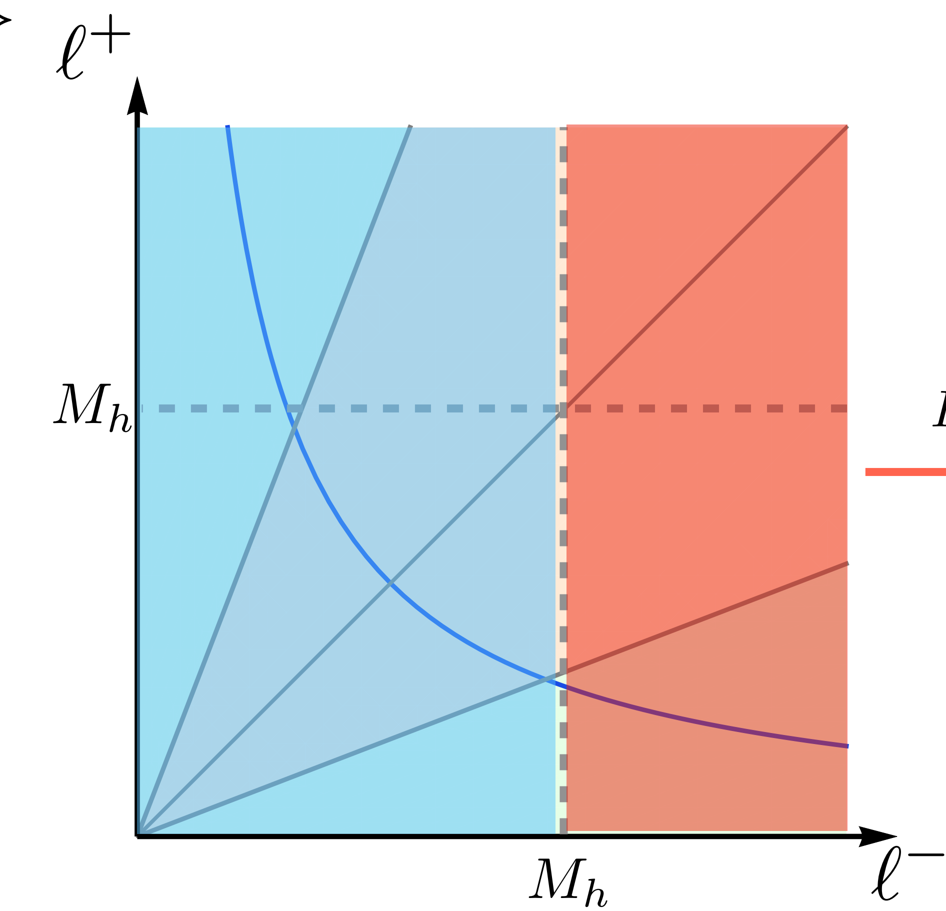
$$[\![\langle\gamma\gamma|\,O_2^{(0)}(z)\,|h\rangle]\!] = -\frac{g_\perp^{\mu\nu}}{2}\int_0^\infty\frac{d\ell_+}{\ell_+}\,J^{(0)}(-M_h\ell_+)\,S^{(0)}(zM_h\ell_+)$$

Result in two same integrands between H_2O_2 and H_3O_3

Cancellation of Rapidity Divergence

$$\int_0^a dz\frac{1}{z} + \int_b^{+\infty} dz\frac{1}{z} = \int_0^a dz\frac{1}{z^{1-\eta}} + \int_b^{+\infty} dz\frac{1}{z^{1-\eta}} = \frac{1}{\eta}(a^\eta - b^\eta) = \ln\frac{a}{b}$$

$$\int_0^a \frac{dz}{z} f(z) + \int_b^{+\infty} \frac{dz}{z} f(z) \to \int_b^a \frac{dz}{z} f(z)$$



$$\int_0^1 dz [\![H_2(z)]\!][\![O_2(z)]\!]$$

$$H_3\left(\int_{M_h}^{\infty} \frac{d\ell^-}{\ell^-}\int_0^{\infty} \frac{d\ell^+}{\ell^+} + \int_0^{M_h}\frac{d\ell^-}{\ell^-}\int_0^{\infty}\frac{d\ell^+}{\ell^+}\right) J(M_h\ell^-)J(M_h\ell^+)S(\ell^+\ell^-) = 0$$

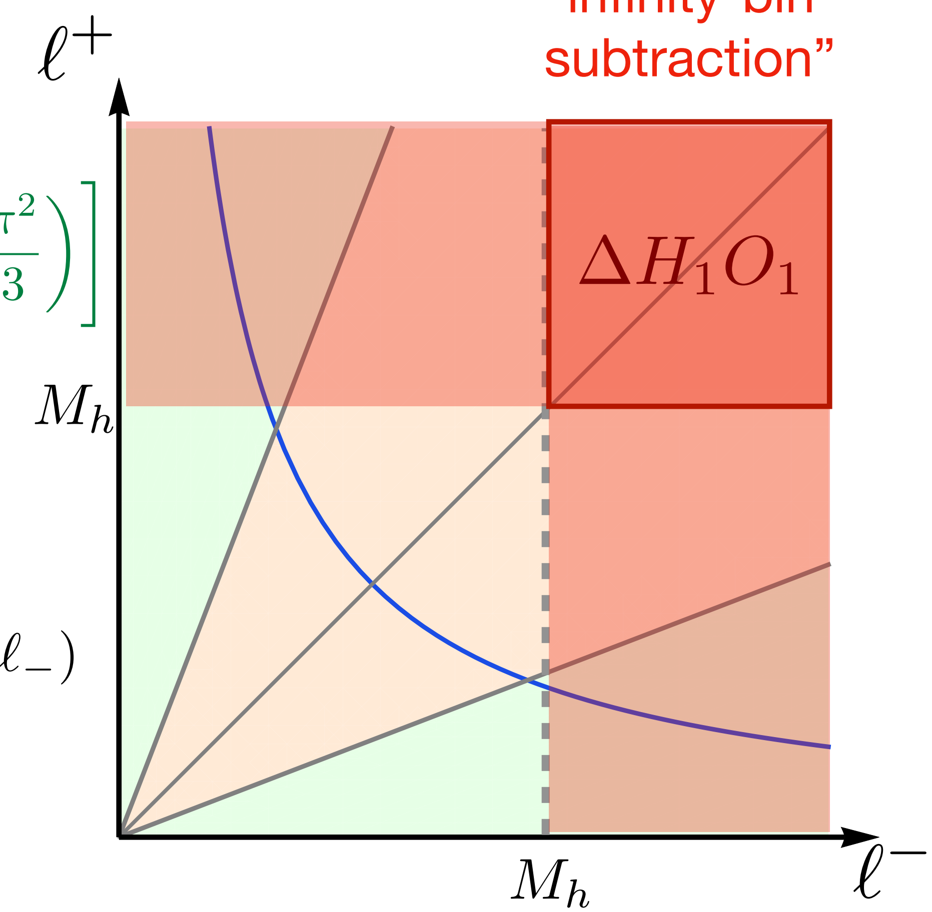
rapidity divergences cancel explicitly !!!

Rearrangement

$$S_\infty(\ell^+\ell^-) = \lim_{\ell^\pm\to\infty} S(\ell^+\ell^-, m_b^2)$$

$$S_\infty(\ell_+\ell_-) \sim m_b \left(\frac{\ell_+\ell_-}{\mu^2}\right)^{-\epsilon}\left[\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} + \frac{C_F\alpha_s}{4\pi}\left(\frac{\ell_+\ell_-}{\mu^2}\right)^{-\epsilon}\left(-\frac{2}{\epsilon^2}+\frac{6}{\epsilon}+12-\frac{\pi^2}{3}\right)\right]$$

$S_\infty(\ell^+\ell^-)$ is not at soft scale, but at hard scale



$$\Delta H_1 O_1 = -H_3 \lim_{\sigma\to -1}\int_{M_h}^\infty \frac{d\ell_-}{\ell_-}\int_{\sigma M_h}^\infty \frac{d\ell_+}{\ell_+} J(M_h\ell_-)J(-M_h\ell_+)S_\infty(\ell_+\ell_-)$$
$$+(\ell_+\leftrightarrow\ell_-)$$

$$\mathcal{M}(h\to\gamma\gamma) = (H_1+\Delta H_1)\,O_1$$
$$+\,4\lim_{\delta\to 0}\int_\delta^1 dz\left[H_2(z)\,O_2(z) - [\![H_2(z)]\!]\,[\![O_2(z)]\!]\right]$$
$$+\,H_3\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-)\,J(-M_h\ell_+)\,S(\ell_+\ell_-)+(\ell_+\leftrightarrow\ell_-)$$

get rid of rapidity regulator !!!   no endpoint divergences any more !!!

Rearrangement

$$\mathcal{M}(h\to\gamma\gamma) = (H_1+\Delta H_1)\,O_1 \quad \cdots\cdots\cdots\cdots \quad T_1$$

$$+\,4\lim_{\delta\to 0}\int_\delta^1 dz\left[H_2(z)\,O_2(z) - [\![H_2(z)]\!]\,[\![O_2(z)]\!]\right] \quad \cdots\cdots\cdots \quad T_2$$

$$+\,H_3\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-)\,J(-M_h\ell_+)\,S(\ell_+\ell_-) \cdots\cdots \quad T_3$$

$$T_1 = -2+\frac{C_F\alpha_s}{4\pi}\left[-\frac{\pi^2}{3\epsilon^2}+\frac{1}{\epsilon}\left(\frac{2\pi^2}{3}L_h-10\zeta_3\right)-\frac{2\pi^2}{3}L_h^2+(12+20\zeta_3)\,L_h-36-\frac{7\pi^4}{30}\right],$$

$$T_2 = \frac{C_F\alpha_s}{4\pi}\left[\frac{\pi^2}{3\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{2\pi^2}{3}L_h+2\zeta_3\right)+\frac{\pi^2}{3}\left(L_h^2+2L_hL_m-L_m^2\right)-4\zeta_3\,L_h+8\zeta_3+\frac{13\pi^4}{90}\right],$$

$$T_3 = \frac{L^2}{2}+\frac{C_F\alpha_s}{4\pi}\left[\frac{8\zeta_3}{\epsilon}-\frac{L^4}{12}-L^3+\left(4-\frac{\pi^2}{3}\right)L^2-\left(8-\frac{2\pi^2}{3}\right)L\right.$$
$$\left.+\,L_m\left(-3L^2+6L-16\zeta_3\right)-4\zeta_3-\frac{\pi^4}{9}\right]$$

- NLO amplitude is exactly reproduced !!!
- T1 only depends on hard log
- no $1/\epsilon^4$ poles any more
- LL and NLL only appear in T3
- T3 is "almost" RG-invariant

Resummation at LL

Leading Log only appears in the "almost" RG-invariant T_3

Sudakov exponent of the integrand:

$$H_3(\mu)\, J(M_h\ell_-,\mu)\, J(-M_h\ell_+,\mu)\, S(\ell_+\ell_-,\mu) \sim m_b\, \theta(\ell_+\ell_- - m_b^2)\, \exp\Big[-\frac{C_F\alpha_s}{2\pi}\ln\frac{M_h}{\ell_-}\ln\frac{-M_h - i0}{\ell_+} + \dots\Big]$$

Performing convolution with cutoff:

$$\mathcal{M}_b(h\to\gamma\gamma)\big|_{\rm LL} = \lim_{\sigma\to -1}\mathcal{M}_0\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\,\theta(\ell_+\ell_- - m_b^2)\,\exp\Big[-\frac{C_F\alpha_s}{2\pi}\ln\frac{M_h}{\ell_-}\ln\frac{\sigma M_h}{\ell_+}\Big]$$

$$= \mathcal{M}_0\,\frac{L^2}{2}\,{}_2F_2\Big(1,1;\frac{3}{2},2;-\frac{C_F\alpha_s}{8\pi}L^2\Big)$$

T. Liu, Penin, '17 J. Wang, '19

Mismatch in Cutoff Scheme

T3 is "RG-invariant":

$$T_3 = H_3 \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} J(M_h \ell_-) J(-M_h \ell_+) S(\ell_+ \ell_-)$$ bare

$$= H_3(\mu) \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} J(M_h \ell_-, \mu) J(-M_h \ell_+, \mu) S(\ell_+ \ell_-, \mu)$$ renormalized

This can be employed to determine Z-factor of soft function, Talk by Xiangpeng
but the convolution is not well defined!

When cutoff is involved:

$$T_3 = H_3 \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{M_h} \frac{d\ell_+}{\ell_+} J(M_h \ell_-) J(-M_h \ell_+) S(\ell_+ \ell_-)$$ bare

$$\neq H_3(\mu) \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{M_h} \frac{d\ell_+}{\ell_+} J(M_h \ell_-, \mu) J(-M_h \ell_+, \mu) S(\ell_+ \ell_-, \mu)$$ renormalized

$$\delta T_3 \sim \int_0^{\sigma M_h} \frac{d\ell'_+}{\ell'_+} \left(\int_0^\infty d\ell_- \int_0^{M_h} d\ell'_- - \int_0^{M_h} d\ell_- \int_0^\infty d\ell'_- \right) \theta(\ell'_+ \ell'_- - m_b^2)$$
$$\times (\ell'_+ \ell'_- - m_b^2)^{-\epsilon} \left[\frac{\theta(\ell'_- - \ell_-)}{\ell'_-(\ell'_- - \ell_-)} + \frac{\theta(\ell_- - \ell'_-)}{\ell_-(\ell_- - \ell'_-)} \right]_+ \neq 0$$

A Short Summary

- Factorization theorem is built at bare level
- NLO amplitude is reproduced exactly
- Non-trivial soft-quark soft function is derived
- Refactorization helps us to regulate the rapidity divergences
- Endpoint divergences are removed by rearrangement

How to deal with the mismatch between renormalization and convolutions with cutoffs?

Renormalized Factorization Theorem at Subleading Power in SCET

ZLL, Neubert, 2003.03393 (JHEP)

ZLL, Mecaj, Neubert, Wang, Fleming, 2005.03013 (JHEP)

ZLL, B. Mecaj, M. Neubert, X. Wang

2009.04456 (PRD) & 2009.06779 (JHEP)

Renormalized Factorization Theorem

T1: Absorb the contributions from infinity-bin and mismatch

T2: Divergences at the endpoints are subtracted

$$\mathcal{M}_b = H_1(\mu)\,\langle O_1(\mu)\rangle$$
$$+\,2\int_0^1 dz\left[H_2(z,\mu)\,\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}\,[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\,[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right]$$
$$+\,g_\perp^{\mu\nu}\,H_3(\mu)\,\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\,J(M_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu)\Big|_{\text{leading power}}$$

T3: Cutoff is involve to regulated endpoint divergences

- Renormalization does NOT commute with the cutoff → Mismatch
- Cutoff arises from rearrangement — Nature !
- Both of the contributions from infinity-bin and mismatch only depend on hard scale — Proved to all order in α_s
- Renomalization of O_1 and O_2 is mixing

Renormalization

$$H_1(\mu) \sim H_1^{(0)} Z_{11}^{-1} + H_2^{(0)}(z) \otimes Z_{21}^{-1}(z)$$

$$O_1(\mu) = Z_{11} O_1^{(0)}\,, \qquad Z_{11} = Z_m^{-1} = \frac{m_b(\mu)}{m_{b,0}}$$

$$H_2(z,\mu) = \int_0^1 dz'\, H_2^{(0)}(z')\, Z_{22}^{-1}(z',z)$$

$$O_2(z,\mu) = \int_0^1 dz'\, Z_{22}(z,z')\, O_2^{(0)}(z') + Z_{21}(z)\, O_1^{(0)}$$

$$\frac{[\![\bar{H}_2(z,\mu)]\!]}{z} = \int_0^{\infty} dz'\, \frac{[\![\bar{H}_2^{(0)}(z')]\!]}{z'}\, [\![Z_{22}^{-1}(z',z)]\!]$$

$$[\![O_2(z,\mu)]\!] = \int_0^{\infty} dz'\, [\![Z_{22}(z,z')]\!]\, [\![O_2^{(0)}(z')]\!] + [\![Z_{21}(z)]\!]\, O_1^{(0)}$$

$$H_3(\mu) = H_3^{(0)} Z_{33}^{-1}$$

$$J(\pm M_h\ell,\mu) = \int_0^{\infty} d\ell'\, Z_J(\pm M_h\ell, \pm M_h\ell')\, J^{(0)}(\pm M_h\ell')$$

$$S(w,\mu) = \int_0^{\infty} dw'\, Z_S(w,w';\mu)\, S^{(0)}(w')$$

Bodwin, Ee, Lee, Wang, 2101.04872, 2107.07941

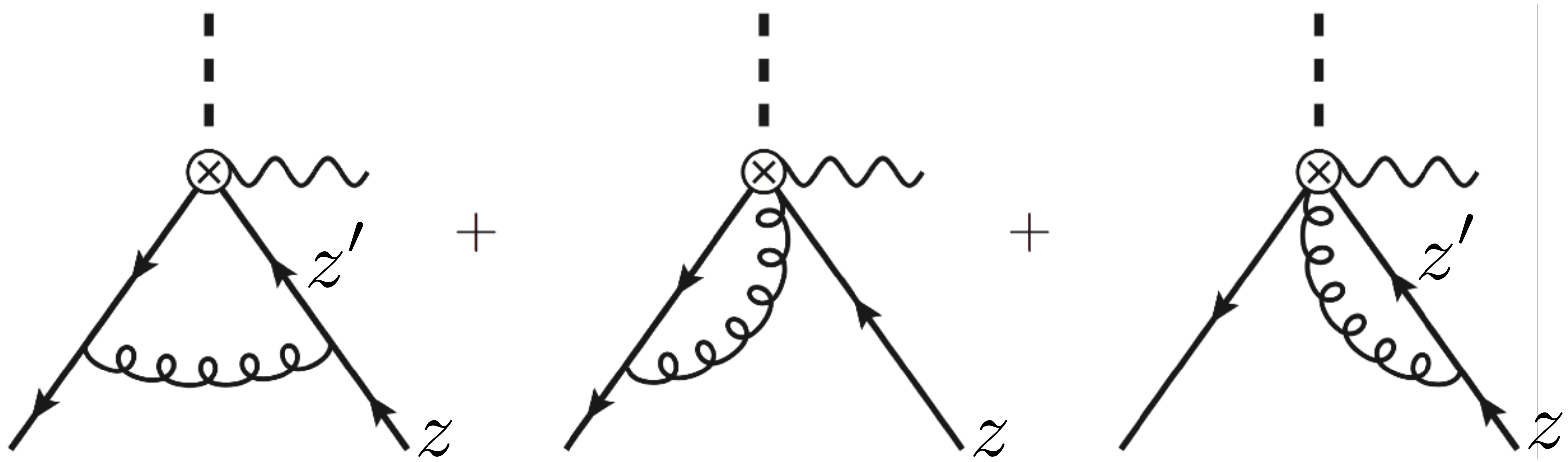
The red integral intervals are incompatible with subtraction/cutoff → **Mismatch**

$$Z_S(w,w') = \frac{w}{w'} Z_{33} \int_0^{\infty} \frac{dx}{x}\, Z_J^{-1}\Big(\frac{M_h w'}{x\ell_+}, \frac{M_h w}{\ell_+}\Big)\, Z_J^{-1}(-xM_h\ell_+, -M_h\ell_+)\,,$$

determined by the RG invariance of T3

Renormalization

Calculation of Z_{22} : UV divergences of matrix element of O_2



$$Z_{22}(z,z') = \left(1+\frac{C_F\alpha_s}{4\pi\epsilon}\right)\delta(z-z') - \frac{C_F\alpha_s}{2\pi\epsilon}\frac{1}{z'(1-z')}\left[z(1-z')\frac{\theta(z'-z)}{z'-z} + z'(1-z)\frac{\theta(z-z')}{z-z'}\right]_+ + \mathcal{O}(\alpha_s^2)$$

Brodsky-Lepage Kernel

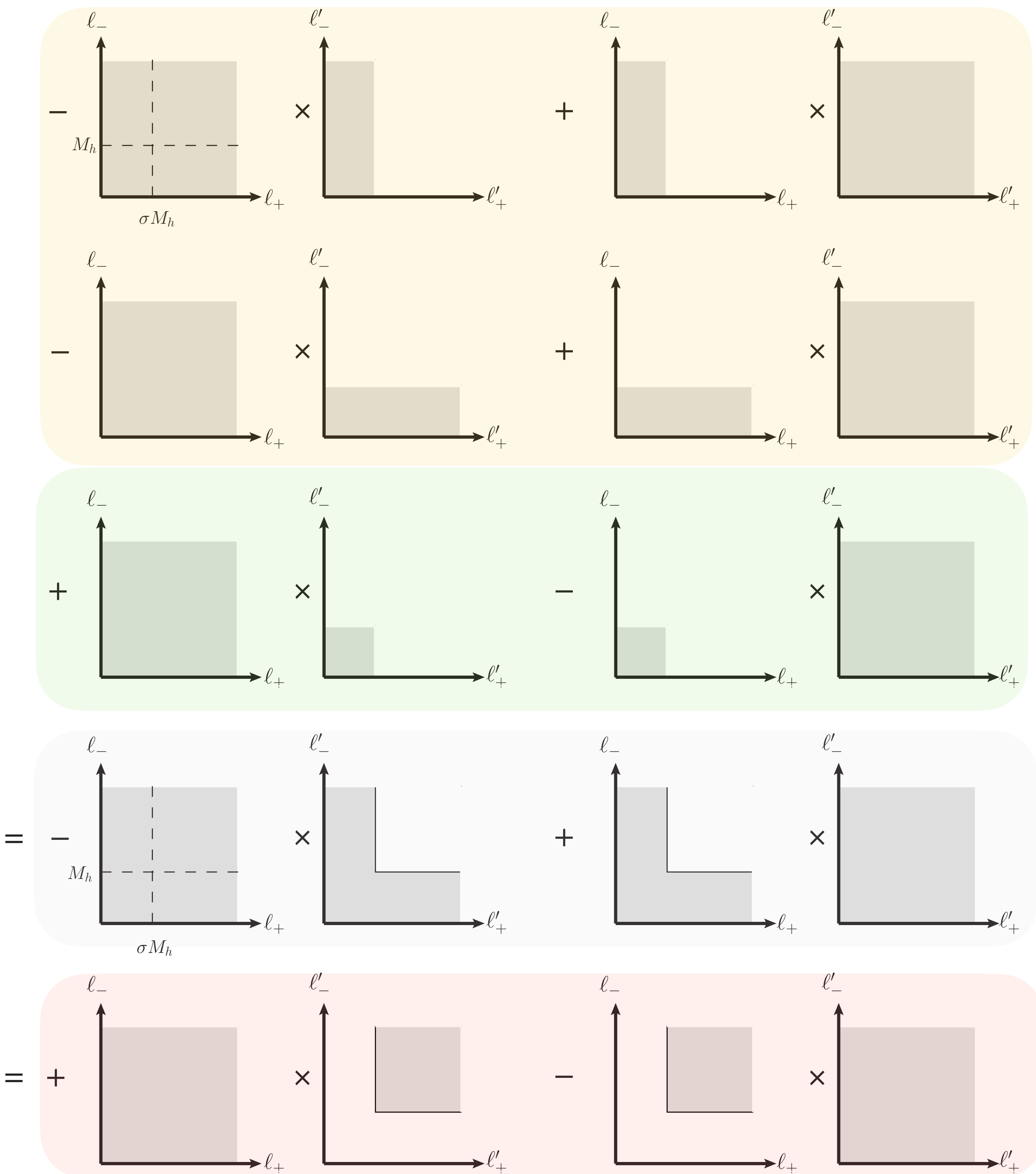
$z\,,\,z' \to 0$

$$[\![Z_{22}(z,z')]\!] = \left[1-\frac{C_F\alpha_s}{4\pi\epsilon}\left(2\ln z+3\right)\right]\delta(z-z') - \frac{C_F\alpha_s}{2\pi\epsilon}\,z\left[\frac{\theta(z'-z)}{z'(z'-z)} + \frac{\theta(z-z')}{z(z-z')}\right]_+ + \mathcal{O}(\alpha_s^2)$$

$$Z_{33}\,[\![Z_{22}^{-1}(z,z')]\!] = M_h\,\frac{z}{z'}\,Z_J(z'M_h^2, zM_h^2)$$

consistent with refactorization condition of H_2

Mismatch Terms in Renormalization



δT_2

$$-2\left(\int_0^1 dz \int_0^\infty dz' - \int_0^\infty dz \int_0^1 dz'\right) [\![H_2(z,\mu)]\!] [\![Z_{22}(z,z')]\!] [\![\langle O_2(z')\rangle]\!]$$

$$+ (c \leftrightarrow \bar{c})$$

different integral intervals but same integrands by refactorization conditions

δT_3

$$H_3(\mu)\left[\int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J(M_h\ell_-,\mu)\, J(-M_h\ell_+,\mu)\, S(\ell_+\ell_-,\mu)\right.$$

$$\left. - \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J^{(0)}(M_h\ell_-)\, J^{(0)}(-M_h\ell_+)\, S^{(0)}(\ell_+\ell_-)\right]$$

integrals over the entire planes are scaleless

Sum of mismatch can be rearranged to two integrals purely in hard region

Renormalized Matching Coefficients

Final renormalized H_1 :

Infinity-bin

Mismatch

$$H_1(\mu) = \left(H_1^{(0)} + \Delta H_1^{(0)} - \delta H_1^{(0),\,\mathrm{tot}}\right) Z_{11}^{-1}$$
$$+\, 2\lim_{\delta\to 0}\int_\delta^{1-\delta} dz \left[H_2^{(0)}(z)\, Z_{21}^{-1}(z) - \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z}\, [\![Z_{21}^{-1}(z)]\!] - \frac{[\![\bar{H}_2^{(0)}(\bar{z})]\!]}{\bar{z}}\, [\![Z_{21}^{-1}(\bar{z})]\!] \right]$$

Mixing from O_2

Final results at NLO:

$$H_1(\mu) = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \left\{ -2 + \frac{C_F \alpha_s}{4\pi} \left[-\frac{\pi^2}{3} L_h^2 + (12 + 8\zeta_3) L_h - 36 - \frac{2\pi^2}{3} - \frac{11\pi^4}{45} \right] + \mathcal{O}(\alpha_s^2) \right\}$$

$$H_2(z,\mu) = \frac{y_b(\mu)}{\sqrt{2}} \frac{1}{z(1-z)} \left\{ 1 + \frac{C_F \alpha_s}{4\pi} \left[2L_h \big(\ln z + \ln(1-z)\big) + \ln^2 z + \ln^2(1-z) - 3 \right] + \mathcal{O}(\alpha_s^2) \right\},$$

$$[\![\bar{H}_2(z,\mu)]\!] = \frac{y_b(\mu)}{\sqrt{2}} \left[1 + \frac{C_F \alpha_s}{4\pi} \left(2L_h \ln z + \ln^2 z - 3 \right) + \mathcal{O}(\alpha_s^2) \right],$$

$$H_3(\mu) = \frac{y_b(\mu)}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_s}{4\pi} \left(L_h^2 + 2 - \frac{\pi^2}{6} \right) + \mathcal{O}(\alpha_s^2) \right]$$

$$L_h = \ln \frac{-M_h^2}{\mu^2}$$

Renormalized Matrix Elements

Final renormalized matrix elements at NLO:

$$L_m = \ln\frac{m_b^2}{\mu^2} \qquad \hat{w} = \frac{w}{m_b^2}$$

$$\langle O_1(\mu)\rangle = m_b(\mu)\, g_\perp^{\mu\nu}$$

$$\langle O_2(z,\mu)\rangle = \frac{N_c\alpha_b}{2\pi}\, m_b(\mu)\, g_\perp^{\mu\nu}\left\{ -L_m + \frac{C_F\alpha_s}{4\pi}\left[L_m^2\left(\ln z + \ln(1-z) + 3\right)\right.\right.$$
$$\left.\left. - L_m\left(\ln^2 z + \ln^2(1-z) - 4\ln z\ln(1-z) + 11 - \frac{2\pi^2}{3}\right) + F(z) + F(1-z)\right] + \mathcal{O}(\alpha_s^2)\right\}$$

$$[\![\langle O_2(z,\mu)\rangle]\!] = \frac{N_c\alpha_b}{2\pi}\, m_b(\mu)\, g_\perp^{\mu\nu}\left\{ -L_m + \frac{C_F\alpha_s}{4\pi}\left[L_m^2\left(\ln z + 3\right) - L_m\left(\ln^2 z + 11 - \frac{2\pi^2}{3}\right)\right.\right.$$
$$\left.\left. + \frac{\ln^3 z}{6} - \frac{\ln z}{2} + 11 - \frac{\pi^2}{3} - 2\zeta_3\right] + \mathcal{O}(\alpha_s^2)\right\}$$

$$J(p^2,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[\ln^2\left(\frac{-p^2 - i0}{\mu^2}\right) - 1 - \frac{\pi^2}{6}\right] + \mathcal{O}(\alpha_s^2)$$

$$S(w,\mu) = -\frac{N_c\alpha_b}{\pi}\, m_b(\mu)\left[S_a(w,\mu)\,\theta(w - m_b^2) + S_b(w,\mu)\,\theta(m_b^2 - w)\right]$$

$$S_a(w,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + 2\,\mathrm{Li}_2\left(\frac{1}{\hat{w}}\right) - 4\ln\left(1 - \frac{1}{\hat{w}}\right)\left(L_m + 1 + \ln\left(1 - \frac{1}{\hat{w}}\right) + \frac{3}{2}\ln\hat{w}\right)\right]$$

$$S_b(w,\mu) = \frac{C_F\alpha_s}{\pi}\,\ln(1-\hat{w})\left[L_m + \ln(1-\hat{w})\right]$$

Renormalization-Group Equations

Hard functions

$$\frac{d}{d\ln\mu} H_1(\mu) = ?$$

$$\frac{d}{d\ln\mu} H_2(z,\mu) = \int_0^1 dz'\, H_2(z',\mu)\, \gamma_{22}(z',z)$$

$$\frac{d}{d\ln\mu} [\![\bar{H}_2(z,\mu)]\!] = \int_0^\infty dz'\, [\![\bar{H}_2(z',\mu)]\!]\, \frac{z}{z'}\, [\![\gamma_{22}(z',z)]\!]$$

$$\frac{d}{d\ln\mu} H_3(\mu) = \gamma_{33}\, H_3(\mu)$$

Matrix elements

$$\frac{d}{d\ln\mu} \langle O_1(\mu)\rangle = -\gamma_{11}\, \langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu} \langle O_2(z,\mu)\rangle = -\int_0^1 dz'\, \gamma_{22}(z,z')\, \langle O_2(z',\mu)\rangle - \gamma_{21}(z)\, \langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu} [\![\langle O_2(z,\mu)\rangle]\!] = -\int_0^\infty dz'\, [\![\gamma_{22}(z,z')]\!]\, [\![\langle O_2(z',\mu)\rangle]\!] - [\![\gamma_{21}(z)]\!]\, \langle O_1(\mu)\rangle$$

$$\frac{d}{d\ln\mu} J(p^2,\mu) = -\int_0^\infty dx\, \gamma_J(p^2, xp^2)\, J(xp^2,\mu)$$

$$\frac{d}{d\ln\mu} S(w,\mu) = -\int_0^\infty dx\, \gamma_S(w, w/x)\, S(w/x,\mu)$$

★ RGE of H_1 is highly non-trivial because of the contributions from infinity-bin subtraction and mismatch in renormalization

Radiative Jet Function

Using the RG invariance of decay amplitude $B^- \to \gamma \ell^- \bar{\nu}$

$$\mathcal{M}(B^- \to \gamma \ell^- \bar{\nu}) \propto F_B(\mu)\, H(m_b, 2E_\gamma, \mu) \int_0^\infty \frac{d\omega}{\omega}\, J(-2E_\gamma \omega, \mu)\, \phi_+^B(\omega, \mu)$$

and the two-loop RGE of B meson LCDA, Braun, Ji, Manashov, '19

we can derive the two-loop anomalous dimension of jet function:

$$\gamma_J(p^2, xp^2; \mu) = \left[\Gamma_{\rm cusp}(\alpha_s)\, \ln \frac{-p^2}{\mu^2} - \gamma'(\alpha_s)\right] \delta(1-x) + \Gamma_{\rm cusp}(\alpha_s)\, \Gamma(1, x)$$

$$+ C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\theta(1-x)}{1-x}\, h(x) + \mathcal{O}(\alpha_s^3)$$

Bell, Feldmann, Wang, Yip, '13

The solution of RGE of jet function at two loops is — jet function in dual space

$$J(p^2, \mu) = \exp\left[-2S(\mu_j, \mu) - a_{\gamma'}(\mu_j, \mu)\right] \tilde{\jmath}(\partial_\eta, \mu_j) \left(\frac{-p^2 e^{-2\gamma_E}}{\mu_j^2}\right)^{\eta + a_\Gamma(\mu_j, \mu)} \frac{\Gamma\big(1 - \eta - a_\Gamma(\mu_j, \mu)\big)}{\Gamma\big(1 + \eta + a_\Gamma(\mu_j, \mu)\big)}$$

$$\times \exp\left[C_F \int_{\alpha_s(\mu_j)}^{\alpha_s(\mu)} \frac{d\alpha}{2\pi} \left[\mathcal{H}\big(\eta + a_\Gamma(\mu_j, \mu_\alpha)\big) + \mathcal{O}(\alpha)\right]\right]\Bigg|_{\eta=0}$$

ZLL, Neubert, 2003.03393 (JHEP)

Soft-Quark Soft Function

RGE of soft-quark soft function: ZLL, Mecaj, Neubert, Wang, Fleming, 2005.03013 (JHEP)

$$\frac{d}{d\ln\mu} S(w,\mu) = -\int_0^\infty dw'\, \gamma_S(w,w';\mu)\, S(w',\mu)$$

γ_S can be determined by RG invariance of T_3

$$\gamma_S(w,w';\mu) = -\left[\Gamma_{\rm cusp}(\alpha_s)\, \ln\frac{w}{\mu^2} - \gamma_s(\alpha_s)\right] \delta(w-w') - 2\Gamma_{\rm cusp}(\alpha_s)\, w\, \Gamma(w,w')$$
$$- 2C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{w\,\theta(w'-w)}{w'(w'-w)}\, h\!\left(\frac{w}{w'}\right) + \mathcal{O}(\alpha_s^3)$$

The solution of RGE gives:

$$S(w,\mu) = U_S(w;\mu,\mu_s) \left[\int_0^\infty \frac{dx}{x}\, S(w/x,\mu_s)\, G_{4,4}^{2,2}\!\left(\begin{matrix} -a,-a,1-a,1-a \\ 1,1,0,0 \end{matrix}\,\middle|\, \frac{1}{x}\right)\right.$$
$$\left. - m_b \frac{C_F\alpha_s(\mu_s)}{\pi} \int_0^1 \frac{dx}{1-x}\, \frac{h(x)}{\beta_0}\, \frac{r^{1+\frac{2C_F}{\beta_0}\ln x} - 1}{1+\frac{2C_F}{\beta_0}\ln x}\, G_{4,4}^{2,2}\!\left(\begin{matrix} -a,-a,1-a,1-a \\ 0,1,0,0 \end{matrix}\,\middle|\, \frac{x m_b^2}{w}\right) + \mathcal{O}(\alpha_s^2)\right]$$

Meijer G-Function

Anomalous Dimensions

Anomalous dimensions:

$$\gamma_{ij} = \left(2\alpha_b \frac{\partial}{\partial \alpha_b} + 2\alpha_s \frac{\partial}{\partial \alpha_s}\right) Z_{ij}^{(1)}$$

$$\gamma_{11} = \frac{3C_F\alpha_s}{2\pi}$$

$$\gamma_{21}(z) = -\frac{N_c\alpha_b}{\pi}\left\{1 + \frac{C_F\alpha_s}{4\pi}\left[\ln^2 z + \ln^2(1-z) - 4\ln z\ln(1-z) + 11 - \frac{2\pi^2}{3}\right]\right\}$$

$$[\![\gamma_{21}(z)]\!] = -\frac{N_c\alpha_b}{\pi}\left\{1 + \frac{C_F\alpha_s}{4\pi}\left(\ln^2 z + 11 - \frac{2\pi^2}{3}\right)\right\}$$

$$\gamma_{22}(z,z') = -\frac{C_F\alpha_s}{\pi}\left\{\left[\ln z + \ln(1-z) + \frac{3}{2}\right]\delta(z-z') + z(1-z)\left[\frac{1}{z'(1-z)}\frac{\theta(z'-z)}{z'-z} + \frac{1}{z(1-z')}\frac{\theta(z-z')}{z-z'}\right]_+\right\}$$

$$[\![\gamma_{22}(z,z')]\!] = -\frac{C_F\alpha_s}{\pi}\left\{\left(\ln z + \frac{3}{2}\right)\delta(z-z') + z\left[\frac{\theta(z'-z)}{z'(z'-z)} + \frac{\theta(z-z')}{z(z-z')}\right]_+\right\}$$

$$\gamma_{33} = \frac{C_F\alpha_s}{\pi}\left(L_h - \frac{3}{2}\right)$$

Determine RGE of H_1

RGE of H_1 has general formula:

$$\frac{d\,H_1(\mu)}{d\ln\mu} = H_1(\mu)\gamma_{11}(\mu) + 2\int_0^1 dz\Big(H_2(z,\mu)\gamma_{21}(z,\mu) - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\gamma_{21}(z,\mu)]\!] - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\gamma_{21}(z,\mu)]\!]\Big) + H_3(\mu)\gamma_{31}(\mu)$$

RGE be determined by RG invariance:

$$\frac{d}{d\ln\mu}\mathcal{M}_b = 0 = \left[\left(\frac{d}{d\ln\mu} - \gamma_{11}\right)H_1(\mu)\right]\langle O_1(\mu)\rangle + \frac{d}{d\ln\mu}T_2(\mu) + \frac{d}{d\ln\mu}T_3(\mu)$$

The unknown quantity can be expressed as

non-zero due to cutoff

mismatch in T_2

$$H_3(\mu)\gamma_{31}(\mu)\langle O_1(\mu)\rangle = -\frac{d\,T_3(\mu)}{d\ln\mu} + 4\left(\int_0^1 dz\int_0^\infty dz' - \int_0^\infty dz\int_0^1 dz'\right)\frac{[\![\bar{H}_2(z',\mu)]\!]}{z'}[\![\gamma_{22}(z',z,\mu)]\!][\![\langle O_2(z,\mu)\rangle]\!]$$

each of them depends on hard and soft scales,
but their combination is purely hard

Determine RGE of H1

With the two refactorization conditions:

$$2\left(\int_0^1 dz \int_0^\infty dz' - \int_0^\infty dz \int_0^1 dz'\right) \frac{[\![\bar{H}_2(z',\mu)]\!]}{z'} [\![\gamma_{22}(z',z,\mu)]\!][\![\langle O_2(z,\mu)\rangle]\!]$$

$$= H_3(\mu)\int_0^\infty dx\, K(x)\Bigg\{\int_{M_h}^{M_h/x}\frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+} J(x\, M_h\ell_-,\mu;\epsilon)J(-M_h\ell_+,\mu;\epsilon)S(\ell_+\ell_-,\mu;\epsilon)$$

$$-2\int_{M_h}^{M_h/x}\frac{d\ell_-}{\ell_-}J(x\,M_h\ell_-,\mu;\epsilon)[\![Z_{21}\,(\ell_-/M_h,\mu)]\!]\,Z_{11}^{-1}(\mu)\,\langle O_1(\mu)\rangle\Bigg\}$$

To regulate the endpoint divergences, one need to keep the full dependence on ϵ in renormalized jet and soft functions before performing convolution

On the other hand:

$$\frac{dT_3(\mu)}{d\ln\mu} = H_3(\mu)\int_0^\infty dx\, K(x,\mu)\int_{M_h}^{M_h/x}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}J(xM_h\ell_-,\mu)\,J(-M_h\ell_+,\mu)\,S(\ell_+\ell_-,\mu) + (\ell_+\leftrightarrow\ell_-)$$

with $K(x,\mu) = \Gamma_{\text{cusp}}(\alpha_s)\,\Gamma(1,x) + C_F\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\theta(1-x)}{1-x}h(x) + \mathcal{O}(\alpha_s^3)$

Combination of the two mismatch terms gives an integral in purely hard region

RG Evolution

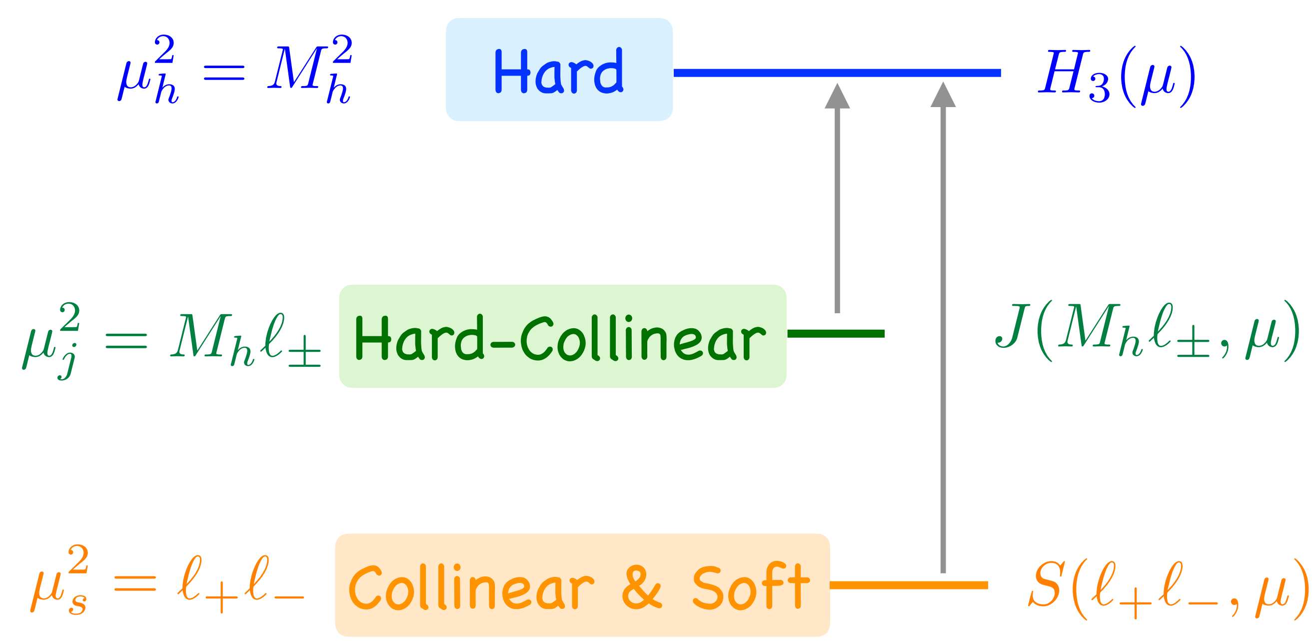
Last piece of anomalous dimensions:

$$\gamma_{31}(\mu) = \frac{N_c \alpha_b}{\pi} \sum_{n=0}^{\infty} \gamma_{31}^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^{n+1}$$

$$\gamma_{31}^{(0)} = 16\zeta_3 C_F \,,$$

$$\gamma_{31}^{(1)} = C_F^2 \left[-32\zeta_3 L_h^2 + \left(\frac{8\pi^4}{15} - 48\zeta_3\right) L_h + \frac{8\pi^2\zeta_3}{3} - 32\zeta_5 + \frac{4\pi^4}{5} + 168\zeta_3 \right]$$
$$+ C_F C_A \left(-\frac{176\zeta_3}{3} L_h - \frac{16\pi^2\zeta_3}{3} - \frac{44\pi^4}{45} + \frac{1072\zeta_3}{9} \right)$$
$$+ C_F n_f T_F \left(\frac{64\zeta_3}{3} L_h + \frac{16\pi^4}{45} - \frac{320\zeta_3}{9} \right)$$

Dynamical scale choice in T3



- Double log term in anomalous dimension complicates the scale evolution of H1
- Evolving the matrix elements up to hard scale is accessible
- Dynamical scale setting is necessary to adapt to the convolution

Resummation at NLL

Evolved quantities in T3 with dynamical scale setting

$$H_{3,\mathrm{LO}}(\mu) = -\frac{y_b(M_h)}{\sqrt{2}} e^{2S_\Gamma(-M_h^2,\mu^2)-2a_{\gamma_q}(-M_h^2,\mu^2)},$$

$$J_{\mathrm{LO}}(M_h\ell_-,\mu) = e^{-2S_\Gamma(-M_h\ell_-,\mu^2)-a_{\gamma'}(-M_h\ell_-,\mu^2)-2\gamma_E\, a_-}\frac{\Gamma(1-a_-)}{\Gamma(1+a_-)},$$

$$J_{\mathrm{LO}}(-M_h\ell_+,\mu) = e^{-2S_\Gamma(M_h\ell_+,\mu^2)-a_{\gamma'}(M_h\ell_+,\mu^2)-2\gamma_E\, a_+}\frac{\Gamma(1-a_+)}{\Gamma(1+a_+)},$$

$$S_{\mathrm{LO}}(\ell_+\ell_-,\mu) = -\frac{N_c\alpha_b}{\pi} m_b e^{2S_\Gamma(\ell_+\ell_-,\mu^2)+a_{\gamma_s}(\ell_+\ell_-,\mu^2)+4\gamma_E\, a_s}$$

$$\times\left(1-\frac{\alpha_s(\ell_+\ell_-)}{8\pi}\gamma_{m,0}\ln\frac{m_b^2}{\ell_+\ell_-}\right) G^{2,2}_{4,4}\left(\begin{matrix}-a_s,-a_s,1-a_s,1-a_s\\ 0,1,0,0\end{matrix}\,\middle|\,\frac{m_b^2}{w}\right)$$

$$\to\ \theta(\ell_+\ell_- - m_b^2)\frac{\Gamma^2(1+a_s)}{\Gamma^2(1-a_s)} + \mathcal{O}\left(\frac{m_b^2}{\ell_+\ell_-}\right)$$

$$S_\Gamma(\nu,\mu) = \frac{\alpha_s(\nu)}{4\pi}\left(-\frac{\Gamma_0}{8}\ln^2\frac{\mu^2}{\nu^2}\right) + \left(\frac{\alpha_s(\nu)}{4\pi}\right)^2\left(\frac{\beta_0\Gamma_0}{12}\ln^3\frac{\mu^2}{\nu^2}+\cdots\right)+\cdots$$

$$a_\Gamma(\nu,\mu) = \frac{\alpha_s(\nu)}{4\pi}\left(-\frac{\Gamma_0}{2}\ln\frac{\mu^2}{\nu^2}\right)+\cdots$$

$$\alpha_s(\mu) = \alpha_s(\nu)\left(1-\frac{\alpha_s(\nu)}{4\pi}\beta_0\ln\frac{\mu^2}{\nu^2}+\cdots\right)$$

$$a_- = a_\Gamma(-M_h\ell_-,\mu^2)$$
$$a_+ = a_\Gamma(M_h\ell_+,\mu^2)$$
$$a_s = a_\Gamma(\ell_+\ell_-,\mu^2)$$

NLL Resummation for decay amplitude

$$\mathcal{M}_b^{\mathrm{NLL}} = \frac{N_c\alpha_b}{\pi}\frac{y_b(M_h)}{\sqrt{2}} m_b\,\varepsilon_\perp^*(k_1)\cdot\varepsilon_\perp^*(k_2)\,\frac{L^2}{2}\left\{\sum_{n=0}^\infty\frac{2\Gamma(n+1)}{\Gamma(2n+3)}\left(-\frac{C_F\alpha_s(M_h)}{2\pi}L^2\right)^n - \frac{1}{L}\sum_{n=1}^\infty\frac{\Gamma(n+1)}{\Gamma(2n+2)}\left(-\frac{C_F\alpha_s(M_h)}{2\pi}L^2\right)^n\left[3-\beta_0\frac{\alpha_s(M_h)}{2\pi}L^2\frac{n(n+1)}{(2n+2)(2n+3)}\right]\right\}$$

Logarithms at Three Loops

Solving the RGEs iteratively in perturbation theory, one can predict the large logarithms in three-loop decay amplitude down to $\alpha\alpha_s^2 L^3$

$$\mathcal{M}_b = \frac{N_c \alpha_b}{\pi} \frac{m_b^2}{v} \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2)$$

$$\times \left\{ \frac{L^2}{2} - 2 + \frac{C_F \alpha_s(\hat{\mu}_h)}{4\pi} \left[-\frac{L^4}{12} - L^3 - \frac{2\pi^2}{3} L^2 + \left(12 + \frac{2\pi^2}{3} + 16\zeta_3 \right) L - 20 + 4\zeta_3 - \frac{\pi^4}{5} \right] \right.$$

$$\left. + C_F \left(\frac{\alpha_s(\hat{\mu}_h)}{4\pi} \right)^2 \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} - \frac{\beta_0}{30} \right) L^5 + d_4^{\rm OS} L^4 + d_3^{\rm OS} L^3 + \dots \right] \right\}$$

agree with numerical result of Niggetiedt, 2020

Strong evidence for the correctness of our factorization !!!

with

$$d_4^{\rm OS} = \left(\frac{3}{2} + \frac{\pi^2}{18} \right) C_F + \left(-\frac{91}{27} + \frac{\pi^2}{36} \right) C_A + \frac{32}{27} T_F n_f \,,$$

$$d_3^{\rm OS} = \left(-\frac{1}{2} + \frac{7\pi^2}{9} + \frac{20}{3} \zeta_3 \right) C_F + \left(-\frac{199}{18} - \frac{22\pi^2}{27} - 4\zeta_3 \right) C_A + \left(\frac{22}{9} + \frac{8\pi^2}{27} \right) T_F n_f$$

Is conventional QCD approach valid beyond leading logarithmic accuracy?

Resummation beyond NLL

ZLL, B. Mecaj, M. Neubert, X. Wang, in preparation

- Difficulties to resum large logs at NNLL and beyond

evolution from lower to higher scale

1. Soft-quark soft function is expressed by Meijer G-Function
2. RGE of $\langle O_2(z,\mu) \rangle$ is non-local and has mixing

3. Higher-order logs in higher-order anomalous dimension of $H_1(\mu)$

✓ evolution from higher to lower scale

Evolution of H1

RGE of hard coefficient H1 is given by

$$\frac{d\,H_1(\mu)}{d\ln\mu} = H_1(\mu)\gamma_{11}(\mu) + 4\int_0^1 dz\Big(H_2(z,\mu)\gamma_{21}(z,\mu) - [\![H_2(z,\mu)]\!][\![\gamma_{21}(z,\mu)]\!]\Big) + H_3(\mu)\gamma_{31}(\mu)$$

with

$$\gamma_{31}(\mu) = \frac{N_c\alpha_b}{\pi}\sum_{n=0}^{\infty}\gamma_{31}^{(n)}\left(\frac{\alpha_s}{4\pi}\right)^{n+1}$$

$$\gamma_{31}^{(0)} = 16\zeta_3 C_F\,,$$

$$\gamma_{31}^{(1)} = C_F^2\Bigg[\boxed{-32\zeta_3 L_h^2} + \left(\frac{8\pi^4}{15} - 48\zeta_3\right)L_h + \frac{8\pi^2\zeta_3}{3} - 32\zeta_5 + \frac{4\pi^4}{5} + 168\zeta_3\Bigg] + C_F C_A\left(-\frac{176\zeta_3}{3}L_h - \frac{16\pi^2\zeta_3}{3} - \frac{44\pi^4}{45} + \frac{1072\zeta_3}{9}\right) + C_F n_f T_F\left(\frac{64\zeta_3}{3}L_h + \frac{16\pi^4}{45} - \frac{320\zeta_3}{9}\right)$$

We need to resum the large logs in γ_{31} before solve RGE of H1 !

Resummation for the anomalous dimension

$\gamma_{31}(\mu)$ is defined by

$$\gamma_{31}(\mu) = \lim_{\sigma\to -1} 2\int_0^\infty dx\, K(x,\mu) \int_{M_h}^{M_h/x} \frac{d\ell_-}{\ell_-} J(xM_h\ell_-,\mu) \times \left[\int_{\sigma M_h}^\infty \frac{d\ell_+}{\ell_+} J^{(\epsilon)}(-M_h\ell_+,\mu) \frac{S^{(\epsilon)}(\ell_+\ell_-,\mu)}{m_b(\mu)} - 2[\![Z_{21}(\ell_-/M_h,\mu)]\!]\, Z_{11}^{-1}(\mu) \right]$$

divergent at endpoint, dimensionally regulated

only poles in MSbar scheme

Technically we need to introduce two "renormalized" functions

$$J^{(\epsilon)}(-M_h\ell_+,\mu) = \exp\left[-2S_\Gamma^{(\epsilon)}(M_h\ell_+,\mu^2) - a_{\gamma'}^{(\epsilon)}(M_h\ell_+,\mu^2)\right] \left[1 + \sum_{n=1}^\infty \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \sum_{m=0}^\infty \epsilon^m c_{n,m}^J\right]$$

$$\frac{S_\infty^{(\epsilon)}(\ell_+\ell_-,\mu)}{m_b(\mu)} = \exp\left[2S_\Gamma^{(\epsilon)}(\ell_+\ell_-,\mu^2) + a_{\gamma_s}^{(\epsilon)}(\ell_+\ell_-,\mu^2) + a_{\gamma_m}^{(\epsilon)}(\ell_+\ell_-,\mu^2)\right] \times \left(\frac{\ell_+\ell_-}{\mu^2}\right)^{-\epsilon} \left[1 + \sum_{n=1}^\infty \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \sum_{m=0}^\infty \epsilon^m c_{n,m}^S\right]$$

remain constant terms in higher order of ϵ

Evolution in D dimension

Generalize the solution from 4D to $(4-2\epsilon)$ D

$$S_\Gamma(\mu_i,\mu) \rightarrow S_\Gamma^{(\epsilon)}(\mu_i,\mu) = -\int_{\alpha_s(\mu_i)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)-2\epsilon\alpha} \int_{\alpha_s(\mu_i)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')-2\epsilon\alpha'}$$

$$\begin{aligned} S_\Gamma^{(\epsilon)}(\mu_i,\mu) = &\frac{\alpha_s(\mu)}{4\pi}\Gamma_0\left[\frac{1}{4\epsilon^2}\left(1-\left(\frac{\mu_i}{\mu}\right)^{-2\epsilon}\right)-\frac{1}{2\epsilon}\ln\left(\frac{\mu_i}{\mu}\right)\right] \\ &+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left[\beta_0\Gamma_0\left(\frac{1}{\epsilon^3}\left(-\frac{1}{16}\left(\frac{\mu_i}{\mu}\right)^{-4\epsilon}+\frac{1}{4}\left(\frac{\mu_i}{\mu}\right)^{-2\epsilon}-\frac{3}{16}\right)+\frac{1}{4\epsilon^2}\ln\left(\frac{\mu_i}{\mu}\right)\right)\right. \\ &\left.+\Gamma_1\left(\frac{1}{16\epsilon^2}\left(1-\left(\frac{\mu_i}{\mu}\right)^{-4\epsilon}\right)-\frac{1}{4\epsilon}\ln\left(\frac{\mu_i}{\mu}\right)\right)\right] \end{aligned}$$

The poles cancel after expanding in ϵ

The full dependence on ϵ helps to regulate endpoint singularities

Running Cutoff

The amplitude doesn't depend on the cutoff ν

$$\mathcal{M}_b(h\to\gamma\gamma) = H_1(\mu,\nu)\langle O_1(\mu)\rangle$$
$$+\,4\lim_{\delta\to 0}\left[\int_\delta^1 dz\, H_2(z,\mu)\,\langle\, O_2(z,\mu)\rangle - \int_\delta^{\nu/M_h} dz\, [\![H_2(z,\mu)]\!]\,[\![\langle O_2(z,\mu)\rangle]\!]\right]$$
$$+\, g_\perp^{\mu\nu} H_3(\mu)\lim_{\sigma\to -1}\int_0^{\nu}\frac{d\ell_-}{\ell_-}\int_0^{\sigma\nu}\frac{d\ell_+}{\ell_+}\, J(M_h\ell_-,\mu)\, J(-M_h\ell_+,\mu)\, S(\ell_+\ell_-,\mu)$$

The hard coefficient H_1 depends on cutoff

$$H_1(\mu,\nu) = H_1^{(0)} Z_{11}^{-1} + 4\lim_{\delta\to 0}\left[\int_\delta^1 dz\, H_2^{(\epsilon)}(z,\mu) Z_{21}(z) - \int_\delta^{\nu/M_h} dz\, [\![H_2^{(\epsilon)}(z,\mu)]\!][\![Z_{21}(z)]\!]\right] Z_{11}^{-1}$$
$$-\, H_3^{(\epsilon)}(\mu)\lim_{\sigma\to 1}\int_\nu^\infty \frac{d\ell_-}{\ell_-}\int_{\sigma\nu}^\infty\frac{d\ell_+}{\ell_+} J^{(\epsilon)}(M_h\ell_-,\mu) J^{(\epsilon)}(-M_h\ell_+,\mu)\frac{S_\infty^{(\epsilon)}(\ell_+\ell_-,\mu)}{m_b(\mu)}$$

We can always use S_∞ instead of S. The difference is at NNLP.

No Meijer G-function. Calculation simplified!

$\nu = M_h$ is still the best choice.

Resummation for T_2

$\langle O_2(z,\mu)\rangle$ and $[\![\langle O_2(z,\mu)\rangle]\!]$ are at scale m_b, no need for evolution

The evolution of the hard coefficient can be expressed by Gegenbauer polynomials

$$
\begin{aligned}
h_{2n}(\mu) &= \int_0^1 dz\, H_2(z,\mu) C_{2n}^{(3/2)}(2z-1) z(1-z) \\
&= \frac{y_b(-M_h^2)}{\sqrt{2}} \left(\frac{\alpha_s(\mu^2)}{\alpha_s(-M_h^2)} \right)^{\frac{3C_F}{\beta_0}} \left[1 + \frac{C_F \alpha_s(-M_h^2)}{4\pi} (4H_{2n+1}^2 - 3) \right] e^{-2a_h H_{2n+1}}
\end{aligned}
$$

RGEs of Gegenbauer moments are local

$$\int_0^1 dz H_2(z,\mu) \langle O_2(z,\mu)\rangle = \sum_{n=0}^{\infty} a_{2n}(\mu) h_{2n}(\mu)$$

$$\langle O_2(z,\mu)\rangle = z(1-z) \sum_{n=0}^{\infty} a_n(\mu) C_n^{(3/2)}(2z-1)$$

Cancellation of endpoint divergences in T2

$$\int_0^1 dz\, H_2(z,\mu)\,\langle O_2(z,\mu)\rangle = \sum_{n=0}^{\infty} a_{2n}(\mu)\, h_{2n}(\mu)$$

$$\supseteq -\frac{N_c\alpha_b}{\pi}\frac{y_b(-M_h^2)}{\sqrt{2}}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(-M_h^2)}\right)^{\frac{3C_F}{\beta_0}}\sum_{n=0}^{\infty}\frac{2(4n+3)}{(2n+1)(n+1)}e^{-2a_h H_{2n+1}}$$

Each moment is well defined, but the sum of series is divergent!

The evolution of subtraction term can be obtained by refactorization

$$[\![H_2(z,\mu)]\!] = -\,H_3(\mu)J(zM_h,\mu)$$

$$= \frac{y_b(-M_h^2)}{\sqrt{2}}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(-M_h^2)}\right)^{\frac{3C_F}{\beta_0}} e^{-2\gamma_E a_h}\frac{\Gamma(1-a_h)}{\Gamma(1+a_h)}\times\left[1+\frac{C_F\alpha_s(-M_h^2)}{4\pi}(\partial_\eta^2-3)\right] z^{-1+\eta+a_h}\bigg|_{\eta=0}$$

$$2\int_0^1 dz\,[\![H_2(z,\mu)]\!][\![\langle O_2(z,\mu)\rangle]\!] \supseteq -\frac{N_c\alpha_b}{\pi}\frac{y_b(-M_h^2)}{\sqrt{2}}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(-M_h^2)}\right)^{\frac{3C_F}{\beta_0}}$$

$$\times\sum_{n=0}^{\infty}\frac{2(4n+3)}{(2n+1)(n+1)}e^{-2\gamma_E a_h}\frac{(2n+2)(2n+1)\Gamma(2n+1-a_h)}{\Gamma(2n+3+a_h)}$$

Identical in $n \to \infty$

Conclusions

- First renormalized factorization theorem at subleading power (with endpoint divergences) is established in SCET
- Two refactorization conditions are proved at operator level
- Nature cutoffs arise from subtractions and rearrangement of endpoint singularities
- Contributions from mismatch between renormalization and convolution with cutoffs are proved to be purely hard
- Operators mix under renormalization

Thanks for your attention!

Backup

Scale evolution: