Confronting $B \to K\pi$ theory predictions with experimental data

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Setting the scene

- $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$ a very important channel
- As discussed yesterday multiple different analyses at LHCb
- Varying treatment of non-local contributions
- Varying treatment of $m(K\pi)$ (and q^2 regions measured)

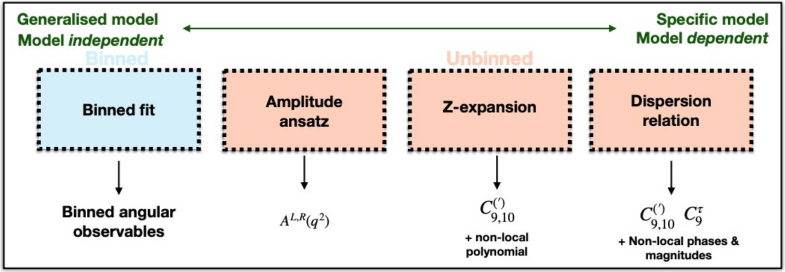


diagram by T. Hadavizadeh

Setting the scene

- The $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$ decay rate is dominated by a P-wave $K^{*0}(892)$ contribution.
- P-wave decay rate of $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$:

$$\frac{\mathrm{d}\Gamma_P}{\mathrm{d}m(K\pi)\mathrm{d}\cos\theta_\ell \mathrm{d}\cos\theta_K \mathrm{d}\phi} = \frac{9}{32\pi}[J_{1s}\sin^2\theta_K + J_{1c}\cos^2\theta_K + J_{2s}\sin^2\theta_K\cos 2\theta_\ell + J_{2c}\cos^2\theta_K\cos 2\theta_\ell$$
$$+J_3\sin^2\theta_K\sin^2\theta_\ell\cos 2\phi + J_4\sin 2\theta_K\sin 2\theta_\ell\cos\phi$$
$$+J_5\sin 2\theta_K\sin\theta_\ell\cos\phi + J_{6s}\sin^2\theta_K\cos\theta_\ell$$
$$+J_{6c}\cos^2\theta_K\cos\theta_\ell + J_7\sin 2\theta_K\sin\theta_\ell\sin\phi + J_8\sin 2\theta_K\sin 2\theta_\ell\sin\phi$$
$$+J_9\sin^2\theta_K\sin^2\theta_\ell\sin 2\phi] \times |BW_P(m(K\pi))|^2$$

- The angular coefficients J_i terms being bilinear combinations of amplitudes
- $m(K\pi)$ dependence provided by $\times|BW_P(m(K\pi))|^2$, no $m(K\pi)$ dependence in the rest of the decay rate
- Local form factors effects are contained in the angular coefficients measured

Setting the scene

- The next largest contribution is from the S-wave.
  - *Note*: The importance of this contribution depends on the $m(K\pi)$ window chosen
- The S-wave + interference contributions to the decay rate are as follows:

$$
\begin{aligned}
\frac{\mathrm{d}\Gamma_S}{\mathrm{d}\cos\theta_\ell \mathrm{d}\cos\theta_K \mathrm{d}\phi \mathrm{d}m(K\pi)} = +\frac{1}{4\pi}\Big[& (\tilde{J}^c_{1a} + \tilde{J}^c_{2a}\cos 2\theta_\ell)|BW_S|^2 \\
& + [\tilde{J}^{c,r}_{1b}\mathrm{Re}(BW_S BW_P^*) - \tilde{J}^{c,i}_{1b}\mathrm{Im}(BW_S BW_P^*)]\cos\theta_K \\
& + [\tilde{J}^{c,r}_{2b}\mathrm{Re}(BW_S BW_P^*) - \tilde{J}^{c,i}_{2b}\mathrm{Im}(BW_S BW_P^*)]\cos 2\theta_\ell\cos\theta_K \\
& + [\tilde{J}^{r}_{4}\mathrm{Re}(BW_S BW_P^*) - \tilde{J}^{i}_{4}\mathrm{Im}(BW_S BW_P^*)]\sin 2\theta_l\sin\theta_K\cos\phi \\
& + [\tilde{J}^{r}_{5}\mathrm{Re}(BW_S BW_P^*) - \tilde{J}^{i}_{5}\mathrm{Im}(BW_S BW_P^*)]\sin\theta_l\sin\theta_K\cos\phi \\
& + [\tilde{J}^{r}_{7}\mathrm{Im}(BW_S BW_P^*) + \tilde{J}^{i}_{7}\mathrm{Re}(BW_S BW_P^*)]\sin\theta_l\sin\theta_K\sin\phi \\
& + [\tilde{J}^{r}_{8}\mathrm{Im}(BW_S BW_P^*) + \tilde{J}^{i}_{8}\mathrm{Re}(BW_S BW_P^*)]\sin 2\theta_l\sin\theta_K\sin\phi\Big]
\end{aligned}
$$

- Again there is no $m(K\pi)$ dependence in the angular terms which effectively include local form factor effects.
- Higher partial waves can essentially be ignored in this region of $m(K\pi)$ ([1512.08627])

Setting the scene - three areas of discussion

- S- and P-wave contributions have differing angular structures providing some separating power, however, the $m(K\pi)$ lineshape is still very important
  - Controlling the $m(K\pi)$ lineshape is especially important for gaining sensitivity to S-wave and interference
- To avoid biases the $m_{K\pi}$ lineshape must be fit with an accurate description.

- Multiple P-wave states? $K^*(1410)$?

- At present it is assumed that $m(K\pi)$ factorises with q^2
- Any $m_{K\pi}$ dependence in the $B \to K\pi$ local form factors is not included.
- Can we check the impact of this?

- What S-wave $m(K\pi)$ lineshape parameterisation to use?

What can theory say about local form factors?

$B \to K\pi$ Form factors

Definition of Lorentz-Invariant Form Factors:

$$
\begin{aligned}
i\langle K^-(k_1)\pi^+(k_2)|\bar{s}\gamma^\mu b|\bar{B}^0(q+k)\rangle &= F_\perp\, k_\perp^\mu \\
-i\langle K^-(k_1)\pi^+(k_2)|\bar{s}\gamma^\mu\gamma_5 b|\bar{B}^0(q+k)\rangle &= F_t\, k_t^\mu + F_0\, k_0^\mu + F_\parallel\, k_\parallel^\mu \\
\langle K^-(k_1)\pi^+(k_2)|\bar{s}\sigma^{\mu\nu}q_\nu b|\bar{B}^0(q+k)\rangle &= F_\perp^T\, k_\perp^\mu \\
\langle K^-(k_1)\pi^+(k_2)|\bar{s}\sigma^{\mu\nu}q_\nu\gamma_5 b|\bar{B}^0(q+k)\rangle &= F_0^T\, k_0^\mu + F_\parallel^T\, k_\parallel^\mu
\end{aligned}
$$

Functions $F_i^{(T)}(k^2, q^2, q\cdot\bar{k})$. Partial-wave expansion:

$$
\begin{aligned}
F_{0,t}(k^2,q^2,q\cdot\bar{k}) &= \sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\; F_{0,t}^{(\ell)}(k^2,q^2)\; P_\ell^{(0)}(\cos\theta_K) \\
F_{\perp,\parallel}(k^2,q^2,q\cdot\bar{k}) &= \sum_{\ell=1}^{\infty}\sqrt{2\ell+1}\; F_{\perp,\parallel}^{(\ell)}(k^2,q^2)\; \frac{P_\ell^{(1)}(\cos\theta_K)}{\sin\theta_K}
\end{aligned}
$$

Theory for S and P-wave $B \to \pi K$ form factors

- Two main approaches:
  - Lattice QCD at large q^2
  - Light-cone sum rules at low q^2

- Light-cone sum rule analyses

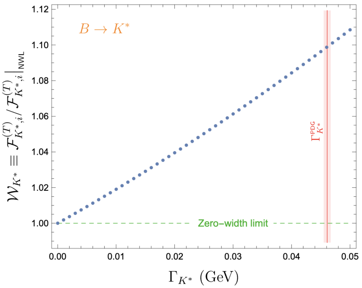
  [J. Virto, A. Khodjamirian, S. Descotes-Genon JHEP 1912, 083 (2019)]

  [J. Virto, A. Khodjamirian, S. Descotes-Genon, KKV JHEP 06, 034 (2023)]

  - Only available in terms of B meson LCDA
  - Improvement over assuming K^* is a stable state
  - Finite width effects in P wave at 20% level for BR
  - Higher resonances large impact → can be constrained by moment analysis

- S wave even more challenging; generally broad resonances

- Relevant for $B \to K^* \ell\ell$, but also $B \to K\pi\pi$!



Light-Cone Sum Rules for S and P-wave $B \to K\pi$ Form Factors

$$\int_{s_{\rm th}}^{s_0} ds\ e^{-s/M^2}\ \ \omega_i(s,q^2)\ \ f^\star_{0,+}(s)\ \ F_i^{(\ell)}(s,q^2) = \Pi_i^{\rm OPE}(q^2,\sigma_0,M^2)$$

- s_0 – Effective threshold
- $\omega_i(s,q^2)$ – (known) kinematic factors
- $\langle K^-(k_1)\pi^+(k_2)|\bar{s}\gamma_\mu d|0\rangle = f_+(k^2)\ \overline{k}_\mu + \frac{m_K^2-m_\pi^2}{k^2} f_0(k^2)\ k_\mu$
- $\Pi_i^{(T),\rm OPE}$ – OPE result for the correlation function (in terms of B-LCDAs)

  → No closed expression for FFs: use sume rules to contrain favourite model

  → $f_{+,0}(k^2)$ from data

  → s_0 from two-point sum rule for $K\pi$ form factor.

QCD to constrain P and S-wave models

- Use QCD sum rules to constrain $B \to (K\pi)_P$ and $B \to (K\pi)_S$ parametrizations/models
- Simple sum of Breit-wigners can be used for P wave, but not for S wave

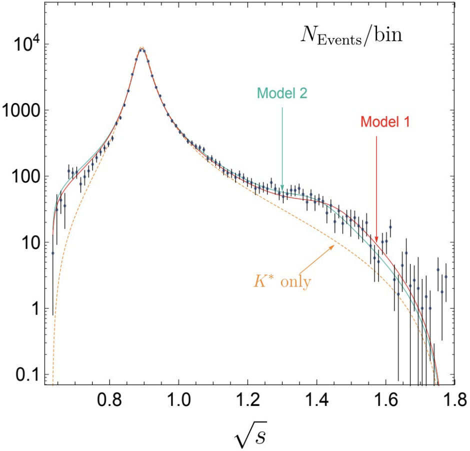
Model requirements:

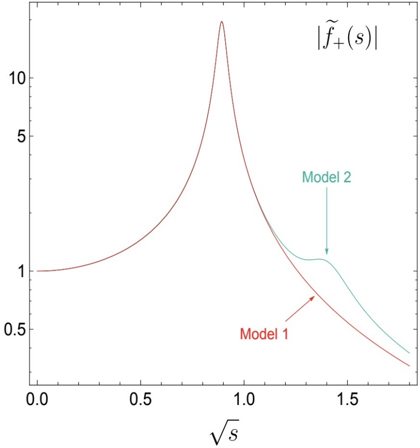
- appropriate analytical properties
- poles corresponding to known resonances
- cuts for the relevant open channels
- simple (linear) dependence on the parameters to be constrained by the sum rules

Light-cone sum rules for P-wave $B \to \pi K$ form factors

[S. Descotes-Genon, A. Khodjamirian, J. Virto, JHEP 1912, 083 (2019)] [arXiv:1908.02267];

$K\pi$ form factor $f_+(s)$ from $\tau \to K\pi\nu_\tau$

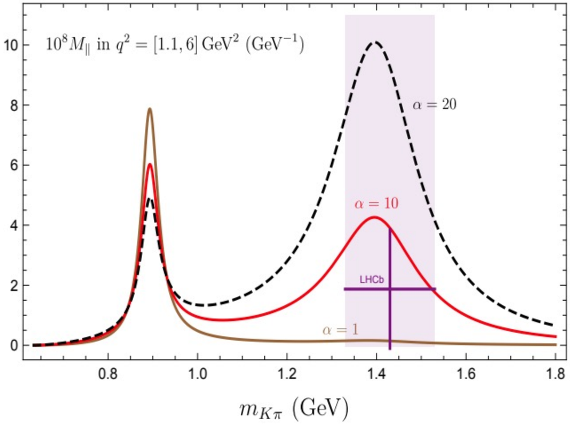




Light-cone sum rules for P-wave $B \to \pi K$ form factors

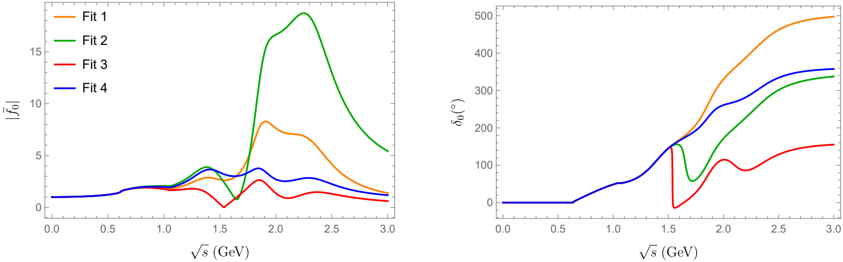
[S. Descotes-Genon, A. Khodjamirian, J. Virto, JHEP 1912, 083 (2019)] [arXiv:1908.02267];

$$\mathcal{F}_{K^*(1410)}(q^2) = \alpha\, \mathcal{F}_{K^*(892)}(q^2) \qquad \alpha \text{ free parameter}$$



Light-cone sum rules for S-wave $B \to \pi K$ form factors

Input: S-wave $K\pi$ form factors



L. Von Detten, F. No el, C. Hanhart, M. Hoferichter and B. Kubis, Eur. Phys. J. C 81,420 (2021) [arXiv:2103.01966 [hep-ph]].

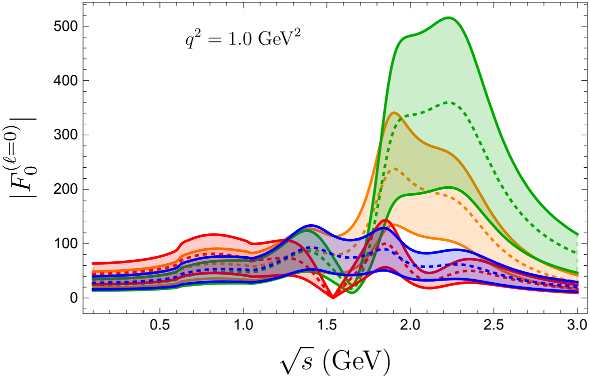
- Uses Omnes coupled channel for $K\pi$ rescattering up to 1 GeV
- Includes source terms for high-mass resonances
- Form factor obtained from fitting to $\tau \to K_S \pi \nu_\tau$ spectrum (simultaneous with P wave)

Light-cone sum rules for S-wave $B \to \pi K$ form factors

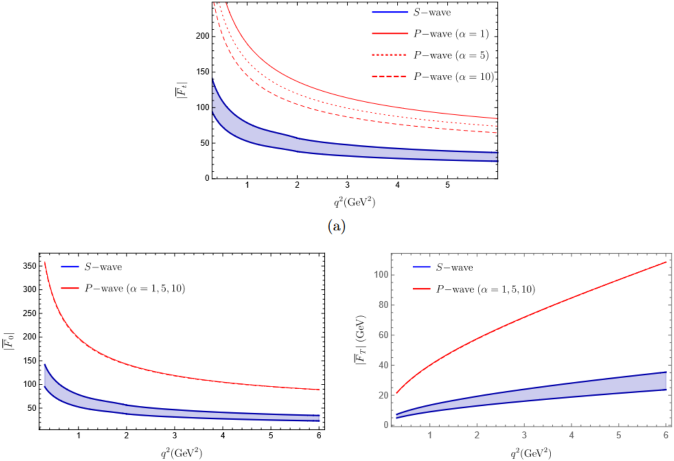
- LCSRs relate the $B \to \pi K$ form factor to the $K\pi$ form factor:

$$F_i^{(\ell=0)}(s, q^2) = \kappa_i(s, q^2)\rho_i(q^2) f_0(s)$$

- ρ determined from the LCSR OPE calculation, κ kinematic factor
- No free parameter, relative contributions of resonances fixed

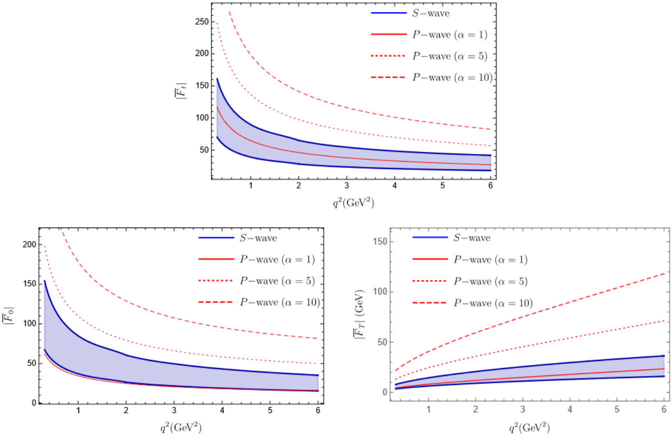


Light-cone sum rules for P versus S-wave (low $m(K\pi)$)



The form factors are integrated over a 100 MeV region around the $K^*(892)$ resonance: $(0.796\ \mathrm{GeV})^2 < s < (0.996\ \mathrm{GeV})^2$.

Light-cone sum rules for P versus S-wave (high $m(K\pi)$)



In the higher s region containing the resonances $K^*(1410)$ and $K_0^*(1430)$:
$(1.33\ \mathrm{GeV})^2 < s < (1.53\ \mathrm{GeV})^2$.

What can data say? – Branching Ratios

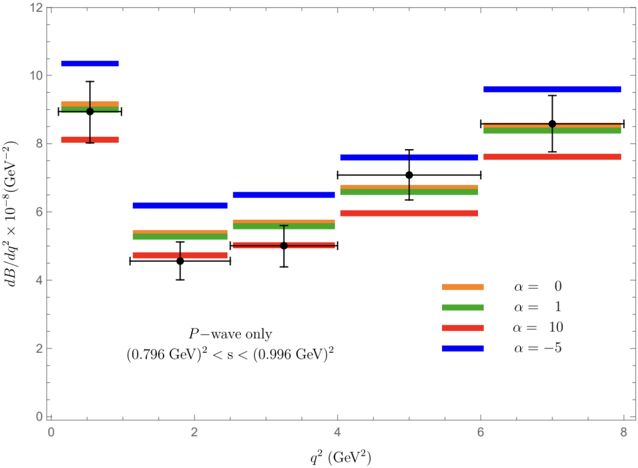
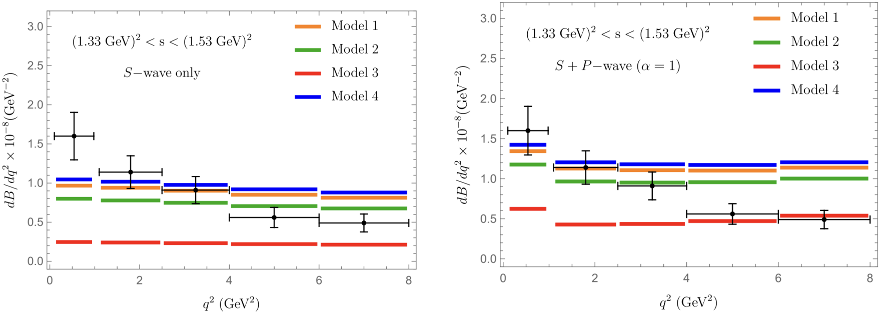


Figure 8: *Theory predictions for the* $B \to (K\pi)_P \ell^+\ell^-$ *branching ratio within the* $K\pi$ *invariant mass bin* $(0.796\ \text{GeV})^2 < s < (0.996\ \text{GeV})^2$*, for different values of* α*, compared to the LHCb measurements of* $B \to K^*\mu^+\mu^-$ *in Ref. [13].*

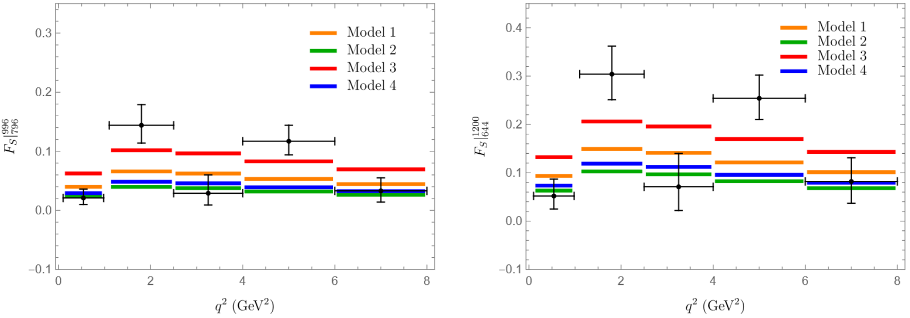
What can data say? – Branching Ratios



What can data say? – S-wave fraction

LHCb [JHEP11(2016)047] [arXiV:1606.04731]

Example: measured scalar contribution in low $m(K\pi)$ shows rapid fluctations?



from: [J. Virto, A. Khodjamirian, S. Descotes-Genon, KKV] [arXiv:2304.02973]

- Predictions for moments at low $m(K\pi)$ available
- Comparison of moments with experiment could give insight into S-wave component
- Currently S-wave is treated as nuisance parameter

What can data say? – Angular Moments

LHCb [JHEP12(2016)065] [arXiV:1609.04736]

Example of use of the data to constrain higher-partial waves:

Moment/τ_B	Amplitude	Exp. Value $\times 10^8$	Theory $\times 10^8$
$-\frac{\sqrt{5}}{2}(\tilde{\Gamma}_3 + 2\tilde{\Gamma}_6)$	$\tau_B\langle \lvert\widehat{S}^L\rvert^2 + \lvert\widehat{S}^R\rvert^2\rangle = \langle M_S\rangle$	2.16 ± 1.62	$[1.12, 4.73]$
$\frac{1}{2}\tilde{\Gamma}_2$	$\tau_B\langle \mathrm{Re}(\widehat{A}_0^L \widehat{S}^{L*} + \widehat{A}_0^R \widehat{S}^{R*})\rangle = \langle M_{0\mathrm{Re}}\rangle$	-0.84 ± 0.29	$[-0.53, -1.24]$
$-\sqrt{\frac{5}{3}}\tilde{\Gamma}_{11}$	$\tau_B\langle \mathrm{Re}(\widehat{A}_{\parallel}^L \widehat{S}^{L*} + \widehat{A}_{\parallel}^R \widehat{S}^{R*})\rangle$	-0.31 ± 0.69	$[-0.23, -0.54]$
$\sqrt{\frac{5}{3}}\tilde{\Gamma}_{15}$	$\tau_B\langle \mathrm{Im}(\widehat{A}_{\perp}^L \widehat{S}^{L*} + \widehat{A}_{\perp}^R \widehat{S}^{R*})\rangle$	0.57 ± 0.69	$[-0.17, -0.36]$
$\frac{1}{\sqrt{3}}\tilde{\Gamma}_{34}$	$\tau_B\langle \mathrm{Re}(\widehat{A}_{\perp}^L \widehat{S}^{L*} - \widehat{A}_{\perp}^R \widehat{S}^{R*})\rangle$	0.35 ± 0.26	$[-0.14, -0.34]$
$-\frac{1}{\sqrt{3}}\tilde{\Gamma}_{38}$	$\tau_B\langle \mathrm{Im}(\widehat{A}_{\parallel}^L \widehat{S}^{L*} - \widehat{A}_{\parallel}^R \widehat{S}^{R*})\rangle$	0.14 ± 0.25	$[-0.29, -0.61]$

from: [J. Virto, A. Khodjamirian, S. Descotes-Genon, KKV] [arXiv:2304.02973]

- Pure S wave moments already disfavor some S-wave models
- Mixed S and P, depend on α and different models for S
- No indication for large D wave contribution

Is it possible to improve the experimental analyses in the future?

Why is multiple P-wave states a problem?

- The angular distribution for each amplitude has, a short-/long-distance component $S(q^2)$, a local form factor component $f(q^2)$ and some $m(K\pi)$ dependence $g(m(K\pi))$.

$$\underbrace{f(q^2)\cdot S(q^2)}_{\propto \text{ Angular terms}} \cdot g(m_K\pi)$$

- With one only one $m(K\pi)$ contribution (say only $K^{*0}(892)$), the above is fine
  - Assuming no $m(K\pi)$ dependence in the angular coefficients: $(f(q^2)\cdot S(q^2))$
- What happens if we have two contributions (g_1 and g_2) to $m(K\pi)$ lineshape?

$$(f_1(q^2)\cdot g_1(m_K\pi) + f_2(q^2)\cdot g_2(m_K\pi))\cdot S(q^2)$$

- The local form factors will be different for each contribution (f_1 and f_2)
- The above factorisation has broken down, and the local form factor effects can no longer be grouped with short-/long-distance components in our angular coefficients.

What options do we have?

Firstly, it is unclear how big this issue might be

- The effect is likely negligible with current statistics and the current $m(K\pi)$ window.

However, this is something to consider going forward, the importance of these effects will only increase with more LHCb data.

If the effect is relevant, the options would be:

- Reduce the size of $m(K\pi)$ window, or at least don't continue to open it further.
  - Note, in the upcoming LHCb binned analysis the $m(K\pi)$ window is widened to increase sensitivity to S-wave/interference observables (now is approx 0.746 to 1.10)
- Extreme solution: Fit a 2nd set of P-wave observables for the $K^{*0}(1410)$

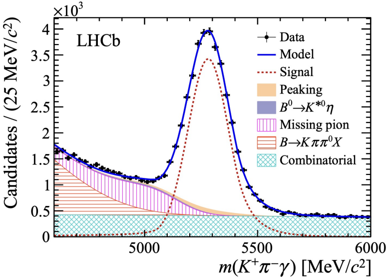
$$f_1(q^2) \cdot S_1(q^2) \cdot g_1(m_K\pi) + f_2(q^2) \cdot S_2(q^2) \cdot g_2(m_K\pi)$$

  - Almost certainly a prohibitively large number of parameters?
  - Moment analysis?
  - What about any interference?

Can we use data to pin down the P-wave $m(K\pi)$ lineshape?

- We hoped to isolate the P-wave lineshape using $B^0 \to K^{*0}\gamma$, where the photon forces the K^{*0} to be P-wave

[1905.06284].

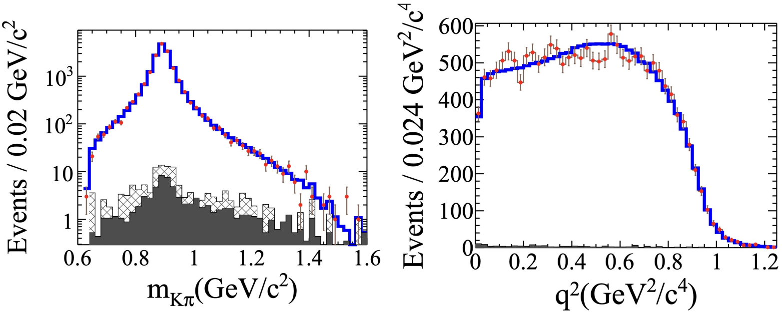


+ *sWeights to isolate $m(K\pi)$ linshape*

- However, this would not quite be the same P-wave lineshape as we expect in $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$
  - Each decay channel has different levels of contributions from different helicity amplitudes, each of which has a different local form factor dependence (thanks to J. Virto for a useful discussion)
  - Not possible to separate different contributions in $B^0 \to K^{*0}\gamma$ as we cannot measure the helicity of the photon.

Testing for $m(K\pi)$ dependence in local form factors

- Floating $m(K\pi)$ dependence may not be possible in $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$ with the current LHCb dataset
- Can we look at other channels to see how big the effect might be? $D^+ \to K^-\pi^+\ell^+\nu_\ell$ for example?
- Large BESIII $D^+ \to K^-\pi^+\ell^+\nu_\ell$ data set and analysis completed [1512.08627]
- This analysis fits the data for $D \to K\pi$ form factor parameters assuming factorisation of $m(K\pi)$
- Can we re-analyse the data with a different model that allows for some $m(K\pi)$ dependence into FFs and compare results?
  - This is the equivalent to adding finite width to K^* state



Testing for $m(K\pi)$ dependence in local form factors

Some definitions for the z parameterization,

$$t_p = (m(B) + m(K) + m(\pi))^2,$$

$$t_m = (m(B) - m(K) - m(\pi))^2,$$

and

$$t_0 = t_p \left(1 - \sqrt{1 - \frac{t_m}{t_p}}\right).$$

Define z as,

$$z(s) = \frac{\sqrt{t_p - s} - \sqrt{t_p - t_0}}{\sqrt{t_p - s} + \sqrt{t_p - t0}},$$

and Δz

$$\Delta z(q^2) = z(q^2) - z(0).$$

Could define a P-wave general form factor expression for a set of coefficients a as,

$$\mathcal{F}_{(p)}(q^2, m_{K\pi}, a) = \frac{BW}{1 - \frac{q^2}{m(R)^2}}(a_0 + a_1 \Delta z(q^2) + a_2 \Delta z(q^2)^2),$$

then add some $m(K\pi)$ dependence into the coefficients a_i:

$$a_N(a, m_{K\pi}) = a[N, 0] + \frac{m_{K\pi}^2}{(m_K + m_\pi)^2} a[N, 1].$$

$$a_{\parallel} = \begin{bmatrix} a_{\parallel}^{0,0} & a_{\parallel}^{0,1} \\ a_{\parallel}^{1,0} & a_{\parallel}^{1,1} \\ a_{\parallel}^{2,0} & a_{\parallel}^{2,1} \end{bmatrix},$$

- Fit the data with $a[N, 1]$ floating and $a[N, 1] = 0$, and compare results

Testing for $m(K\pi)$ dependence in local form factors

- Can we repeat the above $D^+ \to K^-\pi^+\ell^+\nu_\ell$ study with $D^+ \to \pi^-\pi^+\ell^+\nu_\ell$?
- The $D^+ \to \pi^-\pi^+\ell^+\nu_\ell$ channel is CKM suppressed but we can study the $\rho(770) \to \pi^+\pi^-$ resonance
- Including the flexibility in the local form factors for $m(K\pi)$ dependence is equivalent to giving a finite width to the meson
- Finite width effects in P-wave at 20% level for BR of K^{*0}
- The width of the $\rho(770)$ is larger than that of the $K^{*0}(892)$ and so we can expect effects might be larger?

Worth noting:

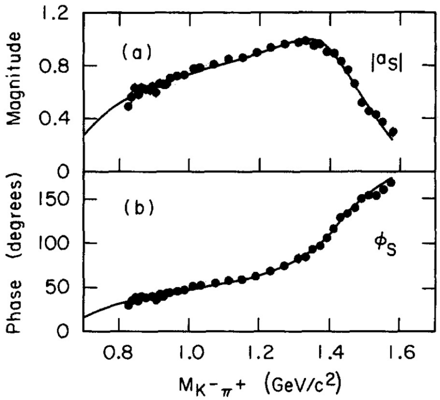
- Detector effects significantly contribute to the width of the experimental mass distribution
- Therefore it might be expected that effects are not quite as large as 20%?

What S-wave parameterisation to use?

- Upcoming LHCb binned analysis uses the LASS paramterstion for the S-wave
  - *Note*: LASS data starts at $m_{K\pi} = 0.825$ GeV, binned $m_{K\pi}$ window starts at 0.746 GeV.
- It is not completely clear what values to use in the LASS parameterisation

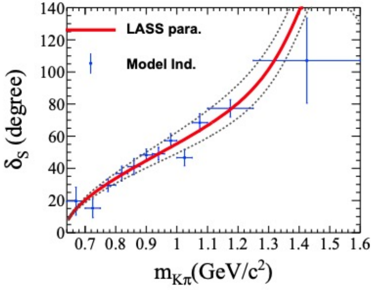
$$\cot \delta_B = \frac{1}{ak} + \frac{rk}{2}$$

- With more data we can look to float these parameters
- or perhaps we should look at model-independent approaches...



Model-independent S-wave fit?

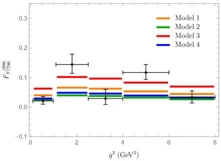
- For $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$ LHCb do not have enough statistics to further split our data set into bins of $m(K\pi)$ and run independent fits.
- Is it possible to fit the S-wave amplitude and phase with a generalised polynomial of some order?
- One could share P-wave information across all $m(K\pi)$ bins
- Using Watson's theorem it would be possible to fix the S-wave phase in each $m(K\pi)$ bin - certainly possible at low $m(K\pi)$?
  - BESIII $D^+ \to K^-\pi^+e^+\nu_e$ use a model-independent approach [1512.08627], they find it the phase agrees with the LASS parameterisation
  - *Using Watson's theorem, for the same isospin and angular momentum the phase as measured in $K\pi$ elastic scattering and a decay channel will be equal in the elastic regime (low $m(K\pi)$)*

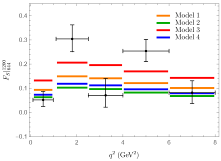


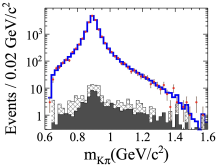
Conclusion

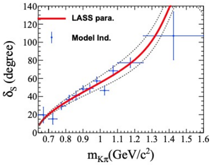
- Discussed light-cone sum rule predictions for the S-wave contribution to $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$
- Highlighted some potential issues for $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$ relating to the $m(K\pi)$ lineshape
  - Some of which may not yet be relevant in comparison to statistical uncertainties
- Made some suggestions of things that could be tried
- Are there any other channels from which we can extract information useful for $B^0 \to K^{*0}(\to K^+\pi^-)\mu^+\mu^-$ analyses?

Thanks for listening









Backup

S-wave $B \to K\pi$ form factors

- Generated by the axial-vector and pseudotensor $b \to s$ transition currents

$$j_A^\mu = \bar{s}\gamma^\mu\gamma_5 b\,, \quad j_T^\mu = \bar{s}\sigma^{\mu\nu}q_\nu\gamma_5 b\,.$$

- Form factors $F_i(k^2, q^2, q\cdot\bar{k})$ defined as

$$\begin{aligned} -i\langle K^-(k_1)\pi^+(k_2)|\bar{s}\gamma^\mu\gamma_5 b|\bar{B}^0(p)\rangle &= F_t\, k_t^\mu + F_0\, k_0^\mu + \ldots\,, \\ \langle K^-(k_1)\pi^+(k_2)|\bar{s}\sigma^{\mu\nu}q_\nu\gamma_5 b|\bar{B}^0(p)\rangle &= F_0^T\, k_0^\mu + \ldots \end{aligned}$$

- S-wave isolated via partial wave expansion:

$$F_{0,t}(k^2, q^2, q\cdot\bar{k}) = F_{0,t}^{(\ell=0)}(k^2,q^2) + \sum_{\ell=1}^{\infty}\sqrt{2\ell+1}\,F_{0,t}^{(\ell)}(k^2,q^2)\;P_\ell^{(0)}(\cos\theta_K)\,,$$

- In progress: LCSR expressions for $B \to (K\pi)_S$ form factors $F_{0,t}^{(\ell=0)}$ and $F_0^{T(\ell=0)}$

- Start with correlation function:

$$\Pi_b(k,q) = i\int d^4x\, e^{ik\cdot x}\langle 0|\mathrm{T}\{j_S(x), j_b(0)\}|\bar{B}^0(q+k)\rangle\,,$$

- Use dispersion relation in the variable k^2:

$$\Pi^{(\mathrm{OPE})}(k^2,q^2) = \frac{1}{\pi}\int\limits_{(m_K+m_\pi)^2}^{\infty} ds\,\frac{\mathrm{Im}\Pi(s,q^2)}{s-k^2}\,.$$

- Obtain spectral density by inserting a full set of states

$$2\,\mathrm{Im}\Pi_b^{(K\pi)}(k,q) = \sum_{K\pi}\int d\tau_{K\pi}\langle 0|j_S\,|K(k_1)\pi(k_2)\rangle^*\langle K(k_1)\pi(k_2)|j_b|\bar{B}^0(q+k)\rangle\,,$$

$$\mathrm{Im}\Pi(s,q^2) = \mathrm{Im}\Pi^{(K\pi)}(s,q^2) + \mathrm{Im}\Pi^{(h)}(s,q^2)\theta(s-s_h)\,.$$

- Assume quark-hadron duality

$$\int_{s_h}^{\infty} ds\, \frac{\mathrm{Im}\prod^{(h)}(s,q^2)}{s-k^2} = \int_{s_0}^{\infty} ds\, \frac{\mathrm{Im}\prod^{(OPE)}(s,q^2)}{s-k^2}\,,$$

- Perform Borel transformation in the variable k^2

$$\frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{s_0} ds\, e^{-s/M^2}\, \mathrm{Im}\prod^{(K\pi)}(s,q^2) = \frac{1}{\pi}\int_{m_s^2}^{s_0} ds\, e^{-s/M^2}\, \mathrm{Im}\prod^{(\mathrm{OPE})}(s,q^2)$$

$$\equiv \prod^{(\mathrm{OPE})}(q^2,s_0,M^2)$$

- $\prod^{\mathrm{OPE}}(q^2,s_0,M^2)$ OPE expression after subtracting the above-threshold contribution from the dispersive integral