Constraining matrix elements for BSM searches with dispersion relations

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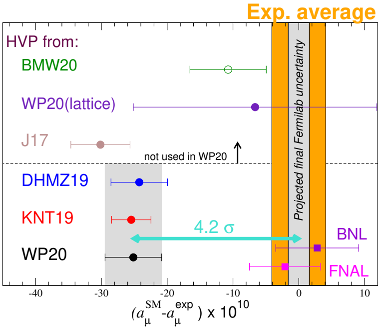
Seminar talk

University of Siegen

Role of hadronic matrix elements at the precision frontier

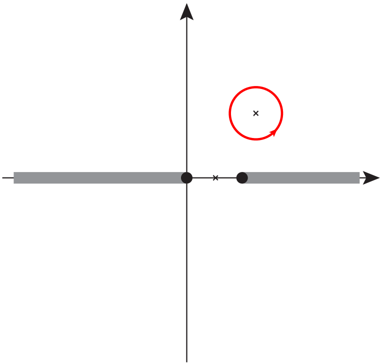
- An obvious point in **low-energy precision observables**: want to constrain quark-level operators, but measure hadrons
- Transition involves **hadronic matrix elements**
  - Effective field theories
  - Lattice QCD
  - **Dispersion relations**
- Examples include
  - Hadronic corrections to $(g-2)_\mu$
  - Direct-detection searches for dark matter (or any other nuclear probe)
  - Flavor physics: B, D, K decays
  - ...



From Cauchy's theorem to dispersion relations

- **Cauchy's theorem**

$$f(s) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{ds'\, f(s')}{s'-s}$$



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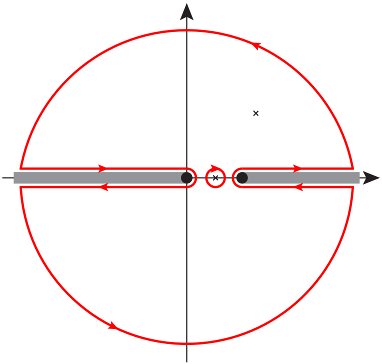
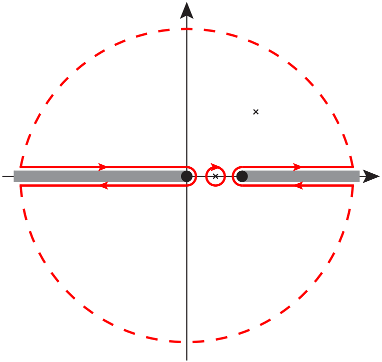


From Cauchy's theorem to dispersion relations

- **Dispersion relation**

$$f(s) = \frac{g}{s - M^2} + \frac{1}{\pi}\int_{\text{cuts}} \frac{ds' \,\text{Im}\, f(s')}{s' - s}$$

$\hookrightarrow$ **analyticity**



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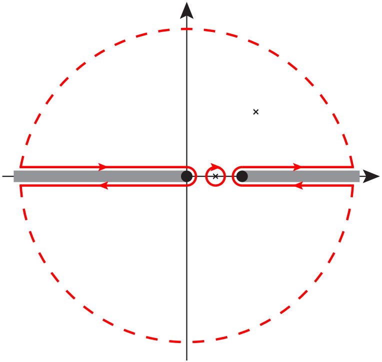
- Subtractions

$$f(s) = \frac{g}{s-M^2} + \underbrace{C}_{f(0)+\frac{g}{M^2}} + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds'\,\text{Im}\, f(s')}{s'(s'-s)}$$



From Cauchy's theorem to dispersion relations

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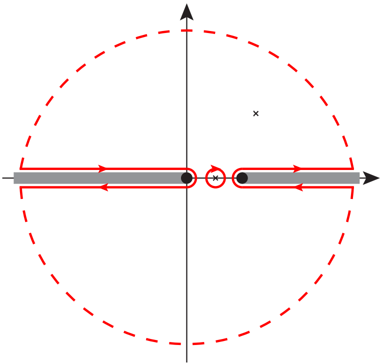
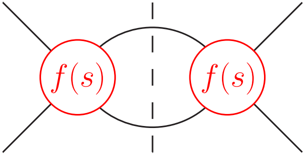
- Imaginary part from **Cutkosky rules**

$\hookrightarrow$ forward direction: **optical theorem**

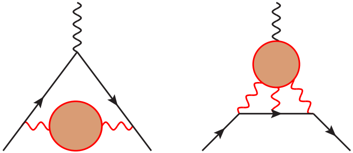
- **Unitarity** for partial waves: $\text{Im}\,f(s) = \rho(s)|f(s)|^2$

- Residue g **reaction-independent**





Hadronic effects in $(g-2)_\mu$



- **Hadronic vacuum polarization**: need hadronic two-point function

$$\Pi_{\mu\nu} = \langle 0 | T\{j_\mu j_\nu\} | 0 \rangle$$

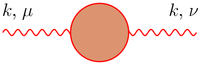
- **Hadronic light-by-light scattering**: need hadronic four-point function

$$\Pi_{\mu\nu\lambda\sigma} = \langle 0 | T\{j_\mu j_\nu j_\lambda j_\sigma\} | 0 \rangle$$

Master formula for HVP contribution to a_μ

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{thr}}}^{\infty} ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s)$$

- General principles yield **direct connection with experiment**
  - **Gauge invariance**



$$= -i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2)$$

  - **Analyticity**

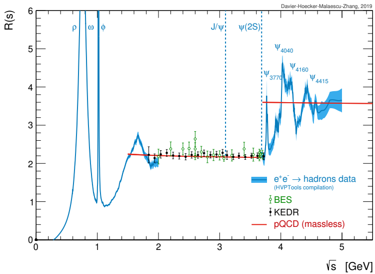
$$\Pi_{\text{ren}} = \Pi(k^2) - \Pi(0) = \frac{k^2}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s(s-k^2)}$$

  - **Unitarity**

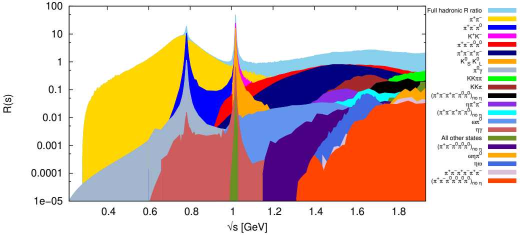
$$\operatorname{Im}\Pi(s) = -\frac{s}{4\pi\alpha}\sigma_{\text{tot}}\left(e^+e^- \to \text{hadrons}\right) = -\frac{\alpha}{3}R_{\text{had}}(s)$$

$\hookrightarrow$ one kinematic variable, one scalar function, no subtractions

Hadronic vacuum polarization from e^+e^- data



Davier, Hoecker, Malaescu, Zhang 2019



Keshavarzi, Nomura, Teubner 2018

- Decades-long effort to measure e^+e^- cross sections
  - cross sections defined photon-inclusively

    $\hookrightarrow$ threshold $s_{\text{thr}} = M_{\pi^0}^2$ due to $\pi^0\gamma$ channel
  - up to about 2 GeV: sum of exclusive channels
  - above: inclusive data + narrow resonances + pQCD
- **Tensions in the data**: most notably between KLOE and BaBar 2π data

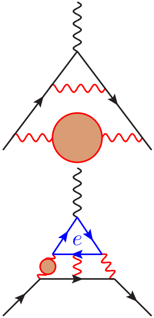
  $\hookrightarrow$ extensive discussion in WP of current status and consequences

Data-driven determination of HVP: our recommendation from WP20

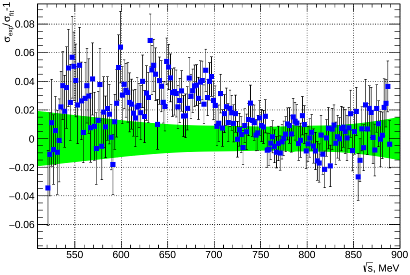
HVP from e^+e^- data

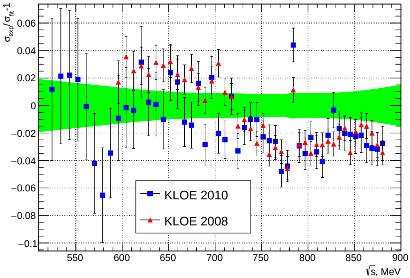
$$a_\mu^{\text{HVP, LO}} = 6931(28)_{\text{exp}}(28)_{\text{sys}}(7)_{\text{DV+QCD}} \times 10^{-11} = 6931(40) \times 10^{-11}$$
$$a_\mu^{\text{HVP}} = 6845(40) \times 10^{-11}$$

- DV+QCD: comparison of inclusive data and pQCD in transition region
- Sensitivity of the data is better than the quoted error
  $\hookrightarrow$ would get $4.2\sigma \to 4.8\sigma$ when ignoring additional systematics
- Systematic effect dominated by [fit w/o KLOE - fit w/o BaBar]/2
- a_μ^{HVP} includes NLO Calmet et al. 1976 and NNLO Kurz et al. 2014 iterations



New data since WP20 (prior to CMD-3)

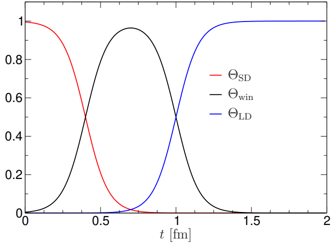


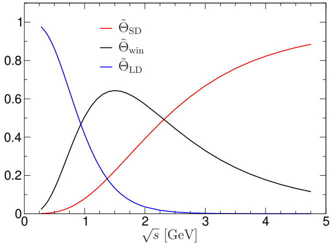


BaBar vs. SND 20 SND 2020 KLOE vs. SND 20

- New data from SND experiment not yet included in WP20 number
  $\hookrightarrow$ lie between BaBar and KLOE
- New data for 3π: BESIII, BaBar
- New data on inclusive region: BESIII (slight tension with pQCD)

Windows in Euclidean time





- BMWc still **only complete calculation** at similar level of precision as e^+e^- data

$$a_\mu^{\text{HVP, LO}}[e^+e^-] = 6931(40)\times 10^{-11} \qquad a_\mu^{\text{HVP, LO}}[\text{BMWc}] = 7075(55)\times 10^{-11}$$

$\hookrightarrow$ globally 2.1σ

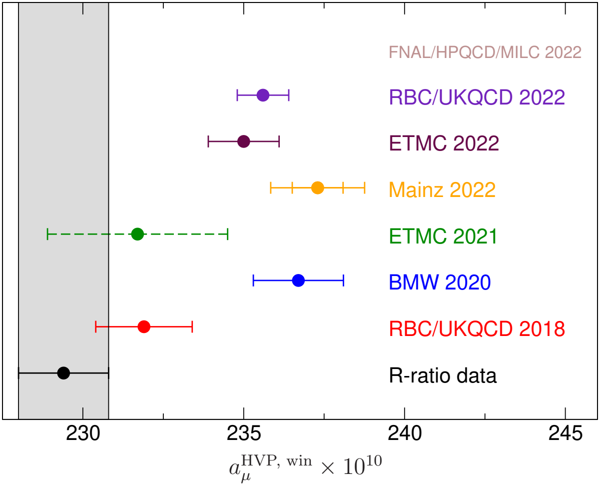
- Idea RBC/UKQCD 2018: define partial quantities

$$a_\mu^{\text{HVP, LO, win}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\rm thr}}^\infty ds \frac{\hat K(s)}{s^2} R_{\rm had}(s)\tilde\Theta_{\rm win}(s)$$

$\hookrightarrow$ smaller systematic errors for same quantity in lattice QCD

$\hookrightarrow$ tool for the comparison to e^+e^- data

A puzzle in the intermediate window: e^+e^- vs. lattice QCD



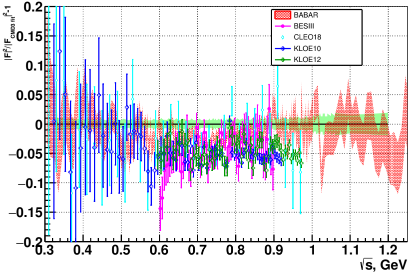
RBC/UKQCD 2022 supersedes RBC/UKQCD 2018

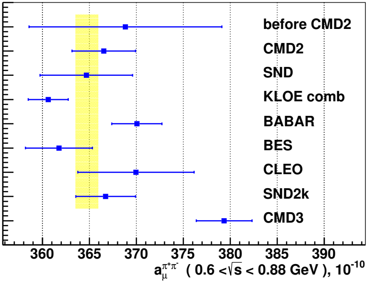
ETMC 2022 supersedes ETMC 2021

FNAL/HPQCD/MILC 2022 agrees for *ud* connected contribution, same for Aubin et al. 2022, χQCD 2022

R-ratio result from Colangelo et al. 2022

A new puzzle: $e^+e^- \to \pi^+\pi^-$ from CMD-3





CMD-3, 2302.08834

generally shows larger pion form factor in the whole energy range under discussion. The most significant difference to other energy scan measurements, including previous CMD-2 measurement, is observed at the left side of ρ-meson ($\sqrt{s} = 0.6 - 0.75$ GeV), where it reach up to 5%, well beyond the combined systematic and statistical errors of the new and previous results. The source of this difference is unknown at the moment.

Where to go from here?

- **Need to understand the details of CMD-3 result**

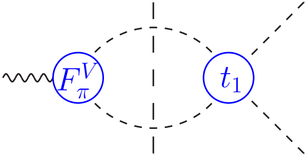
  $\hookrightarrow$ seminar + discussion (online) organized by TI `https://indico.fnal.gov/event/59052/`

- Next plenary meeting in Bern (4–8 Sep 2023) `https://indico.cern.ch/event/1258310/`
- New data on the 2π channel forthcoming:
  - New BaBar and KLOE analyses (a lot more data not analyzed so far)
  - Full statistics of SND
  - New data from BESIII and Belle II
- In addition:
  - **Improved lattice-QCD calculations for full HVP**, more windows
  - Further scrutiny of radiative corrections
  - Potentially τ data to be resurrected as a viable cross check if **progress on isospin breaking** allows (lattice QCD, dispersive)
  - Independent HVP determination from **MuonE**

Back to dispersion relations: the electromagnetic form factor of the pion

- $e^+e^- \to 2\pi$ determined by pion vector form factor F_π^V
- **Unitarity** for **pion vector form factor**

$$\text{Im}\, F_\pi^V(s) = \theta\big(s - 4M_\pi^2\big)\, F_\pi^V(s) e^{-i\delta_1^1(s)} \sin \delta_1^1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_π^V equals $\pi\pi$ P-wave phase δ_1 Watson 1954

- Solution in terms of Omnès function

$$\text{Im}\, F_\pi^V(s) = P(s)\Omega_1^1(s) \qquad \Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

- Implementation in practice
  - Where to get the phase shift δ_1^1 $\Rightarrow$ Roy equations
  - Isospin breaking $\Rightarrow$ ρ–ω mixing
  - Inelastic states $\Rightarrow$ mostly 4π, constrained by Eidelman–Łukaszuk bound

Some comments on CMD-3 from analyticity and unitarity constraints

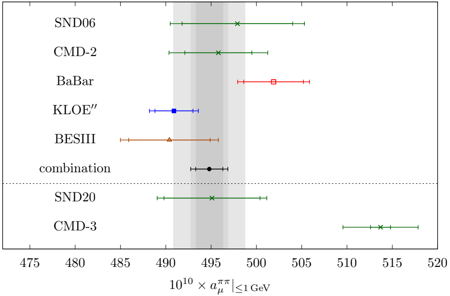
The pion form factor from dispersion relations

$$F_\pi^V(s) = \underbrace{\Omega_1^1(s)}_{\text{elastic } \pi\pi \text{ scattering}} \times \underbrace{G_\omega(s)}_{\text{isospin-breaking } 3\pi \text{ cut}} \times \underbrace{G_{\text{in}}(s)}_{\text{inelastic effects: } 4\pi, \ldots}$$

- $e^+e^- \to \pi^+\pi^-$ cross section subject to strong constraints from **analyticity**, **unitarity**, **crossing symmetry**, leading to dispersive representation with few parameters Colangelo, MH, Stoffer, 2018, 2021, 2022, work in progress
  - **Elastic $\pi\pi$ scattering**: two values of phase shifts
  - **ρ–ω mixing**: ω pole parameters and residue
  - **Inelastic states**: conformal polynomial

$\hookrightarrow$ cross check on data, functional form for all $s \leq 1\,\text{GeV}^2$

Some comments on CMD-3 from analyticity and unitarity constraints

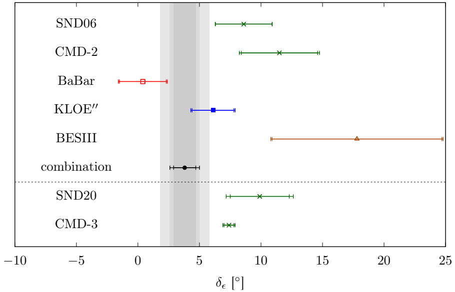


	$a_\mu^{\pi\pi}\big\|_{\leq 1\,\text{GeV}}$	$a_\mu^{\pi\pi}\big\|_{[0.60,0.88]\,\text{GeV}}$	$a_\mu^{\pi\pi}\big\|_{\text{win}}$
SND06	1.7σ	1.8σ	1.7σ
CMD-2	2.0σ	2.3σ	2.1σ
BaBar	2.9σ	3.3σ	3.1σ
KLOE″	4.8σ	5.6σ	5.4σ
BESIII	2.8σ	3.0σ	3.1σ
SND20	2.1σ	2.2σ	2.2σ
comb	3.7σ [5.0σ]	4.2σ [6.1σ]	3.8σ [5.7σ]

- Tensions in $a_\mu^{\pi\pi}\big|_{\leq 1\,\text{GeV}}$ compared to CMD-3:
  - Inner/outer error: experiment/total (also shown: combination + BaBar/KLOE error)
  - Theory error dominated by order in conformal polynomial N
- No red flags for CMD-3 so far, but:
  - Large systematic error from N, correlated/anticorrelated for BaBar/other experiments
  - $\pi\pi$ phase shifts remain reasonable, main change in conformal polynomial

$\hookrightarrow$ suggests that inelastic effects could give a handle on the tension

Some comments on CMD-3 from analyticity and unitarity constraints

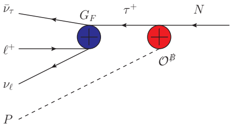


- Can also study consistency of hadronic parameters
  $\hookrightarrow$ **phase of the ρ–ω mixing parameter δ_ϵ**
  - δ_ϵ observable, since defined as a phase of a residue
  - δ_ϵ vanishes in isospin limit, but can be non-vanishing due to $\rho \to \pi^0\gamma, \eta\gamma, \pi\pi\gamma, \ldots \to \omega$
  - Combined-fit $\delta_\epsilon = 3.8(2.0)[1.2]^\circ$ agrees well with narrow-width expectation $\delta_\epsilon = 3.5(1.0)^\circ$, but **considerable spread among experiments**
  - Mass of the ω systematically too low compared to $e^+e^- \to 3\pi$

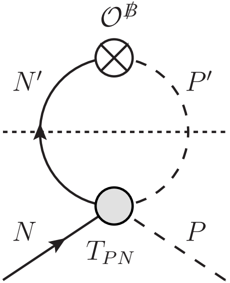
Matrix elements for nucleon decay

- **Operator basis** for nucleon decay in SMEFT

$$
\begin{aligned}
Q_{duq} &= \varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[\left(d_p^{\alpha}\right)^T C u_r^{\beta}\right]\left[\left(q_s^{\gamma j}\right)^T C L_t^k\right] \\
Q_{qqu} &= \varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[\left(q_p^{\alpha j}\right)^T C q_r^{\beta k}\right]\left[\left(u_s^{\gamma}\right)^T C e_t\right] \\
Q_{qqq} &= \varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}\left[\left(q_p^{\alpha j}\right)^T C q_r^{\beta k}\right]\left[\left(q_s^{\gamma m}\right)^T C L_t^n\right] \\
Q_{duu} &= \varepsilon^{\alpha\beta\gamma}\left[\left(d_p^{\alpha}\right)^T C u_r^{\beta}\right]\left[\left(u_s^{\gamma}\right)^T C e_t\right]
\end{aligned}
$$



- For most operators dominant limits from two-body decays
  $\hookrightarrow p \to \pi^0 e^+, \ldots$
- Exception: operators with τ require off-shell processes such as $p \to \pi^0 \ell^+ \nu_\ell \bar{\nu}_\tau$
- Momentum dependence of the form factors from dispersion relations $\Rightarrow$ **pion–nucleon rescattering**



Matrix elements for nucleon decay: normalization

X_i	$W_0^{X_{iL}}(0)$	$W_1^{X_{iL}}(0)$	$W_0^{X_{iR}}(0)$	$W_1^{X_{iR}}(0)$
U_1	0.151(31)	−0.134(18)	−0.159(35)	0.169(37)
S_1	0.043(4)	0.028(7)	0.085(12)	−0.026(4)
S_2	0.028(4)	−0.049(7)	−0.040(6)	0.053(7)
S_3	0.101(11)	−0.075(13)	−0.109(19)	0.080(17)
S_4	−0.072(8)	0.024(6)	−0.044(5)	−0.026(6)
S_{1+2+4}	0.000(0)	0.000(0)	0.000(0)	0.000(0)
S_{2-3-4}	0.000(0)	0.000(0)	0.112(15)	0.000(12)

Yoo et al. 2022

- Normalizations from lattice QCD

$$\langle\pi^0|\left[\bar{u}^c P_A d\right] u_B|p\rangle = \frac{1}{\sqrt{2}}\langle\pi^+|\left[\bar{u}^c P_A d\right] d_B|p\rangle \equiv \frac{1}{\sqrt{2}} U_1^{AB}$$

$$\langle K^0|\left[\bar{u}^c P_A s\right] u_B|p\rangle \equiv S_1^{AB} \quad \langle K^+|\left[\bar{u}^c P_A s\right] d_B|p\rangle \equiv S_2^{AB} \quad \langle K^+|\left[\bar{u}^c P_A d\right] s_B|p\rangle \equiv S_3^{AB} \quad \langle K^+|\left[\bar{d}^c P_A s\right] u_B|p\rangle \equiv S_4^{AB}$$

- Two form factors: $X_i^{AB} = P_B\left[W_0^{X_i^{AB}}(s) + \frac{\not{q}}{m_N} W_1^{X_i^{AB}}(s)\right] u_N(p)$, write $X_{iA} \equiv X_i^{AL}$
- Found two new relations:

$$S_{1A} + S_{2A} + S_{4A} = 0 \text{ (isospin)} \qquad S_{2L} - S_{3L} - S_{4L} = 0 \text{ (Fierz)}$$

Matrix elements for nucleon decay: momentum dependence

- For which scalar functions should one write dispersion relations?
  - Need to **avoid kinematic singularities and zeros**: $W_0(s)$, $W_1(s)$
  - Would like **simple unitary relations**: $W_\pm(s) = W_0(s) \pm \frac{\sqrt{s}}{m_N} W_1(s)$ because

$$\text{Im}\, W_+(s) = W_+(s) e^{-i\delta_{0+}(s)} \sin\delta_{0+}(s) \qquad \text{Im}\, W_-(s) = W_-(s) e^{-i\delta_{1-}(s)} \sin\delta_{1-}(s)$$

    with πN phase shifts $\delta_{\ell\pm}$, $j = \ell \pm 1/2$

  - Further constraint from **baryon-pole diagrams** (from ChPT Aoki et al. 2000)

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  with πN phase shifts $\delta_{\ell\pm}$, $j = \ell \pm 1/2$
  - Further constraint from **baryon-pole diagrams** (from ChPT Aoki et al. 2000)
- Our solution Crivellin, MH 2023

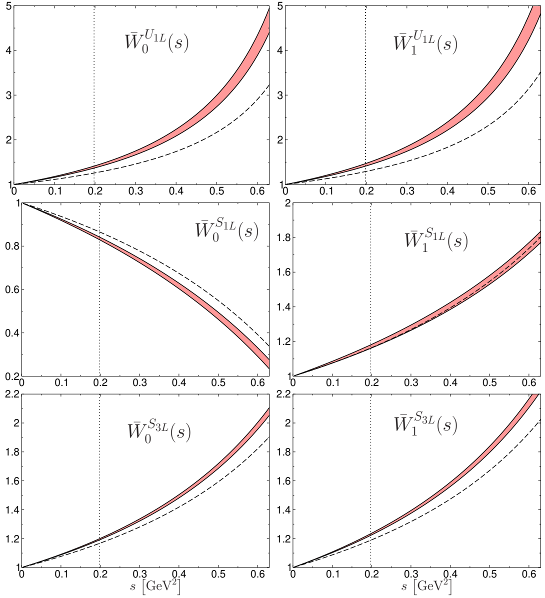
$$W_0(s) = W_0(0)\Big[(1-\alpha)\Omega_{0+}(s) + \alpha \frac{m_B^2}{m_B^2 - s}\Omega_{1-}(s)\Big] \qquad m_B \in \{m_N, m_\Lambda, m_\Sigma\}$$

$$W_+(s)W_-(s) = [W_0(s)]^2 - \frac{s}{m_N^2}[W_1(s)]^2 = [W_0(0)]^2\Omega_{0+}(s)\Omega_{1-}(s)\frac{m_B^2}{m_B^2 - s}(1+\beta s)$$

$$\alpha = -\frac{m_B}{m_N}\frac{W_1(0)}{W_0(0)} \qquad \beta = (1-2\alpha)\Big[\dot{\Omega}_{0+} - \dot{\Omega}_{1-} - \frac{1}{m_B^2}\Big] - \frac{[W_1(0)]^2}{m_N^2[W_0(0)]^2}$$

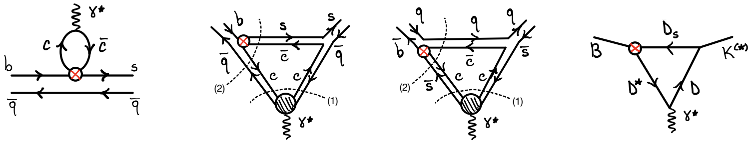
$\hookrightarrow$ implements **normalization, unitarity, and chiral constraints**

Matrix elements for nucleon decay: momentum dependence



- Typical limits:
  - Two-body decays:
    $|C_i| \lesssim (10^{-15}/\,\text{GeV})^2$
  - Four-body decays:
    $|C_i| \lesssim (10^{-10}/\,\text{GeV})^2$
    $\hookrightarrow$ phase space and G_F
- Closes flat directions for τ operators

Matrix elements for $B \to K^{(*)}\gamma^*$



(a) (b) (c)

Ciuchini et al. 2022

- All cases so far: **"normal" thresholds** expected from unitarity

  $\hookrightarrow$ dispersion integral starts at $s = (m_1 + m_2)^2$ for a two-body intermediate state with masses m_1 and m_2

- **Anomalous thresholds** can arise when Landau singularities move onto first Riemann sheet

  $\hookrightarrow$ sufficiently heavy external states, light "left-hand cut"

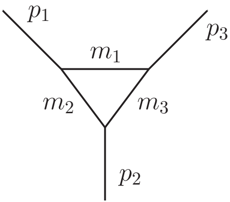
- Recently pointed out in the context of $B \to K^{(*)}\gamma^*$ due to D_s left-hand cut

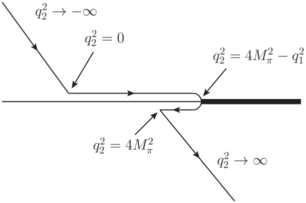
- Here: some vague ideas how one could try to estimate such diagrams

Anomalous thresholds: general case

- Consider the scalar loop function $C_0(s)$, $s = p_2^2$
- Fulfills the dispersion relation

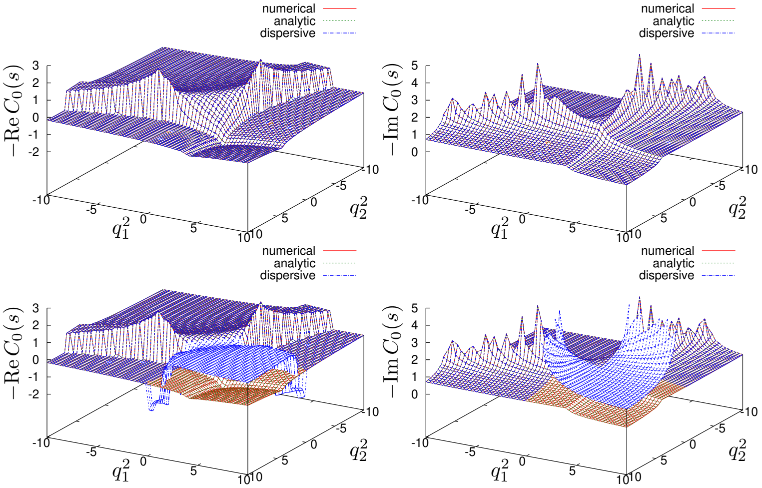
$$C_0(s) = \frac{1}{2\pi i}\int_{(m_2+m_3)^2}^{\infty} ds' \frac{\text{disc}\, C_0(s')}{s'-s}$$
$$+\theta\Big[m_3 p_1^2 + m_2 p_3^2 - (m_2+m_3)(m_1^2 + m_2 m_3)\Big]$$
$$\times \frac{1}{2\pi i}\int_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, C_0(s_x)}{s_x - s}$$
$$s_x = x(m_2+m_3)^2 + (1-x)s_+$$
$$s_+ = p_1^2\frac{m_1^2+m_3^2}{2m_1^2} + p_3^2\frac{m_1^2+m_2^2}{2m_1^2} - \frac{p_1^2 p_3^2}{2m_1^2} - \frac{(m_1^2-m_2^2)(m_1^2-m_3^2)}{2m_1^2}$$
$$+\frac{1}{2m_1^2}\sqrt{\lambda(p_1^2, m_1^2, m_2^2)\,\lambda(p_3^2, m_1^2, m_3^2)}$$





- Anomalous piece parameterizes the contour deformation from threshold to s_+

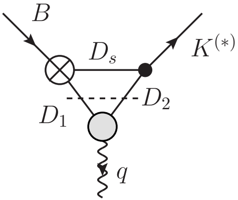
 Anomalous thresholds: an example from HLbL scattering



- Example for $q_1^2 = q_2^2$, $m_1 = m_2 = m_3 = M_\pi$ MH, Colangelo, Procura, Stoffer 2013

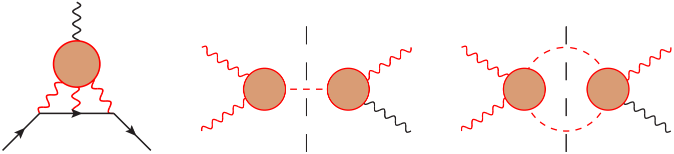
Anomalous thresholds: towards estimates for P_5'

- Observations:
  - Discontinuity in q^2 depends on D-meson form factor $F_D(s)$ and $B \to D\bar{D}K^{(*)}$ P-wave(s)
  - The partial-wave projection of the $B \to D\bar{D}K^{(*)}$ amplitude generates the same logarithm responsible for the anomalous singularities in $C_0(s)$
  - Could evaluate the dispersion relation including anomalous piece if the spectral function of $F_D(s)$ and couplings in D_s exchange were known

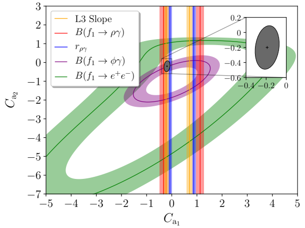


- Questions:
  - What is the relevant dynamical content of $F_D(s)$ and $B \to D\bar{D}K^{(*)}$? How big an error would one make if the decay width were assumed to be saturated by D_s exchange?
  - How would one combine the result with the existing calculations of the $B \to K^{(*)}\gamma^*$ matrix elements while avoiding double counting?

Hadronic light-by-light scattering: data-driven, dispersive evaluations

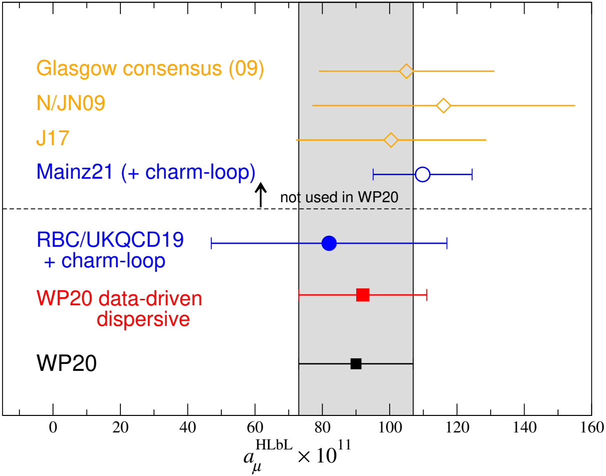


- Organized in terms of **hadronic intermediate states**, in close analogy to HVP Colangelo et al. 2014, ...
- Leading channels implemented with **data input for $\gamma^*\gamma^* \to$ hadrons**, e.g., $\pi^0 \to \gamma^*\gamma^*$
- Uncertainty dominated by subleading channels
  $\hookrightarrow$ **axial-vector mesons** $f_1(1285)$, $f_1(1420)$, $a_1(1260)$
- Transition form factors accessible in e^+e^- collisions
  $\hookrightarrow$ BESIII, Belle II (?)



Zanke, MH, Kubis 2021

Hadronic light-by-light scattering: status



- Lattice QCD Mainz 2021, 2022:

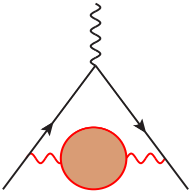
$$a_\mu^{\text{HLbL}}[uds] = 107(15) \times 10^{-11}$$
$$a_\mu^{\text{HLbL}}[c] = 2.8(5) \times 10^{-11}$$

- Preliminary update from RBC/UKQCD 2022 also looks consistent

- Good agreement between lattice QCD and phenomenology at $\simeq 20 \times 10^{-11}$
- Need another factor of 2 for final Fermilab precision work in progress

Summary and outlook

- **Muon $g-2$**: dispersive approaches to HVP and HLbL
  - For HLbL agreement between lattice and phenomenology
    $\hookrightarrow$ another factor 2 looks feasible
  - HVP: puzzles in intermediate window and with CMD-3
  - New e^+e^- data and lattice calculations forthcoming
- Rescattering corrections to **proton-decay matrix elements**
- Some (vague) ideas to estimate the impact of anomalous thresholds on P_5'



Sixth plenary TI workshop

Relation to global electroweak fit

Hadronic running of α

$$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = \frac{\alpha M_Z^2}{3\pi} P \int_{s_{\text{thr}}}^{\infty} ds \frac{R_{\text{had}}(s)}{s(M_Z^2 - s)}$$

- $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ enters as input in **global electroweak fit**

  $\hookrightarrow$ integral weighted more strongly towards high energy Passera, Marciano, Sirlin 2008

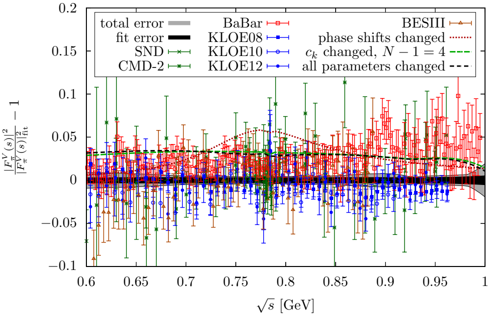
- Changes in $R_{\text{had}}(s)$ have to occur at low energies, $\lesssim 2\,\text{GeV}$ Crivellin et al. 2020, Keshavarzi et al. 2020, Malaescu et al. 2020

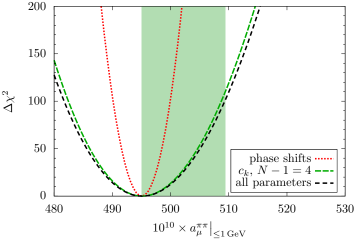
- This seems to happen for BMWc calculation (translated from the space-like), with only moderate increase of tensions in the electroweak fit ($\sim 1.8\sigma \to 2.4\sigma$)

  $\hookrightarrow$ need **large changes in low-energy cross section**

- Similar conclusion from Mainz 2022 calculation of hadronic running

Changing the $\pi\pi$ cross section below 1 GeV



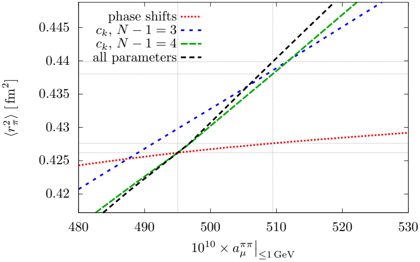


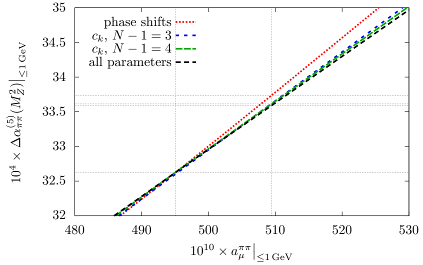
Colangelo, MH, Stoffer 2020

- Changes in 2π cross section **cannot be arbitrary** due to analyticity/unitarity constraints, but increase is actually possible
- Three scenarios:
  1. "Low-energy" scenario: **$\pi\pi$ phase shifts**
  2. "High-energy" scenario: **conformal polynomial**
  3. Combined scenario

$\hookrightarrow$ 2. and 3. lead to uniform shift, 1. concentrated in ρ region

Correlations





Correlations with other observables:

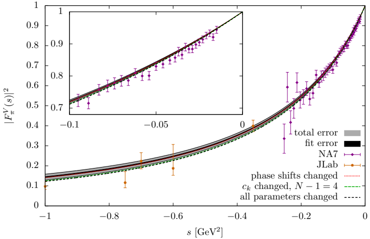
- **Pion charge radius $\langle r_\pi^2 \rangle$**

  ↪ significant change in scenarios 2. and 3.

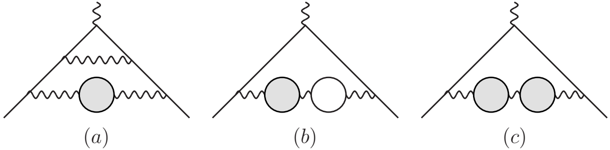
  ↪ can be tested in lattice QCD

- **Hadronic running of α**

- **Space-like pion form factor**



FAQ 1: do e^+e^- data and lattice really measure the same thing?



- Conventions for **bare cross section**
  - Includes radiative intermediate states and final-state radiation: $\pi^0\gamma$, $\eta\gamma$, $\pi\pi\gamma$, ...
  - Initial-state radiation and VP subtracted to avoid double counting
- NLO HVP insertions

$$a_\mu^{\text{HVP, NLO}} \simeq [\underbrace{-20.7}_{(a)} + \underbrace{10.6}_{(b)} + \underbrace{0.3}_{(c)}] \times 10^{-10} = -9.8 \times 10^{-10}$$

$\hookrightarrow$ dominant VP effect from leptons, HVP iteration very small

- Important point: **no need to specify hadronic resonances**

$\hookrightarrow$ calculation set up in terms of decay channels

FAQ 1: do e^+e^- data and lattice really measure the same thing?

- HVP in subtraction determined iteratively (converges with α) and self-consistently

$$\alpha(q^2) = \frac{\alpha(0)}{1 - \Delta\alpha_{\text{lep}}(q^2) - \Delta\alpha_{\text{had}}(q^2)} \qquad \Delta\alpha_{\text{had}}(q^2) = -\frac{\alpha q^2}{3\pi} P \int_{s_{\text{thr}}}^{\infty} ds \frac{R_{\text{had}}(s)}{s(s-q^2)}$$

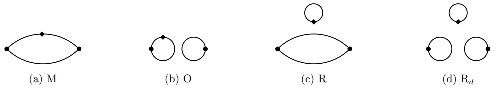
- Subtlety for very narrow $c\bar{c}$ and $b\bar{b}$ resonances (ω and ϕ perfectly fine)

  $\hookrightarrow$ Dyson series does not converge Jegerlehner

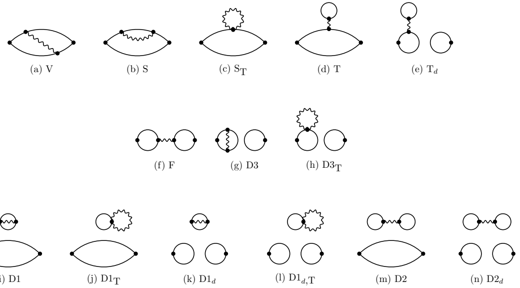
- Solution: take out resonance that is being corrected in R_{had} in VP undressing
- How to match all of this on the lattice?
- Need to calculate all sorts of **isospin-breaking (IB) corrections**

  $\hookrightarrow$ e^2 (QED) and $\delta = m_u - m_d$ (strong IB) corrections

- **Strong isospin breaking** $\propto m_u - m_d$



- **QED effects** $\propto \alpha$



plots from Gülpers et al. 2018

- Diagram (f) F critical for consistent VP subtraction

  $\hookrightarrow$ same diagram without additional gluons is subtracted RBC/UKQCD 2018

FAQ 1: do e^+e^- data and lattice really measure the same thing?

	SD window		int window		LD window		full HVP	
	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$
$\pi^0\gamma$	0.16(0)	–	1.52(2)	–	2.70(4)	–	4.38(6)	–
$\eta\gamma$	0.05(0)	–	0.34(1)	–	0.31(1)	–	0.70(2)	–
ρ–ω mixing	–	0.05(0)	–	0.83(6)	–	2.79(11)	–	3.68(17)
FSR (2π)	0.11(0)	–	1.17(1)	–	3.14(3)	–	4.42(4)	–
M_{π^0} vs. $M_{\pi^\pm}$ (2π)	0.04(1)	–	−0.09(7)	–	−7.62(14)	–	−7.67(22)	–
FSR (K^+K^-)	0.07(0)	–	0.39(2)	–	0.29(2)	–	0.75(4)	–
kaon mass (K^+K^-)	−0.29(1)	0.44(2)	−1.71(9)	2.63(14)	−1.24(6)	1.91(10)	−3.24(17)	4.98(26)
kaon mass $(\bar{K}^0K^0)$	0.00(0)	−0.41(2)	−0.01(0)	−2.44(12)	−0.01(0)	−1.78(9)	−0.02(0)	−4.62(23)
total	0.14(1)	0.08(3)	1.61(12)	1.02(20)	−2.44(16)	2.92(17)	−0.68(29)	4.04(39)
BMWc 2020	–	–	−0.09(6)	0.52(4)	–	–	−1.5(6)	1.9(1.2)
RBC/UKQCD 2018	–	–	0.0(2)	0.1(3)	–	–	−1.0(6.6)	10.6(8.0)
JLM 2021	–	–	–	–	–	–	–	3.32(89)

- Note: error estimates only refer to the effects included

  $\hookrightarrow$ **additional channels missing** (most relevant for SD and int window)

- Reasonable agreement with BMWc 2020, RBC/UKQCD 2018, and James, Lewis, Maltman 2021

  $\hookrightarrow$ if anything, the result would become even larger with pheno estimates

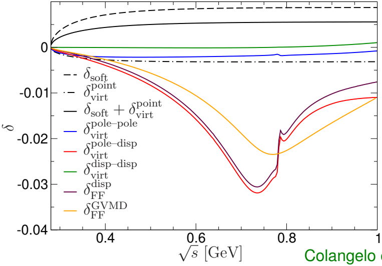
FAQ 2: can we trust radiative corrections/MC generators?

- Typical objection: can we really trust scalar QED in the MC generator?
- Report by Working Group on Radiative Corrections and Monte Carlo Generators for Low Energies

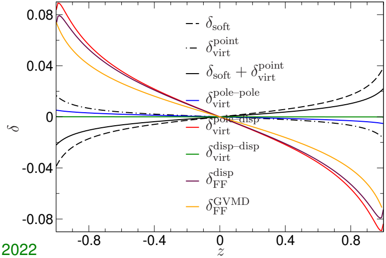
  $\hookrightarrow$ Quest for precision in hadronic cross sections at low energy: Monte Carlo tools vs. experimental data (0912.0749)
- Never just use scalar QED, include pion form factor wherever possible

  $\hookrightarrow$ **FsQED**
- From the point of view of dispersion relations, this captures the **leading infrared enhanced effects**
- Existing NLO calculations do not point to (significant) center-of-mass-energy dependent effects Campanario et al. 2019
- Could there be subtleties in how the form factor is implemented or from pion rescattering?

FAQ 2: can we trust radiative corrections/MC generators?



Colangelo et al. 2022



- Test case: **forward–backward asymmetry** (*C*-odd)
- Large corrections found in GVMD model Ignatov, Lee 2022
- Can be reproduced using dispersion relations

  ↪ effect still comes from **infrared enhanced contributions**
- Relevant effects for the *C*-even contribution?

FAQ 3: what about the τ data?

- Why did people stop using $\tau \to \pi\pi\nu_\tau$ data?
  - Better precision from e^+e^-
  - **IB corrections not under sufficient control**
- If this issue could be solved, would yield very useful cross check

  $\hookrightarrow$ new data at least on spectrum from Belle II
- New developments from the lattice talk by M. Bruno at Edinburgh

  $\hookrightarrow$ re-using HLbL lattice data
- Long-distance QED (G_{EM}) still taken from phenomenology for the time being

  $\hookrightarrow$ dispersive methods?

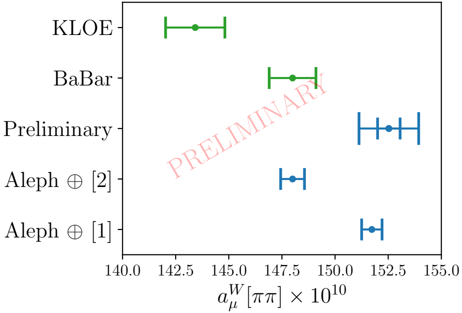
FAQ 3: what about the τ data?

talk by M. Bruno at Edinburgh

Window fever - τ

my PRELIMINARY analysis of exp. + latt. data

- only exp. errs, no attempt at estimating sys. errs for [1] and [2]
- LQCD syst. errs require further investigation/improvements



Isospin-breaking:

[1]: w/o $\rho\gamma$ mixing

[2]: w/ $\rho\gamma$ mixing

What is $\rho\gamma$? too much to say, too little time to explain everything...

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Hadronic running of α and global EW fit

	e^+e^- KNT, DHMZ	EW fit HEPFit	EW fit GFitter	guess based on BMWc
$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) \times 10^4$	276.1(1.1)	270.2(3.0)	271.6(3.9)	277.8(1.3)
difference to e^+e^-		-1.8σ	-1.1σ	$+1.0\sigma$

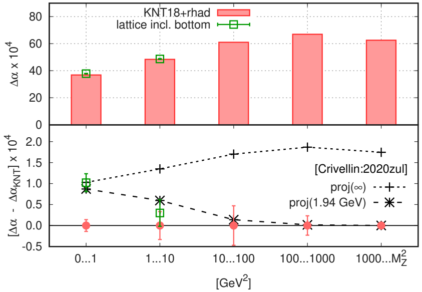
- **Time-like formulation**:

$$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = \frac{\alpha M_Z^2}{3\pi} P \int_{s_{\text{thr}}}^{\infty} ds \frac{R_{\text{had}}(s)}{s(M_Z^2 - s)}$$

- **Space-like formulation**:

$$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = \frac{\alpha}{\pi}\hat{\Pi}(-M_Z^2) + \frac{\alpha}{\pi}\left(\hat{\Pi}(M_Z^2) - \hat{\Pi}(-M_Z^2)\right)$$

- Global EW fit
  - Difference between HEPFit and GFitter implementation mainly treatment of M_W
  - Pull goes into **opposite direction**



BMWc 2020