Intersection Theory for the Computation of Scattering Amplitudes

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1 Introduction

In order to make meaningful comparisons with scattering processes in experiments, precise theoretical predictions are required; the precision of these predictions is increased by computing scattering amplitudes to higher-orders in perturbation theory. These computations are done with the help of multiloop Feynman diagrams, which in turn require the computation of Feynman integrals; in these higher-order corrections, the number of intergrals quickly becomes large. However, it is possible to compute linear relations among these integrals called *integration-by-parts (IBP) identities;* the application of these identities allows one to reduce all integrals in the scattering amplitude to a finite basis of so-called *master integrals.* This process is known as *IBP reduction.*

An IBP identitiy may be obtained from a Feynman integral by inserting a linear combination of loop and external momenta into the integrand and then taking the derivative of this with respect to one of the loop momenta; the result of this is equal to zero as a consequence of dimensional-regularization. The propagators should chosen to be complete in the sense that each scalar product between loop momenta and those between loop and external momenta may be written in terms of a linear combination of propagators. Multiple such relations may be obtained by using different combinations of momenta and by performing integer shifts in the propagator exponents; the result of this is a linear system of equations. Assuming enough such relations are included, one may solve for a given integral in terms of the master integrals.

An issue with this approach is that in cases for which there is a large difference in propagator powers between an integral and the master integrals, many linear relations are required in order to reduce the integral. There is, however, an alternative to this method which avoids this blowup which uses *intersection theory.*

2 Theory

The basic idea of intersection theory follows from linear algebra, where a given vector, $\mathbf{v}$, may be projected onto a basis, $\mathbf{v}_i$, using the formula

$$\mathbf{v} = \sum_i c_i \mathbf{v}_i, \tag{1}$$

with

$$c_i = d_j \mathcal{A}_{ji}^{-1},\ d_i = \mathbf{v} \cdot \mathbf{v}_i,\ \mathcal{A}_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j. \tag{2}$$

Here, the scalar products directly project the vector onto the basis. The same idea could be applied to integrals, provided that there is a suitable definition for a "scalar product" between integrals. This operation should share the properties of bilinearity and nondegeneracy with the vector scalar product, and additionally, it should be invariant under integration-by-parts. This operation is called an *intersection number*, and can be used in an anologous way to Equation 1. Algorithms currently exist for the computation of these intersection numbers for Feynman integrals; however, improving the efficiency of their evaluation to rival the speed of current IBP solvers is still a relatively new area of research.

3 Conclusion

Intersection numbers provide an alternate method for determining linear relations among Feynman integrals; this method has the potential to mitigate issues arising in standard IBP reductions in cases where there is a large difference in numerator powers between an integral and the master integrals. However, this method is not without its own issues: for instance, there is considerable intermediate expression blowup, and the complexity of the problem increases quickly with the number of loops and legs. The improvement of agorithms to avoid these issues is now an active area of research.