

# Feynman Diagrams For Pedestrians

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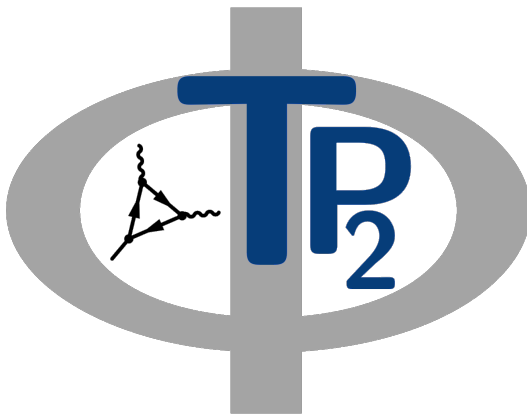
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## Abstract

This set of interleaved lectures and exercises will (re)introduce working experimental particle physicists to the techniques used for computing simple cross sections in the standard model and its extensions. The approach is deliberately pedestrian with an emphasis on real world applications.

After an introduction, gradually more and more time will be spent actually computing stuff in order to build confidence and gain intuition for (new) physics signals in cross sections.

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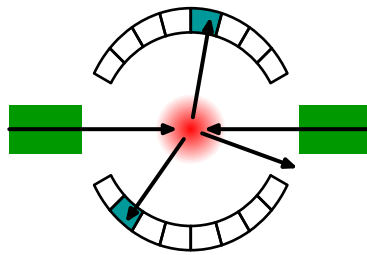
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# 1 Introduction

## 1.1 Scattering Amplitudes

Basic **principle** of quantum mechanics

- an accelerator prepares an **initial state**  $|\text{in}\rangle$
- that is transformed by an **interaction**  $S$
- and a detector measures the overlap of the resulting state with a **final state**  $|\text{out}\rangle$ .



- the **transition probability**  $P$  is given by the absolute square of the **transition amplitude**  $A$

$$A_{\text{in}\rightarrow\text{out}} = \langle \text{out} | S | \text{in} \rangle \quad (1a)$$

$$P_{\text{in}\rightarrow\text{out}} = |A_{\text{in}\rightarrow\text{out}}|^2 \quad (1b)$$

- if the initial and final states  $|\text{in}\rangle$  and  $|\text{out}\rangle$  are not **pure states**, the corresponding **transition probabilities** must be added (e. g. spins and flavors) or integrated (e. g. angles, energies and momenta)

Task(s):

1. describe  $|\text{in}\rangle$  and  $|\text{out}\rangle$ :

- **pure states**: completely polarized electrons, muons, photons  
⇒ **Dirac equation, Klein-Gordon equation, &c.**
- **mixtures**: protons, partially polarized or unpolarized electrons, muons, photons, ...

2. compute  $S$  (i. e. the part of  $S$  that contributes to  $\langle \text{out} | S | \text{in} \rangle$ )

- quantum electro dynamics (QED)
- quantum chromo dynamics (QCD)
- standard model (SM)
- “new physics”, “beyond the SM” (BSM)  
⇒ **Feynman rules**

3. square  $A_{\text{in}\rightarrow\text{out}}$  and integrate  $P_{\text{in}\rightarrow\text{out}}$

- Monte Carlo

## 1.2 Lorentz Transformations

Basic **principle** of special relativity:

- the velocity of light  $c$  is the **same** in each inertial system.

∴ the wavefronts of a spherical light wave is for **every** observer located at

$$|\vec{x}| = ct \quad (2)$$

- introducing the notation  $x_0 = ct$ , this means that the solutions of

$$x_0^2 - \vec{x}^2 = 0 \quad (2')$$

are the same in **every** inertial reference frame

- adding homogeneity and isotropy of space, this means that

$$x^2 = x_0^2 - \vec{x}^2 \quad (3)$$

must be the same in **every** inertial reference frame.

- useful notations:

- **3 vectors**: (**covariant** w. r. t. rotations)

$$\vec{x} = (x^1, x^2, x^3) \quad (4)$$

- **4 vectors**: (**covariant** w. r. t. rotations and **boosts** into a moving inertial frame)

$$x = (x^0; \vec{x}) = (x^0; x^1, x^2, x^3) = (x_0; -x_1, -x_2, -x_3) \quad (5)$$

- introduce **Minkowski metric**

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (6)$$

to shift indices down or up

$$x_\mu = \sum_{\nu=0}^3 g_{\mu\nu} x^\nu, \quad x^\mu = \sum_{\nu=0}^3 g^{\mu\nu} x_\nu \quad (7)$$

- convenient **summation convention**

$$\begin{aligned} xp &= \sum_{\mu=0}^3 x_\mu p^\mu = x_\mu p^\mu = x^\mu p_\mu = g^{\mu\nu} x_\mu p_\nu = g_{\mu\nu} x^\mu p^\nu \\ &= x_0 p_0 - \sum_{i=1}^3 x_i p_i = x_0 p_0 - x_i p_i = x_0 p_0 - \vec{x} \vec{p} \quad (8) \end{aligned}$$

- a Lorentz transformation  $\Lambda$  must leave  $xp$  invariant because  $2xp = (x+p)^2 - x^2 - p^2$ :

$$x_\mu \rightarrow x'_\mu = \Lambda_\mu{}^\nu x_\nu \text{ (with } x'^2 = x^2) \iff g_{\mu\mu'} = \Lambda_\mu{}^\nu \Lambda_{\mu'}{}^{\nu'} g_{\nu\nu'} \quad (9)$$

- derivatives:

$$\frac{\partial}{\partial x_\mu} f(x) = \partial_x^\mu f(x) = \partial^\mu f(x), \quad \frac{\partial}{\partial x^\mu} f(x) = \partial_\mu f(x) \quad (10)$$

for example

$$\partial_x^\mu (xp) = \partial(x_\nu p^\nu) / \partial x_\mu = p^\mu \quad (11)$$

**Problem 1.** Compute the partial derivative w. r. t.  $x$

$$\partial_\mu e^{-ipx}, \quad (a\partial)(b\partial)e^{-ipx}, \quad \partial^2 e^{-ipx} \quad (12)$$

for constant four vectors  $a$ ,  $b$  and  $p$ .

**Problem 2.** Show that

$$\partial_\mu x^\mu = 4 \quad (13a)$$

(NB:  $\partial_\mu x^\mu = \partial x^\mu / \partial x^\mu = g_\mu{}^\mu$  and  $g_\mu{}^\nu = \delta_\mu{}^\nu$ ) and compute

$$\partial^2 e^{-x^2/2} \quad (13b)$$

### 1.3 Schrödinger Equation

- Wave functions satisfy the Schrödinger equation

$$i\hbar \frac{d}{dt} \Psi(t) = H\Psi(t) \quad (14a)$$

with solution (for infinitesimal time intervals)

$$\Psi(t + \delta t) = e^{-iH \cdot \delta t / \hbar} \Psi(t) \quad (14b)$$

$\therefore$  scattering amplitude (infinite time intervals)

$$A_{\text{in} \rightarrow \text{out}} = \langle \text{out} | S | \text{in} \rangle = \lim_{\substack{t_1 \rightarrow -\infty \\ t_2 \rightarrow +\infty}} \langle \text{out}(t_2) | e^{-iH \cdot (t_2 - t_1) / \hbar} | \text{in}(t_1) \rangle \quad (15)$$

- Problems with this approach

- particle production and decay has been observed, but can not be described by wave functions (without “2nd quantization”), because probability is conserved (“unitarity”)
- Schrödinger equation (14) not manifestly Lorentz covariant
- free single particle equation

$$i\hbar \frac{d}{dt} \Psi(t) = \frac{1}{2m} \left( \frac{1}{i\hbar c} \vec{\nabla} \right)^2 \Psi(t) \quad (16)$$

is manifestly not Lorentz covariant!

## 1.4 Units

- from now on, we will use **units** which will give us numbers with **natural** order of magnitude for **quantum mechanics** and **relativistic kinematics**

$$\hbar = c = 1. \quad (17)$$

- **velocities** and **actions** are dimensionless and therefore

$$[\text{energy}] = [\text{momentum}] = [\text{mass}] = \left[ \frac{1}{\text{length}} \right]. \quad (18)$$

- in particular, our **Feynman rules**, will later yield **cross sections** in units of  $[\text{energy}^{-2}]$ , e. g.

$$\sigma = \frac{4\pi\alpha^2}{3E^2} \quad (19)$$

- the relevant conversion factors are

$$\hbar c = 197.327\,053(59) \text{ MeV fm} \quad (20)$$

$$(\hbar c)^2 = 0.389\,379\,66(23) \text{ TeV}^2 \text{ nb} \quad (21)$$

( $\text{TeV}^2 \text{ nb} = \text{GeV}^2 \text{ mb}$ ) and therefore

$$\sigma = \frac{4\pi\alpha^2}{3(E/\text{TeV})^2} 0.39 \text{ nb} \quad (19')$$

## 2 Asymptotic States

- described by **wave equations**, that are
  1. **linear**: **superposition principle** of quantum mechanics
  2. **relativistic**: matrix elements of **observables must** transform under **rotations** and **Lorentz boosts** like **scalars, four vectors, tensors, &c.**
  3. and have the correct **dispersion relation**:  $E^2 = \vec{p}^2 + m^2$
- objects of interest
  - spin-0 particles: **first** particle of this kind observed only recently . . .
    - \* one **invariant** component
  - spin-1/2 particles: **leptons, quarks**
    - \* **at least** two components: **spinor** under rotations
  - spin-1 particles: **gauge bosons**
    - \* massive three components (polarizations)
    - \* massless two components (polarizations)

## 2.1 Klein-Gordon Equation

$$(i\partial_0)^2 \phi(x) = [(-i\vec{\partial})^2 + m^2] \phi(x) \quad (22)$$

- is obviously a **covariant wave equation**, because

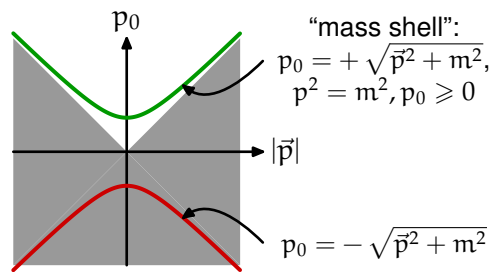
$$(\square + m^2) \phi(x) = 0 \quad (23)$$

- fourier transform

$$\phi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \tilde{\phi}(p), \quad i\partial_\mu \phi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} p_\mu \tilde{\phi}(p), \quad \text{etc} \quad (24)$$

- ∴ **algebraic** equation

$$(p^2 - m^2) \tilde{\phi}(p) = 0 \quad (23')$$



- correct relativistic dispersion relation  $E = +\sqrt{\vec{p}^2 + m^2}$
- but what about the other solution  $E = -\sqrt{\vec{p}^2 + m^2}$  ?

## 2.2 Free Spin-0 Particles

- **general** solution of the **Klein-Gordon equation**

$$\phi(x) = \int \frac{d^4 p}{(2\pi)^4} 2\pi \Theta(p_0) \delta(p^2 - m^2) (\phi^{(+)}(\vec{p}) e^{-ipx} + \phi^{(-)}(\vec{p}) e^{ipx}) \quad (25)$$

$$= \int \frac{d^3 \vec{p}}{(2\pi)^3 2p_0} \Big|_{p_0 = \sqrt{\vec{p}^2 + m^2}} (\phi^{(+)}(\vec{p}) e^{-ipx} + \phi^{(-)}(\vec{p}) e^{ipx}) \quad (26)$$

$$= \int \tilde{d}\vec{p} (\phi^{(+)}(\vec{p}) e^{-ipx} + \phi^{(-)}(\vec{p}) e^{ipx}) \quad (27)$$

- **conserved current**  $\partial_0 j_0(x) - \vec{\nabla} \cdot \vec{j}(x) = \partial_\mu j^\mu(x) = 0$  out of two solutions  $\phi_1$  and  $\phi_2$  of the Klein-Gordon equation with the **same** mass  $m$ :

$$j_\mu(x) = \phi_1^*(x) i \overleftrightarrow{\partial}_\mu \phi_2(x) = \phi_1^*(x) [i\partial_\mu \phi_2(x)] - [i\partial_\mu \phi_1^*(x)] \phi_2(x) \quad (28)$$

- indeed

$$\begin{aligned}
\partial_\mu j^\mu(x) &= \partial^\mu \left( \phi_1^*(x) [i\partial_\mu \phi_2(x)] \right) - \partial^\mu \left( [i\partial_\mu \phi_1^*(x)] \phi_2(x) \right) \\
&= i[\partial^\mu \phi_1^*(x)][\partial_\mu \phi_2(x)] + i\phi_1^*(x)[\partial^2 \phi_2(x)] - i[\partial^2 \phi_1^*(x)]\phi_2(x) \\
&\quad - i[\partial_\mu \phi_1^*(x)][\partial^\mu \phi_2(x)] = -i\phi_1^*(x)m^2\phi_2(x) + im^2\phi_1^*(x)\phi_2(x) = 0 \quad (29)
\end{aligned}$$

- **invariant** and **time independent overlap** out of the conserved current

$$\begin{aligned}
Q &= \int_{x_0=t} d^3\vec{x} j_0(x) = \int_{x_0=t} d^3\vec{x} \phi_1^*(x) i \overleftrightarrow{\partial}_0 \phi_2(x) = \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \frac{1}{2p_0} \left( \phi_1^{(+)*}(\vec{p}) \phi_2^{(+)}(\vec{p}) \cdot e^{ip_0 t} i \overleftrightarrow{\partial}_0 e^{-ip_0 t} + \phi_1^{(-)*}(-\vec{p}) \phi_2^{(+)}(\vec{p}) \cdot e^{-ip_0 t} i \overleftrightarrow{\partial}_0 e^{-ip_0 t} \right. \\
&\quad \left. + \phi_1^{(+)*}(\vec{p}) \phi_2^{(-)}(-\vec{p}) \cdot e^{ip_0 t} i \overleftrightarrow{\partial}_0 e^{ip_0 t} + \phi_1^{(-)*}(-\vec{p}) \phi_2^{(-)}(-\vec{p}) \cdot e^{-ip_0 t} i \overleftrightarrow{\partial}_0 e^{ip_0 t} \right) \\
&= \int \widetilde{d}\vec{p} \left( \phi_1^{(+)*}(\vec{p}) \phi_2^{(+)}(\vec{p}) - \phi_1^{(-)*}(\vec{p}) \phi_2^{(-)}(\vec{p}) \right) \quad (30)
\end{aligned}$$

- normalization **only positive** on the **positive mass shell!**
- ∴  $j_\mu(x)$  **must not** be interpreted as a **probability current!**
- ... anyway: the existence of the **negative mass shell** makes the **energy unbounded** from **below** and **no ground state** exists!
- ∴  $\phi(x)$  **must not** be interpreted as **Schrödinger wave function!**

## 2.3 Anti Particles

Observations:

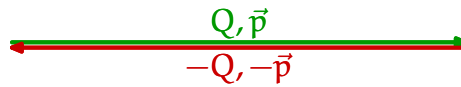
- the **positive** and **negative mass shells** of **free** (i. e. not interacting) particles are independent
- ∴ one can simply **project out** the negative mass shell **in this case** ...
- ... unfortunately, **all local** and **Lorentz invariant** interactions **couple** positive and negative mass shell (see below)
- ... however, **asymptotic states** are assumed to be **noninteracting**
- ∴ we can at least **reinterpret** the negative mass shell

however:

- the amplitude  $\phi^{(+)}(\vec{p})e^{-ipx}$  on the positive mass shell corresponds to the momentum  $+\vec{p}$ , but the amplitude  $\phi^{(-)}(\vec{p})e^{+ipx}$  on the negative mass shell corresponds to the **reversed** momentum  $-\vec{p}$

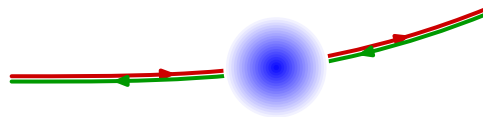


- ∴ the formalism is consistent, if **all quantum numbers** are **reversed** on the negative mass shell
- ∴ in a stationary, i. e. time independent, state, the cases



can **not** be distinguished.

- ∴ the states on the negative mass shell describe **not** particles with “negative energy”, but **anti particles** with opposite quantum numbers instead!
- in **stationary** states, this can be taken a step further: instead of **anti particles** moving **forward** in time ...



- ... one can use **particles** moving **backward** in time, without noticing a difference in the overall balance!
- **caveat**: it is **not** (yet) obvious that this makes sense of interactions are switched on ...
- ... will be shown later.
- **NB**: this is a computationally **convenient** interpretation — there is **no time travel** going on, since we’re in a **steady state**
- The equivalent picture with anti particles moving forward in time requires the full machinery of **quantum field theory**

## 2.4 Dirac Equation

- **task**: find “objects”  $\gamma^\mu$ , such that

$$(\gamma^\mu \partial_\mu)^2 = \partial^2 \quad (31)$$

- because the solutions of the **Dirac equation**

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0 \quad (32)$$

**automagically** satisfy the **Klein-Gordon equation** as well

$$(i\gamma^\mu \partial_\mu + m) (i\gamma^\mu \partial_\mu - m) \psi(x) = (-\partial^2 - m^2) \psi(x) = 0 \quad (33)$$

- the Dirac equation is obviously linear and its solutions satisfy the proper relativistic dispersion relation.
- can we construct “objects”  $\gamma^\mu$ , that satisfy (31)?
- a sufficient condition is

$$[\gamma_\mu, \gamma_\nu]_+ := \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \cdot \mathbf{1} \quad (34)$$

because partial derivatives commute  $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ .

- using a useful and ubiquitous notation, the Feynman slash

$$\not{a} = \gamma_\mu a^\mu = \gamma^\mu a_\mu \quad (35)$$

this reads equivalently  $[\not{a}, \not{b}]_+ := \not{a}\not{b} + \not{b}\not{a} = 2 \cdot \mathbf{a}\mathbf{b} = 2 \cdot a_\mu b^\mu$

## 2.5 Gamma Matrices

- recall the Pauli matrices with the defining property

$$[\sigma^k, \sigma^l] = \sigma^k \sigma^l - \sigma^l \sigma^k = 2i \sum_{m=1}^3 \epsilon^{klm} \sigma^m \quad (36a)$$

$$(\sigma^k)^\dagger = \sigma^k \quad (36b)$$

using the totally antisymmetric tensor  $\epsilon$

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1, \quad \epsilon^{213} = \epsilon^{321} = \epsilon^{132} = -1 \quad (37)$$

- concrete realisation

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (38)$$

- with

$$\sigma^k \sigma^l = \delta^{kl} \mathbf{1} + i \sum_{m=1}^3 \epsilon^{klm} \sigma^m \quad (39)$$

- in particular

$$[\sigma^k, \sigma^l]_+ = 2\delta^{kl} \mathbf{1} \quad (40)$$

- Dirac realisation of the gamma (a. k. a. Dirac) matrices:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (41)$$

- there are (infinitely) many more realisations, but no smaller one

- we can verify the **anti commutation relations** by explicit calculation.

**NB:** **block matrices** are multiplied just like ordinary matrices with **non commuting** matrix elements

$$(\gamma^0)^2 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \mathbf{1} \quad (42a)$$

$$\begin{aligned} [\gamma^0, \gamma^i]_+ &= \gamma^0 \gamma^i + \gamma^i \gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{1} \cdot \sigma^i \\ (-\mathbf{1}) \cdot (-\sigma^i) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (-\sigma^i) \cdot \mathbf{1} \\ \sigma^i \cdot (-\mathbf{1}) & 0 \end{pmatrix} \end{aligned} \quad (42b)$$

$$= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0 \quad (42c)$$

**Problem 3.** Verify the remaining ( $k, l = 1, 2, 3$ ) anti commutation relations (34):

$$[\gamma^k, \gamma^l]_+ = -2\delta^{kl} \cdot \mathbf{1}. \quad (43)$$

- in the Dirac realization (41) we have obviously

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i \quad (44)$$

- however, since  $(\gamma_0)^2 = \mathbf{1}$  and  $(\gamma_i)^2 = -\mathbf{1}$

$\therefore \gamma_0$  **must** have **only real eigenvalues**, and

$\therefore$  all  $\gamma_i$  **must** have **only imaginary eigenvalues**

$\therefore$  this must be true in **all** realizations.

- Another useful and ubiquitous notation is therefore the **Dirac adjoint** for **matrices**

$$\bar{A} = \gamma_0 A^\dagger \gamma_0, \quad \bar{\gamma}_\mu = \gamma_0 \gamma_\mu^\dagger \gamma_0 = \gamma_\mu \quad (45)$$

- **NB:** on the next page we will meet the related, but different **Dirac adjoint** for **column vectors**

$$\bar{v} = v^\dagger \gamma_0 \quad (46)$$

- **Don't mix them up!**

## 2.6 Free Spin-1/2 Particles

- Ansatz:

$$\psi(x) = \int \widetilde{d\mathbf{p}} (\psi^{(+)}(\mathbf{p}) e^{-i\mathbf{p}x} + \psi^{(-)}(\mathbf{p}) e^{i\mathbf{p}x}) \quad (47)$$

$$(i\not{\partial} - m) \psi(x) = 0 \Leftrightarrow \begin{cases} (\not{\mathbf{p}} - m) \psi^{(+)}(\mathbf{p}) = 0 \\ (\not{\mathbf{p}} + m) \psi^{(-)}(\mathbf{p}) = 0 \end{cases} \quad (48)$$

- the adjoint solution

$$\bar{\psi}(x) = \psi(x)^\dagger \gamma_0 = \int \widetilde{d\mathbf{p}} (\bar{\psi}^{(+)}(\mathbf{p}) e^{i\mathbf{p}x} + \bar{\psi}^{(-)}(\mathbf{p}) e^{-i\mathbf{p}x}) \quad (49)$$

satisfies

$$\begin{aligned} \bar{\psi}(x) i \overleftarrow{\not{\partial}} &= i \partial_\mu \bar{\psi}(x) \gamma^\mu = i \partial_\mu \psi(x)^\dagger \gamma_0 \gamma^\mu \gamma_0 \gamma_0 \\ &= i \partial_\mu \psi(x)^\dagger \gamma^{\mu\dagger} \gamma_0 = (-i \partial_\mu \gamma^\mu \psi(x))^\dagger \gamma_0 = \overline{(-i \not{\partial} \psi(x))} = -m \bar{\psi}(x) \end{aligned} \quad (50)$$

∴

$$\bar{\psi}(x) (i \overleftarrow{\not{\partial}} + m) = 0, \quad (51)$$

or

$$\bar{\psi}^{(+)}(\mathbf{p}) (\not{p} - m) = 0 \quad (52a)$$

$$\bar{\psi}^{(-)}(\mathbf{p}) (\not{p} + m) = 0 \quad (52b)$$

- **general** solution of the **Dirac equation**

$$\psi^{(+)}(\mathbf{p}) = \sum_{k=1}^2 u_k(\mathbf{p}) b_k(\mathbf{p}) \quad (53a)$$

$$\psi^{(-)}(\mathbf{p}) = \sum_{k=1}^2 v_k(\mathbf{p}) d_k(\mathbf{p}) \quad (53b)$$

with **four independent** basis solutions  $u_1(\mathbf{p}), u_2(\mathbf{p}), v_1(\mathbf{p}), v_2(\mathbf{p})$  satisfying

$$(\not{p} - m) u_k(\mathbf{p}) = 0 \quad (54a)$$

$$(\not{p} + m) v_k(\mathbf{p}) = 0 \quad (54b)$$

and the corresponding **expansion coefficients**  $b_1(\mathbf{p}), b_2(\mathbf{p}), d_1(\mathbf{p}), d_2(\mathbf{p})$ .

- in the **rest frame** of the particle, i. e. for  $\mathbf{p} = (m, \vec{0})$ , we have  $\not{p} = m\gamma_0$  and the **Dirac equation** simplifies to

$$m(\gamma_0 - \mathbf{1}) u_k(\vec{0}) = \begin{pmatrix} 0 & 0 \\ 0 & -2m \cdot \mathbf{1} \end{pmatrix} u_k(\vec{0}) = 0 \quad (55a)$$

$$m(\gamma_0 + \mathbf{1}) v_k(\vec{0}) = \begin{pmatrix} 2m \cdot \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} v_k(\vec{0}) = 0 \quad (55b)$$

- the independent solutions in the **rest frame** are therefore

$$u_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (56a)$$

$$v_1(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (56b)$$

- from which we can construct solutions for **arbitrary on shell momenta** with  $p^2 = m^2$

$$\mathbf{u}_k(\mathbf{p}) = \frac{\not{p} + m}{\sqrt{p_0 + m}} \mathbf{u}_k(\vec{0}) \quad (57a)$$

$$\mathbf{v}_k(\mathbf{p}) = \frac{\not{p} - m}{\sqrt{p_0 + m}} \mathbf{v}_k(\vec{0}) \quad (57b)$$

- since  $(\not{p} + m)(\not{p} - m) = p^2 - m^2$ , these are obviously solutions of the **Dirac equation** for **on-shell momenta**
- the motivation for the **not obviously covariant normalization** will be apparent after problem 4

**mathematical reminder:**

- compare the **inner product** of a **row vector** with a **column vector**

$$(\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n) \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \sum_{i=1}^n \mathbf{a}_i \mathbf{b}_i \quad (58)$$

to the **outer product** of a **column vector** with a **row vector**

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \otimes (\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_m) = \begin{pmatrix} \mathbf{a}_1 \mathbf{b}_1 & \mathbf{a}_1 \mathbf{b}_2 & \cdots & \mathbf{a}_1 \mathbf{b}_m \\ \mathbf{a}_2 \mathbf{b}_1 & \mathbf{a}_2 \mathbf{b}_2 & \cdots & \mathbf{a}_2 \mathbf{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n \mathbf{b}_1 & \mathbf{a}_n \mathbf{b}_2 & \cdots & \mathbf{a}_n \mathbf{b}_m \end{pmatrix} \quad (59)$$

- these are two **very** different operations
  - the **inner product** produces a **number**
  - the **outer product** produces a **matrix**

**Problem 4.** Determine  $\bar{\mathbf{u}}_k(\mathbf{p})$  und  $\bar{\mathbf{v}}_k(\mathbf{p})$  from the definitions and show that for  $p^2 = m^2$

$$\sum_{k=1}^2 \mathbf{u}_k(\mathbf{p}) \bar{\mathbf{u}}_k(\mathbf{p}) = \not{p} + m, \quad \sum_{k=1}^2 \mathbf{v}_k(\mathbf{p}) \bar{\mathbf{v}}_k(\mathbf{p}) = \not{p} - m \quad (60)$$

**Problem 5.** Compute (always assuming  $p^2 = m^2$ )

$$\bar{\mathbf{u}}_k(\mathbf{p}) \mathbf{u}_l(\mathbf{p}), \quad \bar{\mathbf{v}}_k(\mathbf{p}) \mathbf{v}_l(\mathbf{p}), \quad \bar{\mathbf{u}}_k(\mathbf{p}) \mathbf{v}_l(\mathbf{p}), \quad \bar{\mathbf{v}}_k(\mathbf{p}) \mathbf{u}_l(\mathbf{p}). \quad (61)$$

**Problem 6.** Compute (always assuming  $p^2 = m^2$ )

$$\bar{\mathbf{u}}_k(\mathbf{p}) \gamma_\mu \mathbf{u}_l(\mathbf{p}), \quad \bar{\mathbf{v}}_k(\mathbf{p}) \gamma_\mu \mathbf{v}_l(\mathbf{p}), \quad \bar{\mathbf{u}}_k(\vec{p}) \gamma_0 \mathbf{v}_l(-\vec{p}), \quad \bar{\mathbf{v}}_k(-\vec{p}) \gamma_0 \mathbf{u}_l(\vec{p}). \quad (62)$$

- this is a special case of the **Gordon decomposition** for **arbitrary** solutions of the **Dirac equation**:

$$\begin{aligned}
\bar{u}_k(p)\gamma_\mu u_l(q) &= \bar{u}_k(p) \frac{\not{p}\gamma_\mu + \gamma_\mu \not{q}}{2m} u_l(q) \\
&= \bar{u}_k(p) \frac{p_\nu(g_{\nu\mu} + \frac{1}{2}[\gamma_\nu, \gamma_\mu]_-) + q_\nu(g_{\mu\nu} + \frac{1}{2}[\gamma_\mu, \gamma_\nu]_-)}{2m} u_l(q) \\
&= \frac{p_\mu + q_\mu}{2m} \bar{u}_k(p) u_l(q) + \frac{q_\nu - p_\nu}{2im} \bar{u}_k(p) \frac{i}{2} [\gamma_\mu, \gamma_\nu]_- u_l(q) \quad (63)
\end{aligned}$$

- there are altogether 16 **independent** (anti-)hermitian  $4 \times 4$ -matrices:

$$1 \quad 1 \quad \text{"scalar"} \quad (64a)$$

$$\gamma_\mu \quad 4 \quad \text{"vector"} \quad (64b)$$

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]_- \quad 6 \quad \text{"tensor"} \quad (64c)$$

$$\gamma_5 \gamma_\mu \quad 4 \quad \text{"axial vector"} \quad (64d)$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad 1 \quad \text{"pseudo scalar"} \quad (64e)$$

- **NB:** the bare gamma matrices do **not** transform like vectors, tensors, axial vectors, or pseudo scalars!
- there are additional nontrivial transformations  $L(\Lambda)$  that have to be applied on the left and right, e. g.

$$\gamma_\mu \rightarrow \Lambda_\mu{}^\nu L(\Lambda) \gamma_\nu L^{-1}(\Lambda) \quad (65)$$

- however since

$$\psi(x) \rightarrow L(\Lambda)\psi(\Lambda^{-1}x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(\Lambda^{-1}x)L^{-1}(\Lambda) \quad (66)$$

the  $L(\Lambda)$  compensate each other in **matrix elements** (a. k. a. "sandwiches")

$$\begin{aligned}
\bar{\psi}(x)\gamma_\mu\psi(y) &\rightarrow \bar{\psi}(\Lambda^{-1}x)L^{-1}(\Lambda)\Lambda_\mu{}^\nu L(\Lambda)\gamma_\nu L^{-1}(\Lambda)L(\Lambda)\psi(\Lambda^{-1}x) \\
&= \Lambda_\mu{}^\nu \bar{\psi}(\Lambda^{-1}x)\gamma_\nu\psi(\Lambda^{-1}x) \quad (67)
\end{aligned}$$

$\therefore$  and the  $L(\Lambda)$  can be ignored in the computation of **matrix elements**

$\therefore$  the characterization as **vector**, **tensor**, **axial vector**, or **pseudo scalars** is meaningful

**Problem 7.** Compute  $[\gamma_5, \gamma^\mu]_+$

**Problem 8.** Show the **conservation of the vector current** for two solutions  $\psi_1(x)$  und  $\psi_2(x)$  of the **Dirac equation** (32) and (51) with the same mass  $m$

$$\partial^\mu [\bar{\psi}_1(x)\gamma_\mu\psi_2(x)] = 0. \quad (68)$$

- invariant overlap from conserved current:

$$\begin{aligned}
Q &= \int_{x_0=t} d^3\vec{x} j_0(x) = \int_{x_0=t} d^3\vec{x} \bar{\psi}_1(x) \gamma_0 \psi_2(x) = \int_{x_0=t} d^3\vec{x} \psi_1^\dagger(x) \psi_2(x) \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \frac{1}{2p_0} \left( \psi_1^{(+)\dagger}(\vec{p}) \psi_2^{(+)}(\vec{p}) + \psi_1^{(-)\dagger}(-\vec{p}) \psi_2^{(+)}(\vec{p}) \cdot e^{-2ip_0 t} \right. \\
&\quad \left. + \psi_1^{(+)\dagger}(\vec{p}) \psi_2^{(-)}(-\vec{p}) \cdot e^{2ip_0 t} + \psi_1^{(-)\dagger}(-\vec{p}) \psi_2^{(-)}(-\vec{p}) \right) \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \sum_{k=1}^2 \left( b_{1,k}^\dagger(\vec{p}) b_{2,k}(\vec{p}) + d_{1,k}^\dagger(\vec{p}) d_{2,k}(\vec{p}) \right) \quad (69)
\end{aligned}$$

∴ normalization positive **everywhere!**

**Problem 9.** Show the *Partial Conservation of the Axial Current (PCAC)* for solutions  $\psi_1(x)$  und  $\psi_2(x)$  of the *Dirac equation* with the same mass  $m$

$$\partial^\mu [\bar{\psi}_1(x) \gamma_\mu \gamma_5 \psi_2(x)] = 2im \bar{\psi}_1(x) \gamma_5 \psi_2(x). \quad (70)$$

**Problem 10.** Compute  $\sigma_{kl} = \frac{i}{2} [\gamma_k, \gamma_l]$  for  $k, l = 1, 2, 3$  in the *Dirac realization* of the *Gamma matrices*.

∴ the matrices  $\{\sigma_{23}, \sigma_{31}, \sigma_{12}\}$  can take over the rôle of the **Pauli matrices**  $\{\sigma_1, \sigma_2, \sigma_3\}$  and distinguish **spin up** from **spin down** in the **rest frame**

$$\frac{1}{2} \sigma_{12} u_1(\vec{0}) = +\frac{1}{2} u_1(\vec{0}), \quad \frac{1}{2} \sigma_{12} u_2(\vec{0}) = -\frac{1}{2} u_2(\vec{0}) \quad (71a)$$

$$\frac{1}{2} \sigma_{12} v_1(\vec{0}) = +\frac{1}{2} v_1(\vec{0}), \quad \frac{1}{2} \sigma_{12} v_2(\vec{0}) = -\frac{1}{2} v_2(\vec{0}) \quad (71b)$$

- just as before in the case of scalar particles, we can interpret solutions on the **negative energy mass shell** for spin-1/2 particles as **anti particles** that move in the opposite direction in **space time**:
  - $u_k(\vec{p})$  amplitude for a **particles** in the **initial state**
  - $\bar{u}_k(\vec{p})$  amplitude for a **particle** in the **final state**
  - $v_k(\vec{p})$  amplitude for an **anti particle** in the **final state**
  - $\bar{v}_k(\vec{p})$  amplitude for an **anti particle** in the **initial state**
- in addition to the other quantum numbers, we must **flip the spins** so that the overall balance is maintained

$$\begin{array}{c}
\overleftrightarrow{\hspace{10em}} \\
\begin{array}{c}
Q, \vec{p}, s \\
-\overleftrightarrow{\hspace{10em}} \\
-Q, -\vec{p}, -s
\end{array}
\end{array}$$

## 2.7 Free Spin-1 Particles

- combine  $\vec{E}$  and  $\vec{B}$  into **field strength tensor**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (72)$$

- manifestly covariant **Maxwell equations**

$$\partial^\mu F_{\mu\nu} = j_\nu, \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \quad (73)$$

- current  $j_\mu$  necessarily conserved:  $\partial_\mu j^\mu = 0$**

- equation of motion for the **vector potential  $A_\mu$**

$$(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A_\nu = j^\mu \quad (73')$$

- gauge invariance:**  $F_{\mu\nu}$  does **not** change, if

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \omega(x) \quad (74)$$

- with special **gauge condition**  $\partial_\mu A^\mu = 0$ :  $\partial^2 A_\mu = j_\mu$

- more, but **not** most, general case

$$(g^{\mu\nu} \partial^2 - (1 - \xi) \partial^\mu \partial^\nu) A_\nu = j^\mu \quad (75)$$

- explicit **mass term**

$$\partial^\mu F_{\mu\nu} + M^2 A_\nu = j_\nu \quad (76)$$

or

$$(g^{\mu\nu} (\partial^2 + M^2) - \partial^\mu \partial^\nu) A_\nu = j^\mu \quad (76')$$

- contraction with  $\partial_\mu$

$$M^2 \partial^\nu A_\nu = 0 \quad (77)$$

- massless vector bosons** have **two** degrees of freedom, **massive** ones **three**

- polarization vectors** for **massless vector bosons** with momentum  $k = (k_0; 0, 0, k_0)$

$$\epsilon_\pm = \epsilon_{\mp}^* = \frac{1}{\sqrt{2}} (0; 1, \pm i, 0) \quad (78)$$

- properties

$$\epsilon_\lambda^\mu \epsilon_{\lambda',\mu}^* = -\delta_{\lambda\lambda'} \quad (79a)$$

$$\epsilon_\lambda^\mu k_\mu = 0 \quad (79b)$$

and with  $c = (1; 0, 0, -1)$

$$\sum_{\lambda=-1,+1} \epsilon_\lambda^\mu \epsilon_{\lambda'}^{\nu,*} = -g^{\mu\nu} + \frac{c_\mu k_\nu + c_\nu k_\mu}{ck} \quad (80)$$

- polarization vectors** for **massive vector bosons** with momentum



$k = (k_0; |\vec{k}| \sin \theta \cos \phi, |\vec{k}| \sin \theta \sin \phi, |\vec{k}| \cos \theta)$ :

$$\epsilon_{\pm} = \epsilon_{\mp}^* = \frac{e^{\mp i\phi}}{\sqrt{2}} (0; \cos \theta \cos \phi \mp i \sin \phi, \cos \theta \sin \phi \pm i \cos \phi, -\sin \theta) \quad (81a)$$

$$\epsilon_0 = \frac{k_0}{M} (|\vec{k}|/k_0; \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = \epsilon_0^* \quad (81b)$$

- properties (79) and

$$\sum_{\lambda=-1,0,+1} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu,*} = -g^{\mu\nu} + \frac{k_{\mu} k_{\nu}}{M^2} \quad (82)$$

## 3 Interactions

### 3.1 Propagators

- **Photons** far from all **electrical charges** satisfy

$$\partial^2 A_{\mu}^{(0)}(x) = 0 \quad (83)$$

and we have already seen the solutions.

- In the presence of **electrical charges**, the photons couple to the **electromagnetic current**

$$\partial^2 A_{\mu}(x) = j_{\mu}(x) = -e\bar{\Psi}(x)\gamma_{\mu}\Psi(x) + \dots \quad (84)$$

and the solutions turn out to be more complicated.

- **Assumption**: there is a “**function**”  $D$ , that solves

$$(\partial^2 + m^2)D(x, m) = -\delta^4(x) \quad (85)$$

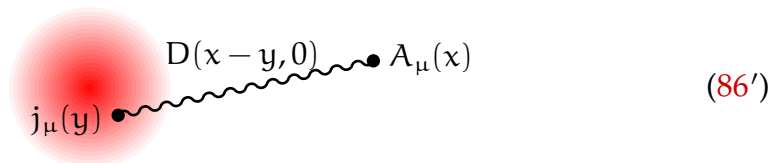
- Then

$$A_{\mu}(x) = A_{\mu}^{(0)}(x) - \int d^4y D(x - y, 0) j_{\mu}(y) \quad (86)$$

is a solution of the **inhomogeneous** equation (84) for **each** solution of the **homogeneous** equation (83), since

$$\begin{aligned} \partial^2 A_{\mu}(x) &= \partial^2 A_{\mu}^{(0)}(x) - \int d^4y [\partial^2 D(x - y, 0)] j_{\mu}(y) \\ &= 0 - \int d^4y [-\delta^4(x - y)] j_{\mu}(y) = j_{\mu}(x) \end{aligned} \quad (87)$$

- **interpretation**: the current  $j_{\mu}(y)$  acts at the space time point  $y$  as a **source** of photons, that are “**propagated**” by the **propagator**  $D(x - y, 0)$  to the space time point  $x$



$$(86')$$

- **NB:** retardation is built in in (86), because we integrate over the **four dimensional space time** and **not** over the **three dimensional space** at a given instant.

**Problem 11.** Show that

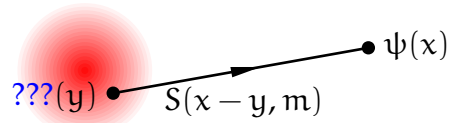
$$S(x, m) = (i\partial + m) D(x, m) \quad (88)$$

is the **Feynman propagator**  $S$  for Dirac particles of mass  $m$ , i. e. that

$$(i\partial - m) S(x, m) = \delta^4(x)$$

if  $(\partial^2 + m^2)D(x, m) = -\delta^4(x)$ , as in (85) above.

- but what is the **source ???** of the **Dirac field**?



$$???(y) \xrightarrow{S(x-y, m)} \psi(x) \quad (89)$$

- consider the **Dirac equation** with (electromagnetic) interaction

$$(i\partial - e\mathcal{A}(x) - m) \psi(x) = 0 \quad (90)$$

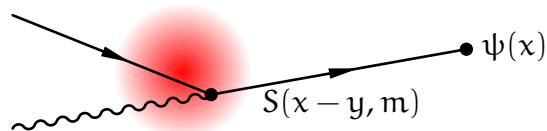
or

$$(i\partial - m) \psi(x) = e\mathcal{A}(x)\psi(x) \quad (90')$$

- with the **formal solution**

$$\psi(x) = \psi^{(0)}(x) + \int d^4y S(x-y, m) e\mathcal{A}(y)\psi(y) \quad (91)$$

that can be represented graphically as

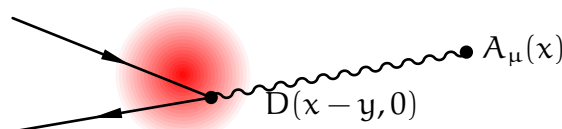


$$\quad (91')$$

- (91) is analogous to (86), when we use the **current**  $j_\mu(y) = -e\bar{\psi}(y)\gamma_\mu\psi(y)$

$$A_\mu(x) = A_\mu^{(0)}(x) - \int d^4y D(x-y, 0) e\bar{\psi}(y)\gamma_\mu\psi(y) \quad (92)$$

which can be represented graphically as



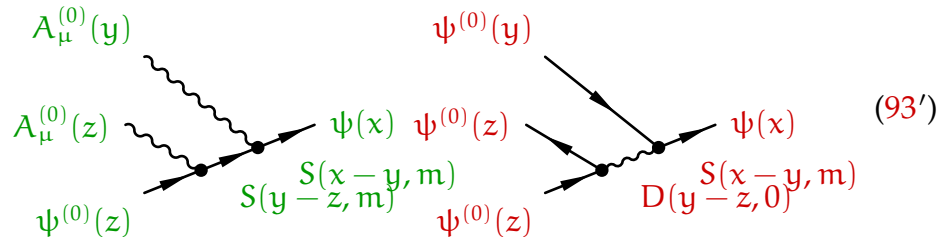
$$\quad (92')$$

- **caveat:** the equations (91) and (92) are **not explicit solutions**, but **coupled integral equations**

- that can be solved **recursively** by **mutual series expansion**

$$\begin{aligned}
\psi(x) &= \psi^{(0)}(x) + \int d^4y S(x-y, m) e \mathcal{A}(y) \psi(y) \\
&= \psi^{(0)}(x) + e \int d^4y S(x-y, m) \mathcal{A}^{(0)}(y) \psi^{(0)}(y) \\
&\quad + e^2 \int d^4y d^4z \left( S(x-y, m) \mathcal{A}^{(0)}(y) S(y-z, m) \mathcal{A}^{(0)}(z) \psi^{(0)}(z) \right. \\
&\quad \left. - S(x-y, m) \gamma^\mu \psi^{(0)}(y) D(y-z, m) \bar{\psi}(z) \gamma_\mu \psi(z) \right) + O(e^3) \quad (93)
\end{aligned}$$

- these **recursive** expansions become very big very fast and a **graphical representation** is useful:



- **remaining open questions:**

- does  $D(x-y, m)$  exist?
- can we **compute** it?

- **fortunately**, we only need to solve

$$(\partial^2 + m^2) D(x, m) = -\delta^4(x) \quad (85')$$

because (most) other **propagators** can be obtained by taking **derivatives**

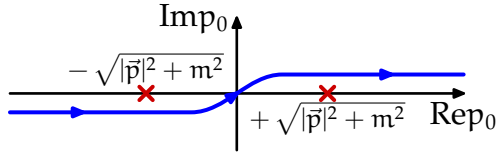
- since the equation is **translation invariant**, we **should** use **Fourier transformation**

- formally (**with  $\epsilon \rightarrow 0+$** )

$$D(x, m) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i\epsilon} \quad (94)$$

$$(\partial^2 + m^2) D(x, m) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{-p^2 + m^2}{p^2 - m^2 + i\epsilon} = - \int \frac{d^4p}{(2\pi)^4} e^{-ipx} = -\delta^4(x) \quad (95)$$

- singularities in the **integral over  $p_0$**  at  $\pm \sqrt{|\vec{p}|^2 + m^2}$  (**+i $\epsilon$**  is a convenient shorthand for the **choice of integration contour**):



- compare

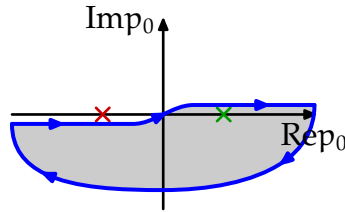
$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{(p_0)^2 - (|\vec{p}|^2 + m^2) + i\epsilon} \stackrel{E = +\sqrt{|\vec{p}|^2 + m^2}}{=} \frac{1}{(p_0)^2 - E^2 + i\epsilon}$$

$$\stackrel{E > 0}{=} \frac{1}{(p_0)^2 - (E - i\epsilon)^2} = \frac{1}{p_0 - E + i\epsilon} \frac{1}{p_0 + E - i\epsilon} = \frac{1}{2E} \left( \frac{1}{p_0 - E + i\epsilon} - \frac{1}{p_0 + E - i\epsilon} \right) \quad (96)$$

- forward in time:

$$x_0 > 0 : \lim_{p_0 \rightarrow -i\infty} e^{-ip_0 x_0} \rightarrow 0 \quad (97a)$$

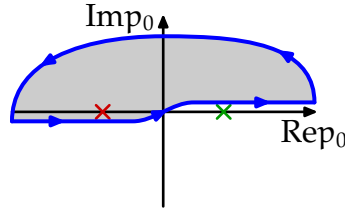
the integration contour in (94) can be closed below



- backward in time:

$$x_0 < 0 : \lim_{p_0 \rightarrow +i\infty} e^{-ip_0 x_0} \rightarrow 0 \quad (97b)$$

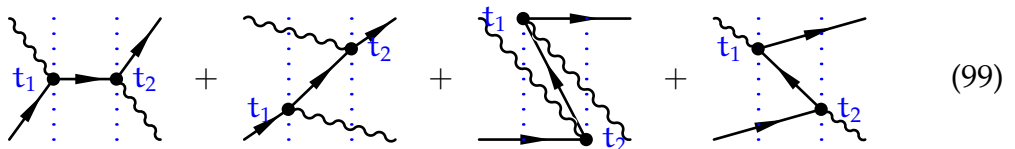
the integration contour in (94) can be closed above



∴ in

$$\Phi'(x) = \int d^4y D(x - y, m) \Phi(y) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i\epsilon} \tilde{\Phi}(p) \quad (98)$$

- the part of  $\tilde{\Phi}(p)$  with  $p_0 = +\sqrt{|\vec{p}|^2 + m^2}$  is propagated into the future
- the part of  $\tilde{\Phi}(p)$  with  $p_0 = -\sqrt{|\vec{p}|^2 + m^2}$  is propagated into the past
- Compton scattering in nonrelativistic perturbation theory contains scattering as well as pair creation and pair annihilation contributions:



- ∴ intermediate states **violate energy conservation** and vertices can have **space like distances**
- ∴ **temporal order** of  $t_1$  and  $t_2$  depends in general on the reference frame, i. e. is **undefined**
- **Feynman's brilliant** (re-)interpretation:
  - **particles** with  $p_0 = +\sqrt{|\vec{p}|^2 + m^2}$  are propagated into the **future**
  - **anti particles** with  $p_0 = -\sqrt{|\vec{p}|^2 + m^2}$  and **opposite charges** are propagated into **past**
  - ∴ **charges** are **conserved** along the arrows in (99)!
  - ∴ the four **nonrelativistic** diagrams in (99) can be combined to two **covariant** expressions using **Feynman propagators**

$$\frac{1}{E - E_0 + i\epsilon} + \frac{1}{E + E_0 + i\epsilon} = \frac{1}{p^2 - m^2 + i\epsilon} \quad (100a)$$

$$\frac{1}{E - E_0 + i\epsilon} + \frac{1}{E + E_0 + i\epsilon} = \frac{1}{p^2 - m^2 + i\epsilon} \quad (100b)$$

- ∴ the **Feynman propagator** allows to extend our interpretation of **external, non-interacting anti particles** as particles traveling **backward** in time to **interacting** particles.

**Problem 12.** Compute the propagator  $S(x, m)$  for *Dirac particles in momentum space!*

- propagator for **massless spin-1 particles**

$$\frac{-ig_{\mu\nu} + i(1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon}}{k^2 + i\epsilon} \quad (101)$$

- the **gauge parameter**  $\xi$  is **arbitrary** and **must cancel** in the **final result**
- **partial** results can depend on  $\xi$
- the propagator for **massive spin-1 particles**

$$\frac{-ig_{\mu\nu} + i \frac{k_\mu k_\nu}{M^2}}{k^2 - M^2 + i\epsilon} \quad (102)$$

is **not** gauge dependent, because (76') can be inverted, contrary to (73')

- **NB:** this is **not** what happens in the **standard model**, where the mass comes solely from the coupling to the **Higgs sector** and a **gauge freedom** remains. However (102) is in lowest order equivalent to the **Higgs mechanism** in **unitarity gauge**
- **NB:** **propagators** and **external states** are **universal** and **models** differ only in the **particle content** and interaction vertices!

### 3.2 Feynman Rules

external spin-1/2 particles:

$$\text{incoming: } \begin{array}{c} \text{p} \\ \text{---} \blacktriangleright \bullet \\ \text{p} \end{array} \iff \dots u(p) \quad (103a)$$

$$\text{outgoing: } \bullet \text{---} \blacktriangleleft \text{p} \iff \bar{u}(p) \dots \quad (103b)$$

external spin-1/2 **anti** particles:

$$\text{incoming: } \begin{array}{c} \text{p} \\ \text{---} \blacktriangleright \bullet \\ \text{p} \end{array} \iff \bar{v}(p) \dots \quad (103c)$$

$$\text{outgoing: } \bullet \text{---} \blacktriangleleft \text{p} \iff \dots v(p) \quad (103d)$$

external spin-1 particles:

$$\text{incoming: } \begin{array}{c} \text{k} \\ \text{~~~~} \blacktriangleright \bullet \\ \text{k} \end{array} \iff \epsilon_\mu(k) \quad (103e)$$

$$\text{outgoing: } \bullet \text{~~~~} \blacktriangleleft \text{k} \iff \epsilon_\mu^*(k) \quad (103f)$$

- internal particles **and** anti particles with **sign of momentum relative to the arrow** (i. e. charge) direction:

$$\text{spin-1/2: } \bullet \text{---} \blacktriangleright \bullet \iff \frac{i}{\not{p} - m + i\epsilon} \quad (104a)$$

$$\text{spin-1 (m = 0): } \bullet \text{~~~~} \blacktriangleright \bullet \iff \frac{-ig_{\mu\nu} + i(1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon}}{k^2 + i\epsilon} \quad (104b)$$

- finally

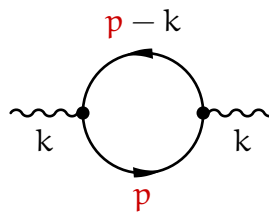
$$\text{spin-0: } \bullet \text{---} \bullet \iff \frac{i}{p^2 - m^2 + i\epsilon} \quad (105)$$

- ∴ the **S**-matrix always contains an **uninteresting diagonal piece** (no interaction) and the **global momentum conservation  $\delta$ -distribution**

$$S = \mathbf{1} + (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2 \dots - \dots q_n) i\mathbf{T} \quad (106)$$

- ∴ we can focus on **T**

- the following Feynman rules produce an expression for  $i\mathcal{T}$ :
  1. draw **all** diagrams using **propagators** and **interaction vertices** that connect the **initial state** with the **final state**
  2. assign **momenta** and **external wave functions** accordingly
  3. use **momentum conservation** at **each vertex** to fix the momenta of the **internal lines**
  4. follow each connected **fermion line against the direction of the arrows** and collect wave functions, propagators and vertices along the way
  5. complete  $i\mathcal{T}$  with the remaining wave functions, propagators and vertices
  6. add **all** diagrams with the **relative signs** such that that sum is **anti symmetric** under the **exchange of identical external (anti-)fermions** (and symmetric for bosons)
- in diagrams with **closed loops not** all momenta are fixed by **momentum conservation**, e. g.:



- in this case, one must **integrate over these loop momenta** with

$$\int \frac{d^4p}{(2\pi)^4} \dots \quad (107)$$

- unfortunately, there are **infinitely** many loop diagrams for each process
- **fortunately**, each loop comes with **additional powers** of the **coupling constants**
- in **weakly interacting theories**, we can **expand** the **scattering amplitudes** in the couplings constants or the **number of loops**

### 3.3 Cross Section

- definition in terms of **observables**

$$\sigma(\Delta\Phi) = \frac{R(\Delta\Phi)}{j} \quad \text{where} \quad (108)$$

$$\Delta\Phi = \text{region of phase space} \quad (109a)$$

$$\sigma(\Delta\Phi) = \text{cross section for scattering into } \Delta\Phi \quad (109b)$$

$$R(\Delta\Phi) = \text{event rate in } \Delta\Phi \quad (109c)$$

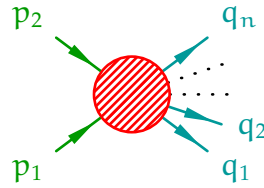
$$j = \text{incoming flux} \quad (109d)$$

- for **fixed target experiments**, the incoming flux  $j$  is just the number of incoming particles per time and area
- **differential cross section**

$$\sigma(\Delta\Phi) = \int_{\Delta\Phi} \frac{d\sigma}{d\Phi}(\Phi) d\Phi \quad (110)$$

with **phase space element**  $d\Phi$ , e. g.  $d\Omega = \sin\theta d\theta d\phi$  for  $2 \rightarrow 2$  processes

- the differential cross section can be **computed** from the **scattering amplitude  $T$**  and the (a. k. a. "**Fermi's golden rule**")
- general formula for  $2 \rightarrow n$  processes



$$d\sigma = \frac{|T|^2}{4 \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \frac{\widetilde{dq}_1 \dots \widetilde{dq}_n}{\prod_i n_i!} (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - \dots - q_n) \quad (111)$$

- where we have used again

$$\widetilde{dp} = \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \Big|_{p_0 = \sqrt{\vec{p}^2 + m^2}} \quad (112)$$

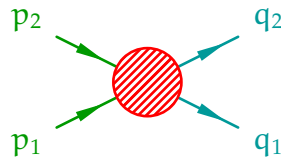
**NB:** in the old days, people use a different normalization for fermions (an additional factor  $2m$ ), that is **only** useful for heavy (**slow**) particles ...

- **symmetry factor**

$$n_i = \begin{cases} \text{number of } \mathbf{identical} \text{ particles} \\ \text{of the species } i \text{ in the} \\ \mathbf{final state} \end{cases} \quad (113)$$

### 3.4 Kinematics

- simplest example:  $2 \rightarrow 2$



- **invariants: Mandelstam variables**

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \quad (\text{total energy}) \quad (114a)$$

$$t = (q_1 - p_1)^2 = (q_2 - p_2)^2 \quad (\text{momentum transfer}) \quad (114b)$$

$$u = (q_1 - p_2)^2 = (q_2 - p_1)^2 \quad (114c)$$



- **momentum conservation:**

$$s + t + u = p_1^2 + p_2^2 + q_1^2 + q_2^2 = \sum_{i=1}^4 m_i^2 \quad (115)$$

- **center of mass system (CMS) at high energies**

$$p_{1/2} = (E; 0, 0, \pm E) \quad (116a)$$

$$q_{1/2} = (E; \pm E \sin \theta \cos \phi, \pm E \sin \theta \sin \phi, \pm E \cos \theta) \quad (116b)$$

$$s = 4E^2, \quad t = -2E^2(1 - \cos \theta), \quad u = -2E^2(1 + \cos \theta) \quad (117)$$

### 3.5 Phase Space

- two particles in the **final state:**

$$\begin{aligned} & \int \frac{d^3 \vec{q}_1}{(2\pi)^3 2E_1} \frac{d^3 \vec{q}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(q_1 + q_2 - P) \\ &= \frac{1}{16\pi^2} \int \frac{|\vec{q}_1|^2 d|q_1| d\Omega_1}{E_1 E_2} \delta(E_1(|\vec{q}_1|) + E_2(|\vec{q}_1|) - E) \\ &= \frac{1}{16\pi^2} \int \frac{|\vec{q}_1| E_1 dE_1 d\Omega_1}{E_1 E_2} \delta(E_1 + E_2(E_1) - E) = \frac{1}{16\pi^2} \int d \cos \theta_1 d\phi_1 \frac{|\vec{q}_1|}{E} \quad (118) \end{aligned}$$

- the **second equality** in (118) follows from  $E^2 = |\vec{q}|^2 + m^2$ , which yields  $|\vec{q}| d|\vec{q}| = E dE$  **independent** of the masses
- in the **third equality** we have used that  $E_2$  depends on  $E_1$  through **momentum conservation** and **dispersion relations**

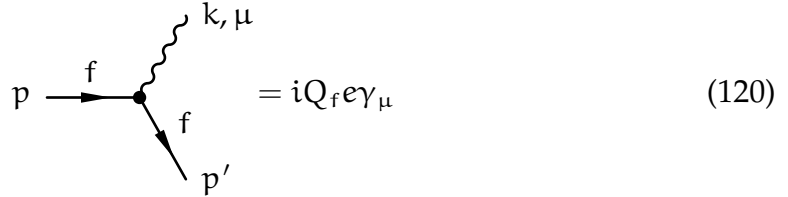
$$\frac{d(E_1 + E_2(E_1) - E)}{dE_1} = 1 + E_1/E_2 = E/E_2 \quad (119)$$

- **special case: high energy limit in the center of mass frame:**  $|\vec{q}_1| = |\vec{q}_2| = E/2 + O(m/|\vec{q}_2|^2)$ .

$$d \cos \theta_1 d\phi_1 / (32\pi^2) \quad (118')$$

## 4 QED

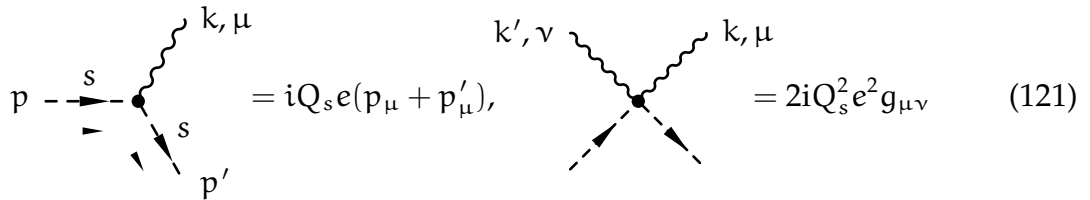
- **Quantum Electro Dynamics**: interacting electrons, positrons and photons (and other charged particles)
- **single interaction vertex for fermions**:



$$= iQ_f e \gamma_\mu \quad (120)$$

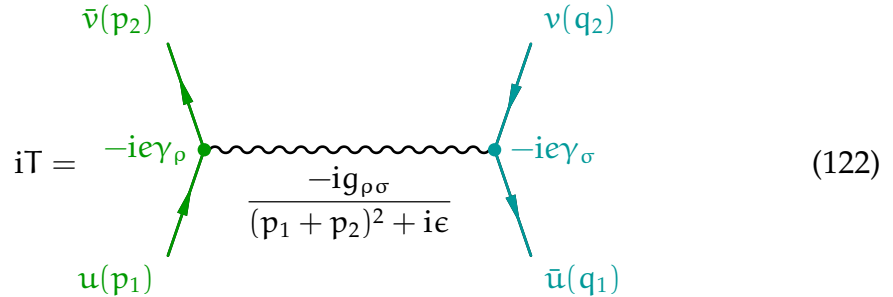
with the **electrical charge**  $Q_f$  of the particle, e. g.  $Q_e = -1$  for electrons

- charged **bosons require** an additional (“sea gull”) vertex



$$= iQ_s e (p_\mu + p'_\mu), \quad = 2iQ_s^2 e^2 g_{\mu\nu} \quad (121)$$

### 4.1 $e^+e^- \rightarrow \mu^+\mu^-$



$$i\mathbb{T} = \frac{-ig_{\rho\sigma}}{(p_1 + p_2)^2 + i\epsilon} \quad (122)$$

$$i\mathbb{T} = \bar{v}(p_2)(-ie\gamma^\rho)u(p_1) \frac{-ig_{\rho\sigma}}{(p_1 + p_2)^2 + i\epsilon} \bar{u}(q_1)(-ie\gamma^\sigma)v(q_2) = i \frac{e^2}{s} [\bar{v}(p_2)\gamma_\rho u(p_1)] [\bar{u}(q_1)\gamma^\rho v(q_2)] \quad (123)$$

$$\mathbb{T}\mathbb{T}^\dagger = \frac{e^2}{s} \frac{e^2}{s} [\bar{v}(p_2)\gamma_{\rho_1} u(p_1) \bar{u}(p_1)\gamma_{\rho_2} v(p_2)] [\bar{u}(q_1)\gamma^{\rho_1} v(q_2) \bar{v}(q_2)\gamma^{\rho_2} u(q_1)] \quad (124)$$

$$\sum_{\text{spins}} \mathbb{T}\mathbb{T}^\dagger = \frac{e^4}{s^2} \text{tr}[(\not{p}_2 - m_e)\gamma_{\rho_1}(\not{p}_1 + m_e)\gamma_{\rho_2}] \text{tr}[(\not{q}_1 + m_\mu)\gamma^{\rho_1}(\not{q}_2 - m_\mu)\gamma^{\rho_2}] = \frac{e^4}{s^2} L_{\rho_2\rho_1}(p_1, p_2, m_e) L^{\rho_1\rho_2}(q_1, q_2, m_\mu) \quad (125)$$

## 4.2 Trace Techniques

- consider a **matrix element**

$$\bar{\mathbf{u}}(\mathbf{p})\Gamma\mathbf{u}(\mathbf{q}) = \sum_{k,l=1}^4 \bar{u}_k(\mathbf{p})\Gamma_{kl}u_l(\mathbf{q}) = \sum_{k,l=1}^4 \Gamma_{kl}u_l(\mathbf{q})\bar{u}_k(\mathbf{p}) \quad (126)$$

- using the **tensor product**

$$\mathbf{u}(\mathbf{q}) \otimes \bar{\mathbf{u}}(\mathbf{p}) = \begin{pmatrix} u_1(\mathbf{q})\bar{u}_1(\mathbf{p}) & \cdots & u_1(\mathbf{q})\bar{u}_4(\mathbf{p}) \\ \vdots & \ddots & \vdots \\ u_4(\mathbf{q})\bar{u}_1(\mathbf{p}) & \cdots & u_4(\mathbf{q})\bar{u}_4(\mathbf{p}) \end{pmatrix} \quad (127)$$

we can write

$$\bar{\mathbf{u}}(\mathbf{p})\Gamma\mathbf{u}(\mathbf{q}) = \sum_{k,l=1}^4 \Gamma_{kl}[\mathbf{u}(\mathbf{q}) \otimes \bar{\mathbf{u}}(\mathbf{p})]_{lk} = \sum_{k=1}^4 (\Gamma[\mathbf{u}(\mathbf{q}) \otimes \bar{\mathbf{u}}(\mathbf{p})])_{kk} \quad (128)$$

- using the **trace** of a matrix

$$\text{tr}(A) = \sum_{k=1}^4 A_{kk} \quad (129)$$

we can express a **matrix element** equivalently as a **trace**

$$\bar{\mathbf{u}}(\mathbf{p})\Gamma\mathbf{u}(\mathbf{p}) = \text{tr}(\Gamma[\mathbf{u}(\mathbf{p}) \otimes \bar{\mathbf{u}}(\mathbf{p})]) \quad (130)$$

- **independent** of the concrete **realization** of the **Dirac matrices**, we can compute their traces using their **anti commutation relations** (34) alone

$$\text{tr}(\mathbf{1}) = 4 \quad (131a)$$

$$\text{tr}(\not{a}\not{b}) = \frac{1}{2} \left( \text{tr}(\not{a}\not{b}) + \text{tr}(\not{b}\not{a}) \right) \stackrel{(34)}{=} \text{tr}(\mathbf{1}) \cdot \mathbf{a} \cdot \mathbf{b} = 4 \cdot \mathbf{a} \cdot \mathbf{b} \quad (131b)$$

$$\text{tr}(\not{a}_1) = \text{tr}(\not{a}_1\not{a}_2\not{a}_3) = \text{tr}(\not{a}_1\not{a}_2 \cdots \not{a}_{2n+1}) \stackrel{(\gamma_5\gamma_5=1)}{=} 0 \quad (131c)$$

$$\text{tr}(\not{a}_1\not{a}_2 \cdots \not{a}_n) = \text{tr}(\not{a}_n \cdots \not{a}_2\not{a}_1) \quad (131d)$$

$$\text{tr}(\gamma_5) = \text{tr}(\gamma_5\not{a}) = \text{tr}(\gamma_5\not{a}\not{b}) = \text{tr}(\gamma_5\not{a}\not{b}\not{c}) = 0 \quad (131e)$$

$$\text{tr}(\gamma_5\not{a}\not{b}\not{c}\not{d}) = 4i \cdot \epsilon(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \quad (131f)$$

only (131d) depends on the existence of a **charge conjugation matrix**

- also from the **anti commutation relations** alone, we can prove **contraction formulae**:

$$\gamma^\mu \not{a} \gamma_\mu = -2 \cdot \not{a} \quad (132a)$$

$$\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu = -2 \cdot \not{c}\not{b}\not{a} \quad (132b)$$

$$\gamma^\mu \gamma_\mu = 4 \quad (132c)$$

$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4 \cdot \mathbf{a} \cdot \mathbf{b} \quad (132d)$$

- it's instructive to prove at least one of the **trace theorems** yourselves, because it helps to memorize the result.

**Problem 13.** Compute

$$\text{tr}(\not{a}\not{b}\not{c}\not{d}) \quad (133)$$

using the **cyclic invariance** of the trace

$$\text{tr}(ABC) = \text{tr}(BCA) \quad (134)$$

and subsequently the **anti commutation relations** (34).

**Problem 14.** Compute the trace  $L_{\mu\nu}(p, q, m) = \text{tr}[(\not{p} + m)\gamma_\mu(\not{q} - m)\gamma_\nu]$ .

**Problem 15.** Compute  $L_{p_2 p_1}(p_1, p_2, 0)L^{p_1 p_2}(q_1, q_2, 0)$  as a function of the **Mandelstam variables** in the **high energy limit**  $s, -t, -u \gg m_i^2$ .

### 4.3 Cross Section

**Problem 16.** Compute the **differential cross section**

$$\frac{d\sigma}{d\Omega}(\cos\theta, E_{CM}) \quad (135)$$

for  $e^+e^- \rightarrow \mu^+\mu^-$  in the region  $s, -t, -u \gg m_e^2$ .

**Problem 17.** Compute the **integrated cross section**

$$\sigma(E_{CM}) \quad (136)$$

for  $e^+e^- \rightarrow \mu^+\mu^-$  in the region  $s \gg m_e^2$ .

### 4.4 FORM

- very efficient computation using programs for **symbolic manipulation**, e. g. FORM.

- declare **variables**

```
1: vector p1, p2, q1, q2;
2: symbol s, t, u, me, mq;
3: indices rho1, rho2;
```

- **expressions**

```
4: local [TT*] =
5:   (g_(1, p2) - me*g_(1)) * g_(1, rho1)
6:   * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
7:   * (g_(2, q1) + mq*g_(2)) * g_(2, rho1)
8:   * (g_(2, q2) - mq*g_(2)) * g_(2, rho2);
```

- traces

```

9:  trace4, 1;
10: trace4, 2;
11: print;
12: .sort;

```

- reduction to Mandelstam variables (114)

$$s = (p_1 + p_2)^2 = 2m_e^2 + 2p_1p_2 \quad (137a)$$

$$t = (q_2 - p_1)^2 = m_q^2 + m_e^2 - 2q_2p_1 \quad (137b)$$

etc.

```

13: id p1.p2 = 1/2 * (s - 2*me^2);
14: id q1.q2 = 1/2 * (s - 2*mq^2);
15: id p1.q1 = - 1/2 * (t - me^2 - mq^2);
16: id p2.q2 = - 1/2 * (t - me^2 - mq^2);
17: id p1.q2 = - 1/2 * (u - me^2 - mq^2);
18: id p2.q1 = - 1/2 * (u - me^2 - mq^2);

```

- human intelligence (i. e. experience, cf. (211)): the result can be written most compactly as a function of **t** und **u**:

```

19: id s = - u - t + 2*me^2 + 2*mq^2;
20: bracket me, mq;
21: print;
22: .sort;

```

running the program:

```
ohl@thopad5:~fdfp$ form mumu.frm
```

```
FORM 4.3 (Jan  4 2023) 64-bits
```

```
Run: Fri Jul 12 15:40:08 2024
```

```

[TT*] =
 64*me^2*mq^2 + 32*p1.p2*mq^2 + 32*p1.q1*p2.q2 + 32*p1.q2*p2.q1 + 32*
q1.q2*me^2;
[TT*] =
 + mq^2 * ( - 32*u - 32*t )
 + mq^4 * ( 48 )
 + me^2 * ( - 32*u - 32*t )
 + me^2*mq^2 * ( 96 )
 + me^4 * ( 48 )
 + 8*u^2 + 8*t^2;
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```

## 4.5 Bhabha Scattering

**Problem 18.** Draw *all Feynman diagrams* for  $e^+e^- \rightarrow e^+e^-$  (“*Bhabha scattering*”)

- do we need to **add** or **subtract** the diagrams?
- more general: what’s the **relative phase** of the diagrams?

$$TT^\dagger = (T_t - T_s)(T_t - T_s)^\dagger = T_t T_t^\dagger - T_t T_s^\dagger - T_s T_t^\dagger + T_s T_s^\dagger \quad (138)$$

- relative signs of the diagrams from permuting the endpoints of the fermion **lines**
- equivalently: relative signs from the number of **closed fermion lines** in **squared diagrams**

$$T_t T_t^\dagger = (-1)^2 \times \text{diagram} , \quad T_t T_s^\dagger = (-1)^1 \times \text{diagram} \quad (139a)$$

$$T_s T_t^\dagger = (-1)^1 \times \text{diagram} , \quad T_s T_s^\dagger = (-1)^2 \times \text{diagram} \quad (139b)$$

## 4.6 FORM

- declaration of variables as above
- expressions

$$T_s = e^2 \frac{1}{s} [\bar{v}(p_2) \gamma_\rho u(p_1)] [\bar{u}(q_1) \gamma^\rho v(q_2)] \quad (140)$$

$$T_t = e^2 \frac{1}{t} [\bar{v}(p_2) \gamma_\rho v(q_2)] [\bar{u}(q_1) \gamma^\rho u(p_1)] \quad (141)$$

- $T_s T_s^\dagger$  just like in  $e^+e^- \rightarrow \mu^+\mu^-$

```

4: local [SS*] =
5:   (g_(1, p2) - me*g_(1)) * g_(1, rho1)
6:   * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
7:   * (g_(2, q1) + me*g_(2)) * g_(2, rho1)
8:   * (g_(2, q2) - me*g_(2)) * g_(2, rho2);

```

- $T_t T_t^\dagger$  similar

```

9: local [TT*] =
10:  (g_(1, q1) + me*g_(1)) * g_(1, rho1)
11:  * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
12:  * (g_(2, p2) - me*g_(2)) * g_(2, rho1)
13:  * (g_(2, q2) - me*g_(2)) * g_(2, rho2);

```

- the **interference terms** are more complicated and contain **traces of eight Dirac matrices**

$$T_s T_t^* = e^4 \frac{1}{s t} [\bar{v}(p_2) \gamma_{\rho_1} u(p_1)] [\bar{u}(p_1) \gamma^{\rho_2} u(q_1)] [\bar{u}(q_1) \gamma^{\rho_1} v(q_2)] [\bar{v}(q_2) \gamma_{\rho_2} v(p_2)] \quad (142)$$

```

14: local [ST*] =
15:     (g_(1, p2) - me*g_(1)) * g_(1, rho1)
16:     * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
17:     * (g_(1, q1) + me*g_(1)) * g_(1, rho1)
18:     * (g_(1, q2) - me*g_(1)) * g_(1, rho2);

```

$$T_t T_s^* = e^4 \frac{1}{s t} [\bar{v}(p_2) \gamma_{\rho_1} v(q_2)] [\bar{v}(q_2) \gamma^{\rho_2} u(q_1)] [\bar{u}(q_1) \gamma^{\rho_1} u(p_1)] [\bar{u}(p_1) \gamma_{\rho_2} v(p_2)] \quad (143)$$

```

19: local [TS*] =
20:     (g_(1, p2) - me*g_(1)) * g_(1, rho1)
21:     * (g_(1, q2) - me*g_(1)) * g_(1, rho2)
22:     * (g_(1, q1) + me*g_(1)) * g_(1, rho1)
23:     * (g_(1, p1) + me*g_(1)) * g_(1, rho2);

```

- FORM does the traces just the same ...

```

24: trace4, 1;
25: trace4, 2;
26: .sort;

```

- reduction to **Mandelstam variables** again as above, but all masses are equal

```

27: id p1.p2 = 1/2 * (s - 2*me^2);
28: id q1.q2 = 1/2 * (s - 2*me^2);
29: id p1.q1 = - 1/2 * (t - 2*me^2);
30: id p2.q2 = - 1/2 * (t - 2*me^2);
31: id p1.q2 = - 1/2 * (u - 2*me^2);
32: id p2.q1 = - 1/2 * (u - 2*me^2);

```

- **human** intelligence and experience: the expression for  $|T_s|^2$  is most **compact** as function of  $t$  and  $u$

```

33: id s = - u - t + 4*me^2;
34: bracket me;
35: print;
36: .sort;

```

- the expression for  $|T_t|^2$  is most **compact** as function of  $s$  and  $u$

– ... two different reductions in the same FORM program require advanced tricks ...

- running the program:

```
ohl@thopad5:~fdfp$ form bhabha.frm
```

```
FORM 4.3 (Jan  4 2023) 64-bits
```

```
Run: Fri Jul 12 15:40:08 2024
```

```
[SS*] =
  + me^2 * ( - 64*u - 64*t )
  + me^4 * ( 192 )
  + 8*u^2 + 8*t^2;
[TT*] =
  + me^2 * ( - 64*u )
  + me^4 * ( 64 )
  + 16*u^2 + 16*t*u + 8*t^2;
[ST*] =
  + me^2 * ( 64*u )
  + me^4 * ( - 96 )
  - 8*u^2;
[TS*] =
  + me^2 * ( 64*u )
  + me^4 * ( - 96 )
  - 8*u^2;
0.00 sec out of 0.00 sec
```

- **NB:**  $|T_s|^2(t, u) = |T_t|^2(s, u)$

- final result **very symmetrical**

$$\sum_{\text{spins}} \text{TT}^* = 8e^4 \left( \frac{t^2 + u^2}{s^2} + \frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} \right) + O(m_e^2) \quad (144)$$

**Problem 19.** Compute the differential cross section

$$\frac{d\sigma}{d\Omega}(\cos \theta) \quad (145)$$

for *Bhabha scattering* in the region  $s, -t, -u \gg m_e^2$ .



# 5 QCD

## 5.1 Feynman Rules

∴ full account requires another full set of lectures

∴ just a few SU(N) formulae

- generators  $T_a$  and totally anti symmetric **structure constants**  $f_{abc}$ :

$$[T_a, T_b] = if_{abc} T_c \tag{146}$$

(summations for  $c = 1, 2, \dots, (N_c^2 - 1)$  implied).

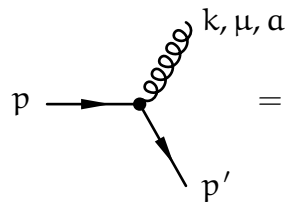
- **normalization** and **contractions**

$$\text{tr}(T_a T_b) = \frac{1}{2} \delta_{ab} \tag{147a}$$

$$T_a T_a = C_F \cdot \mathbf{1} = \frac{N_c^2 - 1}{2N_c} \cdot \mathbf{1} \tag{147b}$$

$$f_{acd} f_{bcd} = C_A \cdot \delta_{ab} \tag{147c}$$

- physical degrees of freedom: **quarks** and **gluons**:



$$= ig\gamma_\mu T_a \tag{148}$$

- **triple gluon couplings**



$$= gf_{a_1 a_2 a_3} g_{\mu_1 \mu_2} (k_{\mu_3}^1 - k_{\mu_3}^2) + gf_{a_1 a_2 a_3} g_{\mu_2 \mu_3} (k_{\mu_1}^2 - k_{\mu_1}^3) + gf_{a_1 a_2 a_3} g_{\mu_3 \mu_1} (k_{\mu_2}^3 - k_{\mu_2}^1)$$

- **quartic gluon couplings**



$$= -ig^2 f_{a_1 a_2 b} f_{a_3 a_4 b} \times (g_{\mu_1 \mu_3} g_{\mu_4 \mu_2} - g_{\mu_1 \mu_4} g_{\mu_2 \mu_3}) - ig^2 f_{a_1 a_3 b} f_{a_4 a_2 b} \times (g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} - g_{\mu_1 \mu_2} g_{\mu_3 \mu_4}) - ig^2 f_{a_1 a_4 b} f_{a_2 a_3 b} \times (g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} - g_{\mu_1 \mu_3} g_{\mu_4 \mu_2})$$

- **Faddeev-Popov ghosts** appear **only** in loop diagrams

## 5.2 3-Jet Production

- quarks are **electrically charged**

$$\begin{array}{c}
 \text{--- } p \text{ ---} \\
 \diagup \\
 \bullet \\
 \diagdown \text{--- } p' \\
 \text{--- } k, \mu, a
 \end{array}
 = -ieQ_q \gamma_\mu$$

and couple in the **standard model** (cf. (176) and (178)) to the  $Z^0$  and  $W^\pm$  as well.

**Problem 20.** Draw *all Feynman diagrams without closed loops*, that follow from the *Feynman rules* of QED und QCD for the electroproduction of *3 jets*

$$e^+ + e^- \rightarrow q + \bar{q} + g$$

(at higher energies, there are additional contributions from  $Z^0$  bosons).

Write down the corresponding algebraic expression for the T-matrix.

**Problem 21.** Verify the *Ward identity* for the gluon

$$T_1|_{\epsilon_\mu^*(k)=k_\mu} + T_2|_{\epsilon_\mu^*(k)=k_\mu} = 0, \quad (152)$$

i. e. the fact that no gluons with *unphysical polarization*  $\epsilon_\mu^*(k) = k_\mu$  are produced. *Hint:* use the *Dirac equation* for the *external* quarks, e. g.

$$\frac{1}{-\not{p}_1 - \not{k} - m + i\epsilon} (\not{k} + \not{p}_1 + m) v(p_1) = -v(p_1) \quad (\text{for } k \neq 0) \quad (153)$$

- down the road, some additional shorthands will be useful

$$T_n = e^2 Q g [\bar{v}(p_2) \gamma_\mu u(p_1)] \frac{1}{s} J_n^\mu(q_1, q_2, k, \epsilon) \quad (154)$$

with the **current matrix element**

$$J^\mu(q_1, q_2, k, \epsilon) = J_1^\mu(q_1, q_2, k, \epsilon) + J_2^\mu(q_1, q_2, k, \epsilon) \quad (155a)$$

$$J_1^\mu(q_1, q_2, k, \epsilon) = \frac{\bar{u}(q_1) T_\alpha \not{\epsilon}^*(k) (\not{q}_1 + \not{k} + m) \gamma^\mu v(q_2)}{(q_1 + k)^2 - m^2 + i\epsilon} \quad (155b)$$

$$J_2^\mu(q_1, q_2, k, \epsilon) = \frac{\bar{u}(q_1) \gamma^\mu (-\not{q}_2 - \not{k} + m) T_\alpha \not{\epsilon}^*(k) v(q_2)}{(q_2 + k)^2 - m^2 + i\epsilon} \quad (155c)$$

- this allows to write

$$\sum_{\text{spins, pol.}} |T_1 + T_2|^2 = \frac{e^4 g^2 Q^2}{s^2} L_{\mu\nu}(p_1, p_2, 0) H^{\mu\nu}(q_1, q_2, k) \quad (156)$$

with the “hadronic tensor”

$$H^{\mu\nu}(q_1, q_2, k) = \sum_{\text{spins}, \epsilon} J^\mu(q_1, q_2, k, \epsilon) J^{\nu,*}(q_1, q_2, k, \epsilon) \quad (157)$$

$\therefore$  with the same trick as in problem 21, we can show that

$$[q_1^\mu + q_2^\mu + k^\mu] J_\mu(q_1, q_2, k, \epsilon) = 0 \quad (158)$$

$\therefore$  using the center of mass momentum  $p = p_1 + p_2 = q_1 + q_2 + k$

$$p_\mu H^{\mu\nu}(q_1, q_2, k) = p_\nu H^{\mu\nu}(q_1, q_2, k) = 0 \quad (159)$$

- angular dependence of  $H^{\mu\nu}(q_1, q_2, k)$  contains a lot of information ...

$\therefore$  ... but the energy dependence is much simpler

$$x_1 = 2q_1 p / p^2, \quad x_2 = 2q_2 p / p^2, \quad x_3 = 2k p / p^2 \quad (160)$$

$\therefore$  integrate over the angles

$$\begin{aligned} \int \widetilde{dq_1} \widetilde{dq_2} \widetilde{dk} (2\pi)^4 \delta^4(q_1 + q_2 + k - p) f(x_1, x_2, x_3) \\ = \frac{s}{128\pi^3} \int dx_1 dx_2 f(x_1, x_2, 2 - x_1 - x_2) \end{aligned} \quad (161)$$

afterwards, the result will depend **only** on  $p$  and the  $x_i$ .

$\therefore$  from (159) we find

$$\int d\tilde{\Omega} H^{\mu\nu}(q_1, q_2, k) = \left( \frac{p^\mu p^\nu}{p^2} - g^{\mu\nu} \right) \tilde{H}(p, x_1, x_2) \quad (162)$$

$\therefore$

$$\tilde{H}(p, x_1, x_2) = -\frac{1}{3} \int d\tilde{\Omega} H^\mu{}_\mu(q_1, q_2, k) \quad (163)$$

- energy conservation

$$x_1 + x_2 + x_3 = \frac{2(q_1 + q_2 + k)p}{p^2} = 2 \quad (164)$$

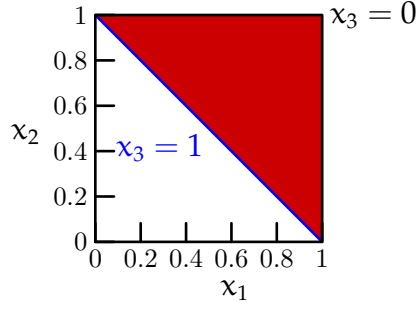
- momentum conservation for **massless** particles

$$x_1 + x_2 \geq x_3 = 2 - x_1 - x_2 \quad (165)$$

with equality for parallel  $q_1$  und  $q_2$

$\therefore$

$$x_1 + x_2 \geq 1 \quad (165')$$



- for **massless** particles

$$\sum_{\text{spins, pol.}} \int d\tilde{\Omega} |\mathbb{T}_1 + \mathbb{T}_2|^2 = \frac{4e^4 g^2 Q^2}{s^2} (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1 p_2 g^{\mu\nu}) \times \left( \frac{p_\mu p_\nu}{s} - g_{\mu\nu} \right) \tilde{\mathbb{H}}(\mathbf{p}, x_1, x_2) = \frac{4e^4 g^2 Q^2}{s} \tilde{\mathbb{H}}(\mathbf{p}, x_1, x_2) \quad (166)$$

- add **phase space factor**

$$\frac{d^2\sigma}{dx_1 dx_2} = \frac{1}{2s} \frac{s}{128\pi^3} \frac{1}{4} \sum_{\text{spins, pol.}} \int d\tilde{\Omega} |\mathbb{T}_1 + \mathbb{T}_2|^2 = \frac{\alpha_s \alpha^2 Q^2}{4s} \tilde{\mathbb{H}}(\mathbf{p}, x_1, x_2) \quad (167)$$

**Problem 22.** Express the invariants  $q_1 q_2$ ,  $q_1 k$  and  $q_2 k$  by  $s$  and the  $x_i$ , assuming all particles to be massless.

**Problem 23.** Compute

$$\frac{d^2\sigma}{dx_1 dx_2}(x_1, x_2) \quad (168)$$

for massless particles.

- start with computing  $H^\mu{}_\mu(q_1, q_2, k)$  as a function of  $q_1 q_2$ ,  $q_1 k$  and  $q_2 k$
- since (152), you may use  $\sum_e \epsilon_\mu \epsilon_\nu^* = -g_{\mu\nu}$
- use the **contraction identities** (132) before calculating the traces!

## 6 Standard Model

			$(\mathbf{C}, \mathbf{T})_Y$	$Q = T_3 + \frac{Y}{2}$
leptons				
$\nu_{e,R}$	$\nu_{\mu,R}$	$\nu_{\tau,R}$	$(\mathbf{1}, \mathbf{1})_0$	0
$e_R$	$\mu_R$	$\tau_R$	$(\mathbf{1}, \mathbf{1})_{-2}$	-1
$\begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu,L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix}$	$(\mathbf{1}, \mathbf{2})_{-1}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
quarks				
$u_R$	$c_R$	$t_R$	$(\mathbf{3}, \mathbf{1})_{4/3}$	$\frac{2}{3}$
$d_R$	$s_R$	$b_R$	$(\mathbf{3}, \mathbf{1})_{-2/3}$	$-\frac{1}{3}$
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$(\mathbf{3}, \mathbf{2})_{1/3}$	$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$
gauge bosons				
$A$	$Z^0$	$W^\pm$	$g$	
Higgs				
	$\Phi$		$(\mathbf{1}, ?)_?$	?

### 6.1 Propagators

- external spin-1 particles

$$\text{incoming: } \begin{array}{c} \xrightarrow{k} \\ \text{---} \\ \xleftarrow{k} \end{array} \iff \epsilon_\mu(k) \quad (169a)$$

$$\text{outgoing: } \begin{array}{c} \xrightarrow{k} \\ \text{---} \\ \xleftarrow{k} \end{array} \iff \epsilon_\mu^*(k) \quad (169b)$$

- internal particles and anti particles

$$\text{Spin-1 (} m \neq 0 \text{): } \begin{array}{c} \xrightarrow{k} \\ \text{---} \\ \xleftarrow{k} \end{array} \iff \frac{-ig_{\mu\nu} + i\frac{k_\mu k_\nu}{M^2}}{k^2 - M^2 + i\Gamma M} \quad (170)$$

- polarization sum:

$$\sum_{\text{pol.}} \epsilon_\mu(k) \epsilon_\nu^*(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \quad (171)$$

- unitarity gauge:  $\partial^\mu D_{\mu\nu} = 0$ , other options

$$\frac{-ig_{\mu\nu} + i(1 - \xi)\frac{k_\mu k_\nu}{k^2 - \xi M^2}}{k^2 - M^2 + i\Gamma M} \quad (172)$$

better for radiative corrections, but larger intermediate terms

- finite width  $\Gamma$ : (tricky for charged particles, but required)

$$|D(p, M)|^2 \propto \frac{1}{(p^2 - M^2)^2 + \Gamma^2 M^2} \quad (173)$$

## 6.2 Feynman Rules

- **vector and axial vector couplings**

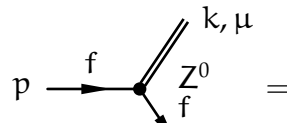
$$g_V = T_3 - 2Q \sin^2 \theta_w \quad (174a)$$

$$g_A = T_3 \quad (174b)$$

- e. g. for **electrons**

$$g_V = -\frac{1 - 4 \sin^2 \theta_w}{2}, \quad g_A = -\frac{1}{2} \quad (175)$$

- **neutral currents**

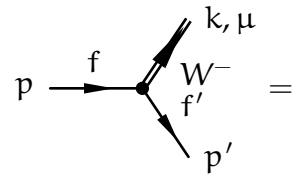


$$= -i \frac{g}{2 \cos \theta_w} (g_V^f \gamma_\mu - g_A^f \gamma_\mu \gamma_5) \quad (176)$$



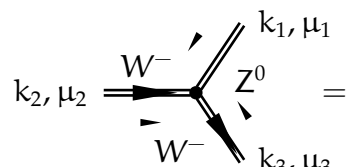
$$= -ie Q_f \gamma_\mu \quad (177)$$

- **charged currents** ( $V_{ff'}$  is the **CKM matrix**)

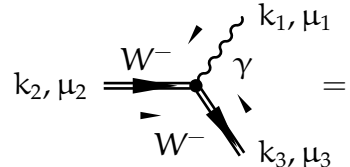


$$= -i \frac{g}{\sqrt{2}} V_{ff'} \tau^+ \gamma_\mu \frac{1 - \gamma_5}{2} \quad (178)$$

- **triple gauge couplings**



$$= ie \cot \theta_w g_{\mu_1 \mu_2} (k_{\mu_3}^1 - k_{\mu_3}^2) + ie \cot \theta_w g_{\mu_2 \mu_3} (k_{\mu_1}^2 - k_{\mu_1}^3) + ie \cot \theta_w g_{\mu_3 \mu_1} (k_{\mu_2}^3 - k_{\mu_2}^1) \quad (179)$$



$$= ieg_{\mu_1 \mu_2} (k_{\mu_3}^1 - k_{\mu_3}^2) + ieg_{\mu_2 \mu_3} (k_{\mu_1}^2 - k_{\mu_1}^3) + ieg_{\mu_3 \mu_1} (k_{\mu_2}^3 - k_{\mu_2}^1) \quad (180)$$

- Yukawa couplings



$$= -i \frac{gm_f}{2M_W} \quad (181)$$



$$= i \frac{gM_Z}{\cos \theta_w} g_{\mu\mu'} \quad (182)$$

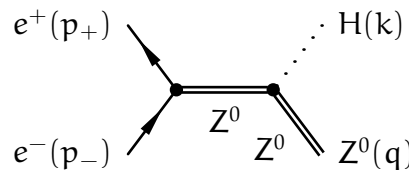


$$= igM_W g_{\mu\mu'} \quad (183)$$

- a **lot** more vertices: **quartic couplings**, **Higgs selfcouplings**, etc.

### 6.3 Higgs Strahlung

**Problem 24.** Compute the scattering amplitude for **Higgs strahlung**  $e^+e^- \rightarrow ZH$



$$(184)$$

ignoring all terms of  $O(m_e/M_Z)$ ,  $O(m_e/M_H)$  and  $O(m_e/\sqrt{s})$ . You may assume that  $\sqrt{s} \gg M_Z$  and ignore the **width** of the Z.

**Problem 25.** Show that the four momenta

$$p_- = (E; 0, 0, E) \quad (185a)$$

$$p_+ = (E; 0, 0, -E) \quad (185b)$$

$$k = (E_H; p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta) \quad (185c)$$

$$q = (E_Z; -p \sin \theta \cos \phi, -p \sin \theta \sin \phi, -p \cos \theta) \quad (185d)$$

with

$$p = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{2\sqrt{s}} \quad (186a)$$

$$E_H = \frac{s + M_H^2 - M_Z^2}{2\sqrt{s}} \quad (186b)$$

$$E_Z = \frac{s + M_Z^2 - M_H^2}{2\sqrt{s}} \quad (186c)$$

and

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \quad (187)$$

satisfy *energy and momentum conservation* and the *mass shell conditions* for  $e^+e^- \rightarrow ZH$  with  $m_e = 0$ . Compute

$$16(p_+q)(p_-q). \quad (188)$$

**Problem 26.** Compute the *differential cross section*

$$\frac{d\sigma}{d\Omega}(\theta_{e^-H}) \quad (189)$$

for *unpolarized Higgs strahlung*  $e^+e^- \rightarrow ZH$ .

**NB:** use the fact that  $\sum \epsilon_\mu \epsilon_\nu^*$  is symmetric in  $\mu \leftrightarrow \nu$ .

**Problem 27.** Compute the *integrated cross section* for *Higgs strahlung*  $e^+e^- \rightarrow ZH$ .



## 7 Solutions

### Solution 1.

$$\partial_\mu e^{-ipx} = -ip_\mu e^{-ipx} \quad (190a)$$

$$(a\partial)(b\partial)e^{-ipx} = -(ap)(bp)e^{-ipx} \quad (190b)$$

$$\partial^2 e^{-ipx} = -i(p\partial)e^{-ipx} = -p^2 e^{-ipx} \quad (190c)$$

### Solution 2.

$$\partial_\mu x^\mu = g_\mu{}^\nu \partial x^\mu / \partial x^\nu = g_\mu{}^\nu \delta^\mu{}_\nu = g_\mu{}^\mu = 4 \quad (191a)$$

$$\begin{aligned} \partial^2 e^{-x^2/2} &= -\partial^\mu \left( x_\mu e^{-x^2/2} \right) = -x_\mu \partial^\mu e^{-x^2/2} - (\partial^\mu x_\mu) e^{-x^2/2} \\ &= x_\mu x^\mu e^{-x^2/2} - g_\mu{}^\mu e^{-x^2/2} = (x^2 - 4) e^{-x^2/2} \end{aligned} \quad (191b)$$

### Solution 3.

$$\begin{aligned} [\gamma^k, \gamma^l]_+ &= \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix} + (k \leftrightarrow l) = \begin{pmatrix} -\sigma^k \sigma^l & 0 \\ 0 & -\sigma^k \sigma^l \end{pmatrix} + (k \leftrightarrow l) \\ &= \begin{pmatrix} -[\sigma^k, \sigma^l]_+ & 0 \\ 0 & -[\sigma^k, \sigma^l]_+ \end{pmatrix} = -2\delta^{kl} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (192a)$$

### Solution 4.

$$\bar{u}_k(p) = u_k^\dagger(p) \gamma_0 = u_k^\dagger(\vec{0}) \gamma^0 \gamma^0 \frac{\not{p} + m}{\sqrt{p_0 + m}} \gamma_0 = \bar{u}_k(\vec{0}) \frac{\not{p} + m}{\sqrt{p_0 + m}} \quad (193a)$$

$$\bar{v}_k(p) = \bar{v}_k(\vec{0}) \frac{\not{p} - m}{\sqrt{p_0 + m}} \quad (193b)$$

Using the definition and multiplying with  $\gamma_0$  from the right

$$\sum_{k=1}^2 u_k(\vec{0}) \bar{u}_k(\vec{0}) = \frac{\gamma_0 + 1}{2}, \quad \sum_{k=1}^2 v_k(\vec{0}) \bar{v}_k(\vec{0}) = \frac{\gamma_0 - 1}{2} \quad (194)$$

Therefore

$$\sum_{k=1}^2 u_k(\mathbf{p}) \bar{u}_k(\mathbf{p}) = \frac{(\not{p} + m)(\gamma_0 + 1)(\not{p} + m)}{2(p_0 + m)} \quad (195a)$$

$$\sum_{k=1}^2 v_k(\mathbf{p}) \bar{v}_k(\mathbf{p}) = \frac{(\not{p} - m)(\gamma_0 - 1)(\not{p} - m)}{2(p_0 + m)} \quad (195b)$$

and (60) follows from

$$\begin{aligned} (\not{p} \pm m)(\gamma_0 \pm 1)(\not{p} \pm m) &= \not{p}\gamma_0\not{p} \pm (m\gamma_0\not{p} + m\not{p}\gamma_0) + m^2\gamma_0 \pm (\not{p} \pm m)^2 \\ &= -\not{p}^2\gamma_0 + 2p_0\not{p} \pm 2p_0m + m^2\gamma_0 + 2m(\not{p} \pm m) = 2(p_0 + m)(\not{p} \pm m). \end{aligned} \quad (196)$$

**Solution 5.**

$$\begin{aligned} \bar{u}_k(\mathbf{p})u_l(\mathbf{p}) &= \bar{u}_k(\vec{0}) \frac{\not{p} + m}{\sqrt{p_0 + m}} \frac{\not{p} + m}{\sqrt{p_0 + m}} u_l(\vec{0}) \\ &= \frac{2m}{p_0 + m} \bar{u}_k(\vec{0})(\not{p} + m)u_l(\vec{0}) \\ &= \frac{2m}{p_0 + m} \bar{u}_k(\vec{0})(p_0 + m)u_l(\vec{0}) = 2m\bar{u}_k(\vec{0})u_l(\vec{0}) = 2m\delta_{kl} \end{aligned} \quad (197)$$

$$\begin{aligned} \bar{v}_k(\mathbf{p})v_l(\mathbf{p}) &= \bar{v}_k(\vec{0}) \frac{\not{p} - m}{\sqrt{p_0 + m}} \frac{\not{p} - m}{\sqrt{p_0 + m}} v_l(\vec{0}) \\ &= \frac{-2m}{p_0 + m} \bar{v}_k(\vec{0})(\not{p} - m)v_l(\vec{0}) \\ &= \frac{2m}{p_0 + m} \bar{v}_k(\vec{0})(p_0 + m)v_l(\vec{0}) = 2m\bar{v}_k(\vec{0})v_l(\vec{0}) = -2m \cdot \delta_{kl} \end{aligned} \quad (198)$$

$$\bar{u}_k(\mathbf{p})v_l(\mathbf{p}) = \bar{u}_k(\vec{0}) \frac{\not{p} + m}{\sqrt{p_0 + m}} \frac{\not{p} - m}{\sqrt{p_0 + m}} v_l(\vec{0}) = 0 \quad (199a)$$

$$\bar{v}_k(\mathbf{p})u_l(\mathbf{p}) = \bar{v}_k(\vec{0}) \frac{\not{p} - m}{\sqrt{p_0 + m}} \frac{\not{p} + m}{\sqrt{p_0 + m}} u_l(\vec{0}) = 0 \quad (199b)$$

**Solution 6.**

$$\begin{aligned}
\bar{u}_k(\mathbf{p})\gamma_\mu u_l(\mathbf{p}) &= \bar{u}_k(\vec{0})\frac{\not{p} + m}{\sqrt{p_0 + m}}\gamma_\mu\frac{\not{p} + m}{\sqrt{p_0 + m}}u_l(\vec{0}) \\
&= \frac{1}{p_0 + m}\bar{u}_k(\vec{0})2p_\mu(\not{p} + m)u_l(\vec{0}) \\
&= \frac{2p_\mu}{p_0 + m}\bar{u}_k(\vec{0})(p_0 + m)u_l(\vec{0}) = 2p_\mu\bar{u}_k(\vec{0})u_l(\vec{0}) = 2p_\mu\delta_{kl} \quad (200a)
\end{aligned}$$

$$\begin{aligned}
\bar{v}_k(\mathbf{p})\gamma_\mu v_l(\mathbf{p}) &= \bar{v}_k(\vec{0})\frac{\not{p} - m}{\sqrt{p_0 + m}}\gamma_\mu\frac{\not{p} - m}{\sqrt{p_0 + m}}v_l(\vec{0}) \\
&= \frac{1}{p_0 + m}\bar{v}_k(\vec{0})2p_\mu(\not{p} - m)v_l(\vec{0}) \\
&= \frac{-2p_\mu}{p_0 + m}\bar{v}_k(\vec{0})(p_0 + m)v_l(\vec{0}) = -2p_\mu\bar{v}_k(\vec{0})v_l(\vec{0}) = 2p_\mu\delta_{kl} \quad (200b)
\end{aligned}$$

$$\begin{aligned}
\bar{u}_k(\vec{p})\gamma_0 v_l(-\vec{p}) &= \bar{u}_k(\vec{0})\frac{\not{p} + m}{\sqrt{p_0 + m}}\gamma_0\frac{\not{p}^\dagger - m}{\sqrt{p_0 + m}}v_l(\vec{0}) \\
&= \bar{u}_k(\vec{0})\frac{\not{p} + m}{\sqrt{p_0 + m}}\frac{\not{p} - m}{\sqrt{p_0 + m}}\gamma_0 v_l(\vec{0}) = 0 \quad (201a)
\end{aligned}$$

$$\bar{v}_k(-\vec{p})\gamma_0 u_l(\vec{p}) = 0 \quad (201b)$$

**Solution 7.**

$$\gamma_5\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = i\gamma^0\gamma^2\gamma^1\gamma^2\gamma^3 = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma_5 \quad (202)$$

since *every*  $\gamma_\mu$  appears exactly once and anti-commutes with the other three, we have for *every*  $\mu$

$$[\gamma_5, \gamma^\mu]_+ = 0 \quad (203)$$

**Solution 8.**

product rule

$$i\partial^\mu [\bar{\psi}_1(x)\gamma_\mu\psi_2(x)] = \bar{\psi}_1(x)\left(i\overleftarrow{\not{\partial}} + i\overrightarrow{\not{\partial}}\right)\psi_2(x) = \bar{\psi}_1(x)(-m + m)\psi_2(x) = 0. \quad (204)$$

**Solution 9.**

product rule and  $[\gamma_5, \gamma_\mu]_+ = 0$

$$\begin{aligned} i\partial^\mu [\bar{\psi}_1(x) \gamma_\mu \gamma_5 \psi_2(x)] &= \bar{\psi}_1(x) \left( i \overleftarrow{\partial} \gamma_5 + i \overrightarrow{\partial} \gamma_5 \right) \psi_2(x) = \bar{\psi}_1(x) \left( i \overleftarrow{\partial} \gamma_5 - \gamma_5 i \overrightarrow{\partial} \right) \psi_2(x) \\ &= \bar{\psi}_1(x) \left( -m \gamma_5 - \gamma_5 m \right) \psi_2(x) = -2m \bar{\psi}_1(x) \gamma_5 \psi_2(x). \end{aligned} \quad (205)$$

**Solution 10.**

$$\begin{aligned} -2i\sigma_{kl} &= \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix} - (k \longleftrightarrow l) \\ &= \begin{pmatrix} -[\sigma^k, \sigma^l]_- & 0 \\ 0 & -[\sigma^k, \sigma^l]_- \end{pmatrix} = -2i\epsilon^{klm} \begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \end{aligned} \quad (206)$$

therefore

$$\sigma_{kl} = \epsilon^{klm} \begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \quad (207)$$

**Solution 11.**

$$(i\partial - m) S(x, m) = (-\partial^2 - m^2) D(x, m) = \delta^4(x) \quad (208)$$

**Solution 12.**

$$\begin{aligned} S(x, m) &= (i\partial + m) \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{\not{p} - m + i\epsilon} \end{aligned} \quad (209)$$

**Solution 13.**

$$\begin{aligned} \text{tr}(\not{a}\not{b}\not{c}\not{d}) &= +\text{tr}(\not{b}\not{c}\not{d}\not{a}) = -\text{tr}(\not{b}\not{c}\not{d}\not{a}) + 2 \cdot \text{ad} \cdot \text{tr}(\not{b}\not{c}) \\ &= +\text{tr}(\not{b}\not{d}\not{c}\not{a}) - 2 \cdot \text{ac} \cdot \text{tr}(\not{b}\not{d}) + 2 \cdot \text{ad} \cdot 4 \cdot \text{bc} \\ &= -\text{tr}(\not{a}\not{b}\not{c}\not{d}) + 2 \cdot \text{ab} \cdot \text{tr}(\not{c}\not{d}) - 2 \cdot \text{ac} \cdot 4 \cdot \text{bd} + 2 \cdot \text{ad} \cdot 4 \cdot \text{bc} \end{aligned}$$

therefore

$$\text{tr}(\not{a}\not{b}\not{c}\not{d}) = 4 \cdot (\text{ab} \cdot \text{cd} - \text{ac} \cdot \text{bd} + \text{ad} \cdot \text{bc}) \quad (133')$$

**Solution 14.**

$$\begin{aligned} L_{\mu\nu} &= \text{tr}[(\not{p} + m)\gamma_\mu(\not{q} - m)\gamma_\nu] = \text{tr}[\not{p}\gamma_\mu\not{q}\gamma_\nu] - m^2 \text{tr}[\gamma_\mu\gamma_\nu] \\ &= 4 \cdot (p_\mu q_\nu - g_{\mu\nu} p q + p_\nu q_\mu) - 4m^2 \cdot g_{\mu\nu} = 4 \cdot (p_\mu q_\nu + p_\nu q_\mu - (p q + m^2)g_{\mu\nu}) \end{aligned} \quad (210)$$

**Solution 15.**

$$\begin{aligned} L_{\rho_2\rho_1} L^{\rho_1\rho_2} &= 16 \cdot (p_{1,\rho_2} p_{2,\rho_1} + p_{1,\rho_1} p_{2,\rho_2} - p_1 p_2 g_{\rho_2\rho_1})(q_1^{\rho_1} q_2^{\rho_2} + q_1^{\rho_2} q_2^{\rho_1} - q_1 q_2 g^{\rho_1\rho_2}) \\ &= 8 \cdot (2(p_1 q_2)2(p_2 q_1) + 2(p_1 q_1)2(p_2 q_2)) = 8 \cdot (u^2 + t^2) \end{aligned} \quad (211)$$

*NB: cross terms cancel:*

$$(p_{1,\rho_2} p_{2,\rho_1} + p_{1,\rho_1} p_{2,\rho_2}) q_1 q_2 g^{\rho_1\rho_2} + p_1 p_2 g_{\rho_2\rho_1} (q_1^{\rho_1} q_2^{\rho_2} + q_1^{\rho_2} q_2^{\rho_1}) = p_1 p_2 g_{\rho_2\rho_1} q_1 q_2 g^{\rho_1\rho_2}$$

**Solution 16.**

$$t = -\frac{s}{2}(1 - \cos \theta), \quad u = -\frac{s}{2}(1 + \cos \theta) \quad (212)$$

therefore

$$\frac{t^2 + u^2}{s^2} = \frac{1 + \cos^2 \theta}{2} \quad (213)$$

$$\begin{aligned} d\sigma &= \frac{1}{4 \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \frac{1}{\prod_i n_i!} \frac{1}{4} |\mathcal{T}|^2 \widetilde{d}q_1 \widetilde{d}q_2 (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) \\ &= \frac{1}{4 \sqrt{(s/2)^2}} \frac{1}{4} |\mathcal{T}|^2 \frac{1}{32\pi^2} d \cos \theta d\phi = \frac{1}{64\pi^2 s} \frac{1}{4} |\mathcal{T}|^2 d\Omega \end{aligned} \quad (214)$$

therefore

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} e^4 (1 + \cos^2 \theta) = \alpha^2 \frac{1}{4s} (1 + \cos^2 \theta) \quad (215)$$

**Solution 17.**

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} 2\pi \int_{-1}^1 d \cos \theta (1 + \cos^2 \theta) = \frac{\alpha^2}{4s} 2\pi \left(2 + \frac{2}{3}\right) = \frac{4\pi\alpha^2}{3s} \quad (216)$$

compare

$$\sigma = \frac{4\pi\alpha^2}{3(\sqrt{s}/\text{TeV})^2} 0.39 \text{ nb} \quad (19'')$$

with

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.0359895(61)} \quad (217)$$

$$\sigma = \frac{87 \text{ fb}}{(\sqrt{s}/\text{TeV})^2} = \frac{8.7 \text{ pb}}{(\sqrt{s}/100 \text{ GeV})^2} \quad (19'')$$

**Solution 18.**

$$i\mathcal{T}_t = \begin{array}{c} e^-(p_1) \quad e^-(q_1) \\ \searrow \quad \swarrow \\ \bullet \\ \gamma[Z^0] \\ \bullet \\ \swarrow \quad \searrow \\ e^+(p_2) \quad e^+(q_2) \end{array} \quad (218a)$$

$$i\mathcal{T}_s = \begin{array}{c} e^+(p_2) \quad e^-(q_1) \\ \swarrow \quad \searrow \\ \bullet \\ \gamma[Z^0] \\ \bullet \\ \swarrow \quad \searrow \\ e^-(p_1) \quad e^+(q_2) \end{array} \quad (218b)$$

**Solution 19.**

$$\therefore t = -\frac{s}{2}(1 - \cos \theta) = -s \sin^2 \left( \frac{\theta}{2} \right) \quad (219a)$$

$$\therefore u = -\frac{s}{2}(1 + \cos \theta) = -s \cos^2 \left( \frac{\theta}{2} \right) \quad (219b)$$

$$\therefore \frac{s^2 + u^2}{t^2} = \frac{1 + \cos^4 \left( \frac{\theta}{2} \right)}{\sin^4 \left( \frac{\theta}{2} \right)} \quad (220a)$$

$$\frac{u^2}{st} = -\frac{\cos^4 \left( \frac{\theta}{2} \right)}{\sin^2 \left( \frac{\theta}{2} \right)} \quad (220b)$$

• finally

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left( \frac{1 + \cos^2 \theta}{2} + \frac{1 + \cos^4 \left( \frac{\theta}{2} \right)}{\sin^4 \left( \frac{\theta}{2} \right)} - 2 \frac{\cos^4 \left( \frac{\theta}{2} \right)}{\sin^2 \left( \frac{\theta}{2} \right)} \right) = \frac{\alpha^2}{4s} \left( \frac{3 + \cos^2 \theta}{1 - \cos \theta} \right)^2 \quad (221)$$

**Solution 20.**

$$i\mathcal{T}_1 = \begin{array}{c} \bar{v}(p_2) \quad v(q_2) \\ \swarrow \quad \swarrow \\ \bullet \quad \bullet \\ \frac{-ig_{\mu\nu}}{(p_1 + p_2)^2 + i\epsilon} \quad \frac{-ieQ\gamma_\nu}{i} \\ \bullet \quad \bullet \\ \frac{i}{q_1 + k - m + i\epsilon} \quad \frac{ie\gamma_\rho}{i} \\ \swarrow \quad \swarrow \\ u(p_1) \quad \bar{u}(q_1) \end{array} \quad (222a)$$

$$i\mathcal{T}_2 = \begin{array}{c} \bar{v}(p_2) \quad v(q_2) \\ \swarrow \quad \swarrow \\ \bullet \quad \bullet \\ \frac{i}{-q_2 - k - m + i\epsilon} \quad \frac{ig\Gamma^a\gamma_\rho}{i} \\ \bullet \quad \bullet \\ \frac{-ig_{\mu\nu}}{(p_1 + p_2)^2 + i\epsilon} \quad \frac{-ieQ\gamma_\nu}{i} \\ \swarrow \quad \swarrow \\ u(p_1) \quad \bar{u}(q_1) \end{array} \quad (222b)$$

$$iT_1 = \left[ \bar{u}(q_1)(igT_a \not{\epsilon}^*(k)) \frac{i}{\not{q}_1 + \not{k} - m + i\epsilon} (-ieQ\gamma^\mu)v(q_2) \right] \times \quad (223a)$$

$$\frac{-i}{(p_1 + p_2)^2 + i\epsilon} [\bar{v}(p_2)(ie\gamma_\mu)u(p_1)]$$

$$iT_2 = \left[ \bar{u}(q_1)(-ieQ\gamma^\mu) \frac{i}{-\not{q}_2 - \not{k} - m + i\epsilon} (igT_a \not{\epsilon}^*(k))v(q_2) \right] \times \quad (223b)$$

$$\frac{-i}{(p_1 + p_2)^2 + i\epsilon} [\bar{v}(p_2)(ie\gamma_\mu)u(p_1)]$$

**Solution 21.**

$$T_1|_{\epsilon_\mu^*(k)=k_\mu} = e^2 Qg [\bar{v}(p_2)\gamma_\mu u(p_1)] \frac{1}{s} \bar{u}(q_1) T_a (\not{k} + \not{q}_1 - m) \frac{1}{\not{q}_1 + \not{k} - m + i\epsilon} \gamma^\mu v(q_2)$$

$$= e^2 Qg [\bar{v}(p_2)\gamma_\mu u(p_1)] \frac{1}{s} \bar{u}(q_1) T_a \gamma^\mu v(q_2) \quad (224)$$

on the other hand

$$T_2|_{\epsilon_\mu^*(k)=k_\mu} = e^2 Qg [\bar{v}(p_2)\gamma_\mu u(p_1)] \frac{1}{s} \bar{u}(q_1) \gamma^\mu \frac{1}{-\not{q}_2 - \not{k} - m + i\epsilon} (\not{k} + \not{q}_2 + m) T_a v(q_2)$$

$$= -e^2 Qg [\bar{v}(p_2)\gamma_\mu u(p_1)] \frac{1}{s} \bar{u}(q_1) T_a \gamma^\mu v(q_2) = -T_1|_{\epsilon_\mu^*(k)=k_\mu} \quad (225)$$

**Solution 22.**

$$2q_1 q_2 = (q_1 + q_2)^2 = (p - k)^2 = s(1 - x_3) \quad (226a)$$

$$2q_1 k = (q_1 + k)^2 = (p - q_2)^2 = s(1 - x_2) \quad (226b)$$

$$2q_2 k = (q_2 + k)^2 = (p - q_1)^2 = s(1 - x_1) \quad (226c)$$

**Solution 23.**

four traces:

$$H^{\mu\nu}(q_1, q_2, k) =$$

$$\sum_\epsilon \frac{\text{tr}[\not{q}_1 T_a \not{\epsilon}^*(q_1 + k) \gamma^\mu \not{q}_2 \gamma^\nu (\not{q}_1 + \not{k}) \not{\epsilon} T_a]}{(2q_1 k)^2} + \sum_\epsilon \frac{\text{tr}[\not{q}_1 T_a \not{\epsilon}^*(q_1 + k) \gamma^\mu \not{q}_2 \not{\epsilon} T_a (-\not{q}_2 - \not{k}) \gamma^\nu]}{(2q_1 k)(2q_2 k)}$$

$$+ \sum_\epsilon \frac{\text{tr}[\not{q}_1 \gamma^\mu (-\not{q}_2 - \not{k}) T_a \not{\epsilon}^* \not{q}_2 \gamma^\nu (\not{q}_1 + \not{k}) \not{\epsilon} T_a]}{(2q_1 k)(2q_2 k)} + \sum_\epsilon \frac{\text{tr}[\not{q}_1 \gamma^\mu (-\not{q}_2 - \not{k}) T_a \not{\epsilon}^* \not{q}_2 \not{\epsilon} T_a (-\not{q}_2 - \not{k}) \gamma^\nu]}{(2q_2 k)^2} \quad (227)$$

color part of the quark traces  $\text{tr}(T_a T_a) = C_F \text{tr}(\mathbf{1}) = C_F N_C$  and polarization sum

$$\begin{aligned} H^{\mu\nu}(q_1, q_2, k) &= 2C_F N_c \frac{\text{tr}[\not{q}_1(\not{q}_1 + \not{k})\gamma^\mu \not{q}_2 \gamma^\nu (\not{q}_1 + \not{k})]}{(2q_1 k)^2} + 2C_F N_c \frac{\text{tr}[\not{q}_1 \not{q}_2 \gamma^\mu (\not{q}_1 + \not{k})(-\not{q}_2 - \not{k})\gamma^\nu]}{(2q_1 k)(2q_2 k)} \\ &+ 2C_F N_c \frac{\text{tr}[\not{q}_1 \gamma^\mu (-\not{q}_2 - \not{k})(\not{q}_1 + \not{k})\gamma^\nu \not{q}_2]}{(2q_1 k)(2q_2 k)} + 2C_F N_c \frac{\text{tr}[\not{q}_1 \gamma^\mu (-\not{q}_2 - \not{k})\not{q}_2 (-\not{q}_2 - \not{k})\gamma^\nu]}{(2q_2 k)^2} \end{aligned} \quad (228)$$

contraction

$$\begin{aligned} H^\mu{}_\mu(q_1, q_2, k) &= -4C_F N_c \frac{\text{tr}[\not{q}_1(\not{q}_1 + \not{k})\not{q}_2(\not{q}_1 + \not{k})]}{(2q_1 k)^2} + 8C_F N_c (q_1 + k)(-q_2 - k) \frac{\text{tr}[\not{q}_1 \not{q}_2]}{(2q_1 k)(2q_2 k)} \\ &+ 8C_F N_c (-q_2 - k)(q_1 + k) \frac{\text{tr}[\not{q}_1 \not{q}_2]}{(2q_1 k)(2q_2 k)} - 4C_F N_c \frac{\text{tr}[\not{q}_1(-\not{q}_2 - \not{k})\not{q}_2(-\not{q}_2 - \not{k})]}{(2q_2 k)^2} \end{aligned} \quad (229)$$

final traces

$$\begin{aligned} H^\mu{}_\mu(q_1, q_2, k) &= -16C_F N_c \frac{2(q_1 k)(q_2 k)}{(2q_1 k)^2} - 64C_F N_c \frac{(q_1 q_2 + q_1 k + q_2 k)(q_1 q_2)}{(2q_1 k)(2q_2 k)} - 16C_F N_c \frac{2(q_1 k)(q_2 k)}{(2q_2 k)^2} \\ &- 8C_F N_c \frac{1 - x_1}{1 - x_2} - 16C_F N_c \frac{(1 - x_3)}{(1 - x_1)(1 - x_2)} - 8C_F N_c \frac{1 - x_2}{1 - x_1} = -8C_F N_c \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \end{aligned} \quad (230)$$

result

$$\frac{d^2\sigma}{dx_1 dx_2} = N_c \frac{4\pi\alpha^2 Q^2}{3s} \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \quad (231)$$

#### Solution 24.

with  $p = p_+ + p_-$ :

$$iT = -i \frac{g}{2 \cos \theta_w} [\bar{v}(p_+) \gamma_\mu (g_V^e - g_A^e \gamma_5) u(p_-)] \frac{-ig^{\mu\nu} + ip^\mu p^\nu / M_Z^2}{p^2 - M_Z^2} (i \frac{g M_Z}{\cos \theta_w} g_{\nu\rho}) \epsilon^{*\rho}(q) \quad (232)$$

and from current conservation  $[\bar{v}(p_+) \gamma_\mu (g_V^e - g_A^e \gamma_5) u(p_-)] p^\mu = -2ig_A^e m_e \bar{v}(p_+) \gamma_5 u(p_-) = O(m_e)$

$$T = -\frac{g^2 M_Z}{2 \cos^2 \theta_w} \frac{1}{s - M_Z^2} [\bar{v}(p_+) \not{\epsilon}^*(q) (g_V^e - g_A^e \gamma_5) u(p_-)] \quad (233)$$

#### Solution 25.

- momentum conservation is *obvious* and energy conservation follows from

$$E_H + E_Z = \sqrt{s} = 2E \quad (234)$$



- the Higgs mass shell follows from

$$E_H^2 - p^2 = \frac{s^2 + M_H^4 + M_Z^4 + 2sM_H^2 - 2sM_Z^2 - 2M_H^2M_Z^2}{4s} - \frac{s^2 + M_H^4 + M_Z^4 - 2sM_H^2 - 2sM_Z^2 - 2M_H^2M_Z^2}{4s} = M_H^2, \quad (235)$$

and the Z mass shell analogously:  $E_Z^2 - p^2 = M_Z^2$ .

- finally

$$4p_{\mp}q = 4E(E_Z \pm p \cos \theta) = s + M_Z^2 - M_H^2 \pm \sqrt{\lambda(s, M_H^2, M_Z^2)} \cos \theta, \quad (236)$$

i. e.

$$\begin{aligned} 16(p_+q)(p_-q) &= (s + M_Z^2 - M_H^2)^2 - \lambda(s, M_H^2, M_Z^2) \cos^2 \theta \\ &= 4sM_Z^2 + \lambda(s, M_H^2, M_Z^2) \sin^2 \theta \end{aligned} \quad (237)$$

### Solution 26.

$$\begin{aligned} \sum_{spins} |T|^2 &= \frac{g^4 M_Z^2}{4 \cos^4 \theta_w} \frac{1}{(s - M_Z^2)^2} \text{tr}(\not{p}_+ \not{\epsilon}^*(q) (g_V^e - g_A^e \gamma_5) \not{p}_- (g_V^e + g_A^e \gamma_5) \not{\epsilon}(q)) \\ &= \frac{g^4 M_Z^2}{4 \cos^4 \theta_w} \frac{1}{(s - M_Z^2)^2} (g_V^{e2} + g_A^{e2}) \text{tr}(\not{p}_+ \not{\epsilon}^*(q) \not{p}_- \not{\epsilon}(q)) \\ &= \frac{g^4 M_Z^2 (g_V^{e2} + g_A^{e2})}{4 \cos^4 \theta_w} \frac{1}{(s - M_Z^2)^2} L^{\mu\nu}(p_+, p_-, 0) \epsilon_\mu^*(q) \epsilon_\nu(q) \end{aligned} \quad (238)$$

polarization sum

$$\sum_{pol.} \epsilon_\mu(q) \epsilon_\nu^*(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2} \quad (239)$$

$$\begin{aligned} \sum_{spins, pol.} |T|^2 &= \frac{g^4 M_Z^2 (g_V^{e2} + g_A^{e2})}{\cos^4 \theta_w} \frac{1}{(s - M_Z^2)^2} (p_+^\mu p_-^\nu + p_-^\mu p_+^\nu - p_+ p_- g^{\mu\nu}) \times \\ &\quad \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2} \right) = \frac{g^4 (g_V^{e2} + g_A^{e2})}{\cos^4 \theta_w} \frac{p_+ p_- M_Z^2 + 2(p_+ q)(p_- q)}{(s - M_Z^2)^2} \\ &= \frac{g^4 (g_V^{e2} + g_A^{e2})}{8 \cos^4 \theta_w} \frac{8sM_Z^2 + \lambda(s, M_H^2, M_Z^2) \sin^2 \theta}{(s - M_Z^2)^2} \end{aligned} \quad (240)$$

phase space (118)

$$\frac{d\sigma}{d\Omega}(\theta_{e^-H}) = \frac{1}{2s} \frac{1}{16\pi^2} \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{2s} \frac{1}{4} \sum_{spins, pol.} |T|^2 \quad (241)$$

i. e.

$$\frac{d\sigma}{d\Omega}(\theta_{e^-H}) = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{s} \frac{\alpha^2(g_V^e{}^2 + g_A^e{}^2)}{128s \sin^4 \theta_w \cos^4 \theta_w} \frac{8sM_Z^2 + \lambda(s, M_H^2, M_Z^2) \sin^2 \theta}{(s - M_Z^2)^2} \quad (242)$$

### Solution 27.

with

$$\int d\Omega (a + b \sin^2 \theta) = 4\pi a + \frac{8\pi}{3} b \quad (243)$$

follows

$$\sigma = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{s} \frac{\pi \alpha^2 (g_V^e{}^2 + g_A^e{}^2)}{48s \sin^4 \theta_w \cos^4 \theta_w} \frac{12sM_Z^2 + \lambda(s, M_H^2, M_Z^2)}{(s - M_Z^2)^2} \quad (244)$$

or

$$\sigma = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{s} \frac{\pi \alpha^2 (1 + (1 - 4 \sin^2 \theta_w)^2)}{192s \sin^4 \theta_w \cos^4 \theta_w} \frac{12sM_Z^2 + \lambda(s, M_H^2, M_Z^2)}{(s - M_Z^2)^2} \quad (245)$$

