Karlsruher Institut für Technologie

Particle Physics Phenomenology after the Higgs Discovery

Precision physics of the Standard Model

Maria Laach School on high energy physics, 03/09/24-12/09/24

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Outline

1) The Standard Model: Lagrangian, fields, interactions and all that

2) Input parameters of the Standard Model and what does it take to get them right

3) The muon decay and the Fermi constant “problem”

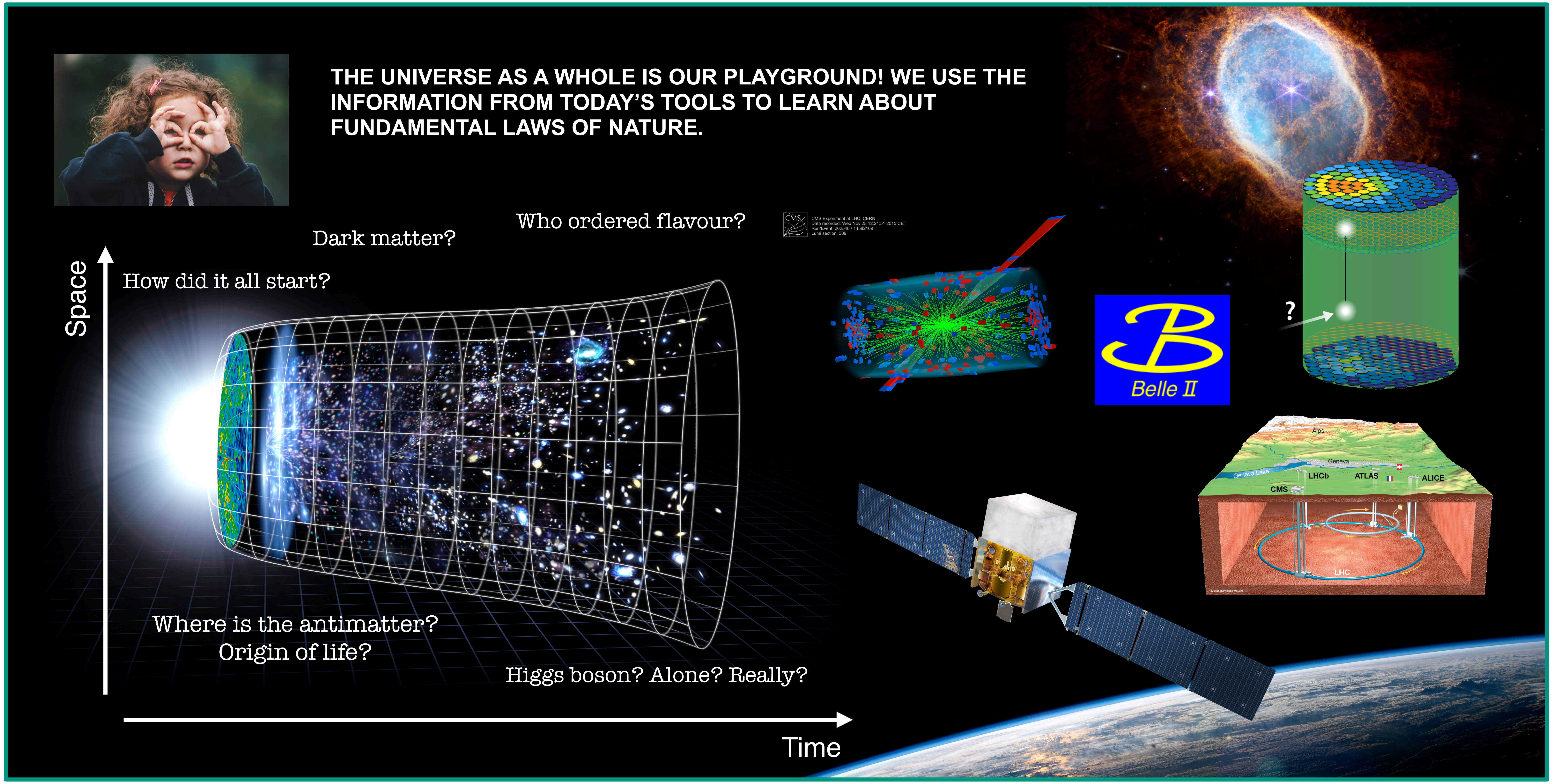
4) The muon anomalous moment saga

5) How to check the unitarity of the CKM matrix

6) Precision physics at the LHC: the anatomy of Higgs production in gluon fusion, electroweak Sudakov logarithms and why the interpretation of the top quark mass measurement is challenging

The Standard Model

In these lectures, I will discuss the Standard Model of particle physics. This (so-called) "model" is not a model — it is a modern theory of strong, electromagnetic and weak interactions, a pinnacle of our understanding of fundamental laws of Nature. And if we add Einstein's gravity to the SM, we get a theory of everything (well, almost), turning the whole Universe into a laboratory where this theory can be studied.

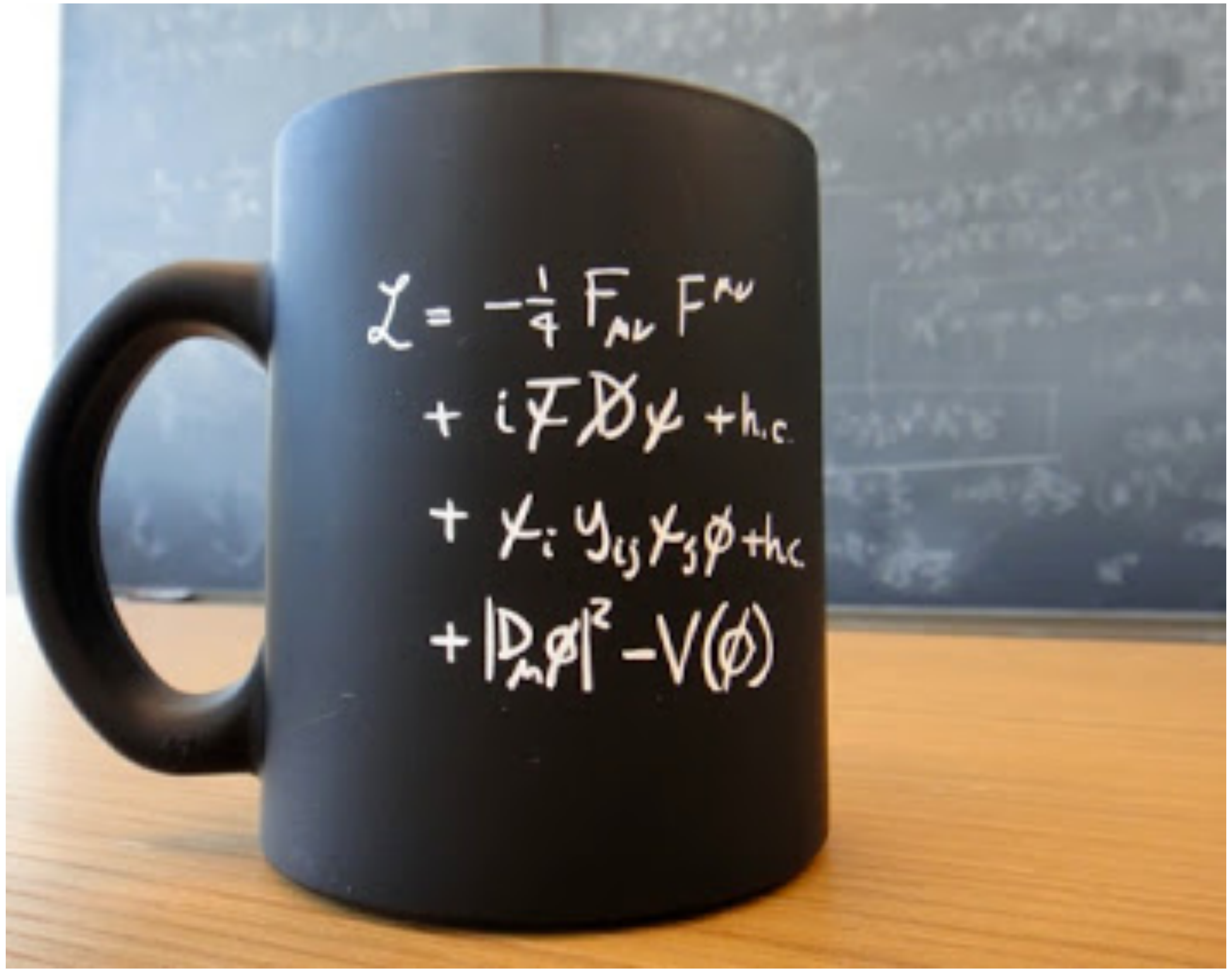


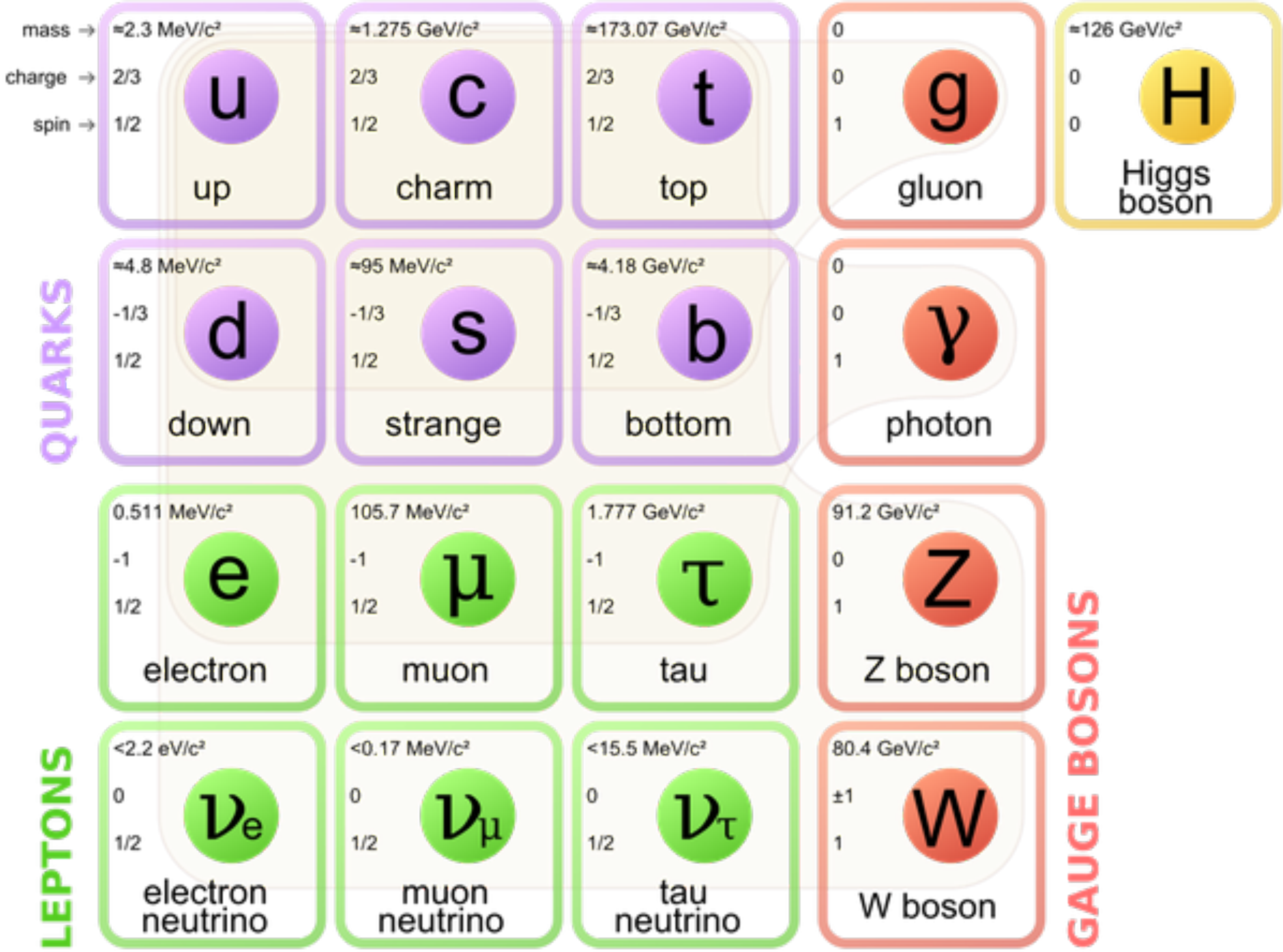
The Standard Model is based on three principal ideas:

- the gauge principle, which forces us to describe electromagnetic, weak and strong interactions by introducing ``compensating'' fields that allow us to turn the global symmetries of the theory into the local ones.
- the idea that the gauge symmetry is broken by the vacuum expectation value of an elementary scalar field: it makes electroweak gauge bosons massive and allows us to provide masses to quarks and leptons without explicitly breaking the gauge invariance in the Lagrangian.
- the requirement of the renormalisability which strongly restricts the number of admissible terms and free parameters in the Lagrangian and, eventually, makes the Standard Model an absolutely predictive theory (at least in principle).

The gauge group of the Standard Model is $SU(3) \times SU(2)_L \times U(1)_Y$. The first gauge group is responsible for the physics of strong interactions; the second and third ones — for weak and electromagnetic ones. The subscript L means "left" and the subscript Y means "hypercharge".

The theory contains matter fields (quarks and leptons). The gauge symmetry is broken by an elementary scalar field (the Higgs field). The theory is renormalisable.





With this, the Lagrangian of the Standard Model is the sum of the following terms:

- the kinetic terms of the gauge fields: they are fully-fixed by the selected gauge groups;
- the kinetic terms of the matter fields: they are fully fixed once quantum numbers of the matter fields (i.e. the way they change under gauge transformations) are specified;
- the kinetic term of the scalar field: it is also fixed once the quantum numbers of the scalar field are fixed;
- the symmetry breaking term of the scalar field that drives the vacuum expectation value of the scalar field to be non-vanishing. Its form is fixed by the requirement of the renormalisability of the theory and the gauge invariance.
- the Yukawa terms: they describe the interaction of the scalar field with the matter fields. They give masses to matter fields after the spontaneous symmetry breaking. The form of the Yukawa terms is fixed by the gauge invariance of the Lagrangian and by the renormalisability of the theory.

The kinetic terms for the gauge fields read:

$$\mathcal{L}_{\text{kin}} = \mathcal{L}^{\text{kin}}_{QCD} + \mathcal{L}^{\text{kin}}_{SU(2)} + \mathcal{L}^{\text{kin}}_{U(1)}$$

$$\mathcal{L}^{\text{kin}}_{\text{QCD}} = -\frac{1}{2}\text{Tr}\left[\hat{G}_{\mu\nu}\hat{G}^{\mu\nu}\right] \qquad \mathcal{L}^{\text{kin}}_{\text{SU(2)}} = -\frac{1}{2}\text{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \qquad \mathcal{L}^{\text{kin}}_{U(1)} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

$$\hat{G}^{\mu\nu} = \partial^\mu\hat{G}^\nu - \partial^\nu\hat{G}^\mu - ig_s[\hat{G}^\mu, \hat{G}^\nu] \qquad \hat{W}^{\mu\nu} = \partial^\mu\hat{W}^\nu - \partial^\nu\hat{W}^\mu - ig[\hat{W}^\mu, \hat{W}^\nu] \qquad B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

The vector potentials are matrices in the corresponding "Lie algebra" space, written as the linear combinations of generators of the corresponding algebra

$$\hat{G}^\mu = \sum_1^8 G^{a,\mu}T^a_s \qquad \hat{W}^\mu = \sum_1^3 W^{i,\mu}T^i$$

Generators are normalised in a standard way and satisfy the standard Lie algebra commutation relations

$$\text{Tr}[T^a, T^b] = \frac{1}{2}\delta^{ab} \qquad [T^i, T^j] = i\epsilon^{ijk}T^k \qquad [T^a_s, T^b_s] = if^{abc}T^c$$

To add matter terms, we need to distinguish between left and right fermion fields. This is necessary since “left” fields participate in the (charged) weak interactions and both “left” and “right” fields are involved in the electromagnetic ones.

The left and right fields are constructed using the projection operators that involve the Dirac matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\psi_L = \frac{1-\gamma_5}{2}\,\psi \qquad \psi_R = \frac{1+\gamma_5}{2}\,\psi \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Since left and right fields transform differently under gauge transformations, mass terms for matter fields are forbidden. Hence, all matter particles in the SM Lagrangian are originally massless.

$$m\bar{\psi}\psi = \frac{m}{2}\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right)$$

We will only consider left-handed neutrinos. Three generations of leptons and of up-type and down-type quarks are included into the theory. Left-handed fields are $SU_L(2)$ doublets; right-handed fields are singlets of the $SU(2)_L$ gauge group. Both left fields and right fields transform under $U(1)_Y$.

$$L_{j,L} = \begin{pmatrix} \nu_j \\ l_j \end{pmatrix}_L, \quad l_{j,R}, \quad \Psi_{j,L} = \begin{pmatrix} U_j \\ D_j \end{pmatrix}_L, \quad U_{j,R}, \quad D_{j,R}$$

The kinetic term for leptons and quarks reads

$$L_{\rm kin} = \sum_j \bar{L}_{j,L} i D_\mu \gamma^\mu L_{j,L} + \sum_j \bar{l}_{j,R} i D_\mu \gamma^\mu l_{j,R} + \sum_j \bar{\Psi}_{j,L} i D_\mu \gamma^\mu \Psi_{j,L} \quad + \sum_j \bar{U}_{j,R} i D_\mu \gamma^\mu U_{j,R} + + \sum_j \bar{D}_{j,R} i D_\mu \gamma^\mu D_{j,R}$$

where the covariant derivative reads

$$D_\mu = \partial_\mu - ig\hat{W} - ig'\hat{Y} B_\mu - ig_s\hat{G} \qquad \hat{W}^\mu = \sum_1^3 W^{i,\mu} T^i \qquad \hat{G}^\mu = \sum_1^8 G^{a,\mu} T_s^a$$

The way the covariant derivative acts on the matter fields follows from the formulas below where $\vec{\tau}$ are the Pauli matrices and $\vec{\lambda}$ are the Gell-Mann matrices.

$$SU_L(2) \qquad \vec{T} L_{j,L} = \frac{\vec{\tau}}{2} L_{j,L}, \quad \vec{T} l_{j,R} = 0, \quad \vec{T}\Psi_{j,L} = \frac{\vec{\tau}}{2}\Psi_{j,L}, \quad \vec{T} U_{j,R} = \vec{T} D_{j,R} = 0,$$

$$U_Y(1) \quad \hat{Y} L_{j,L} = -\frac{1}{2} L_{j,L}, \quad \hat{Y} l_{j,R} = -l_{j,R}, \quad \hat{Y}\Psi_{j,L} = \frac{1}{6}\Psi_{j,L}, \quad \hat{Y} U_{j,R} = \frac{2}{3} U_{j,R}, \quad \hat{Y} D_{j,R} = -\frac{1}{3} D_{j,R},$$

$$SU(3) \qquad \vec{T}_s L_{j,L} = \vec{T}_s l_{j,R} = 0, \quad \vec{T}_s \Psi_{j,L} = \frac{\vec{\lambda}}{2}\Psi_{j,L}, \quad \vec{T}_s U_{j,R} = \frac{\vec{\lambda}}{2} U_{j,R}, \quad \vec{T}_s D_{j,R} = \frac{\vec{\lambda}}{2} D_{j,R}.$$

The kinetic term of the scalar field and the symmetry breaking term read

$$L_{\text{Higgs}} = (D_\mu \varphi)^\dagger (D^\mu \varphi) - \lambda \left(\varphi^\dagger \varphi - \frac{v^2}{2} \right)^2 \qquad \varphi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}$$

$$D_\mu = \partial_\mu - ig\hat{W}^i_\mu - ig' B_\mu \hat{Y} \qquad \vec{T}\phi = \frac{\vec{\tau}}{2}\phi, \quad \hat{Y}\phi = \frac{1}{2}\phi$$

Gauge transformations allow us to remove three real fields from the Higgs doublet

$$\varphi \to U(x)\varphi(x), \quad U(x) = e^{iT^i\theta^i(x)} \qquad \varphi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} \Rightarrow \varphi(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}$$

The Lagrangian becomes $$L_{\text{Higgs}} = (D_\mu \varphi)^\dagger (D^\mu \varphi) - \lambda v^2 h^2 \left(1 + \frac{h}{2v} \right)^2$$

The mass of the Higgs boson reads $$m_h^2 = 2\lambda v^2$$

Triple and quartic Higgs boson couplings are fully determined once the vacuum expectation value and the Higgs mass are known.

The Higgs boson kinetic term plays a very important role in the Standard Model as it allows us to generate masses for gauge fields.

$$L_{\text{Higgs}} = (D_\mu \varphi)^\dagger (D^\mu \varphi) - \lambda \left(\varphi^\dagger \varphi - \frac{v^2}{2} \right)^2 \qquad D_\mu = \partial_\mu - i g \hat{W}^i_\mu - i g' B_\mu \hat{Y}$$

By definition, mass terms are quadratic in the corresponding fields; to find the mass terms of the gauge fields, we need to replace the Higgs field with its vacuum expectation value.

$$\varphi(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} \quad \Rightarrow \quad \varphi_{\text{vac}} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

From the kinetic term, we find

$$(D_\mu \varphi)^\dagger (D^\mu \varphi) \to \frac{v^2 g^2}{8} \left(W^1_\mu W^{1,\mu} + W^2_\mu W^{2,\mu} \right) + \frac{v^2 (g^2 + g'^2)}{8} \left(\cos\theta_W W^3_\mu - \sin\theta_W B_\mu \right)^2$$

where cosine and sine of the weak mixing angle are defined through the gauge coupling constants

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}$$

Written in terms of the W^3 and B fields, the mass term is not diagonal. This is inconvenient.

$$(D_\mu\varphi)^\dagger (D^\mu\varphi) \to \frac{v^2g^2}{8}\left(W^1_\mu W^{1,\mu} + W^2_\mu W^{2,\mu}\right) + \frac{v^2(g^2+g'^2)}{8}\left(\cos\theta_W W^3_\mu - \sin\theta_W B_\mu\right)^2$$

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$$

To take care of this problem, we define new fields which diagonalize the mass matrix. They read

$$Z_\mu = \cos\theta_W\, W^{(3)}_\mu - \sin\theta_W\, B_\mu, \quad A_\mu = \sin\theta_W\, W^{(3)}_\mu + \cos\theta_W\, B_\mu \qquad W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right)$$

The masses of the gauge bosons are: $m_W = \frac{vg}{2}, \quad m_Z = \frac{vg}{2\cos\theta_W}, \quad m_A = 0$

There is an important relation between W and Z masses and the weak mixing angle

$$m_W = m_Z \cos\theta_W$$

Since the field A is massless, it is a candidate to describe the electromagnetic field.

Interactions of matter fields with gauge bosons are hidden in the kinetic term for the matter fields, more precisely in the covariant derivatives.

$$L_{\text{kin}} = \sum_j \bar{L}_{j,L} i D_\mu \gamma^\mu L_{j,L} + \sum_j \bar{l}_{j,R} i D_\mu \gamma^\mu l_{j,R} + \sum_j \bar{\Psi}_{j,L} i D_\mu \gamma^\mu \Psi_{j,L} \qquad D_\mu = \partial_\mu - ig\hat{W} - ig'\hat{Y} B_\mu - ig_s\hat{G}$$

We rewrite the covariant derivative in terms of the physical (mass eigenstate) gauge fields.

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}\left(W_\mu^- T^- + W_\mu^+ T^+\right) - iZ_\mu (g\cos\theta_W T^3 - g'\sin\theta_W Y) - ig\sin\theta_W A_\mu \ (T_3 + Y)$$

$$T^\pm = T_1 \pm iT_2$$

In QED, the coupling of a fermion with the charge Qe (e > 0) to a photon field reads

$$i\bar{\psi}\gamma^\mu \left(\partial^\mu - iQeA_\mu\right)\psi$$

Comparing this expression with the above covariant derivative, we find

$$e = g\sin\theta_W \qquad\qquad Q = T_3 + Y$$

Standard results for electric charges of neutrinos (Q = 0), leptons (Q=-1), up-quarks (Q=+2/3) and down-quarks (Q = -1/3) easily follow from the weak isospin and hypercharge assignments discussed earlier.

Before we continue with the discussion of weak interactions of the matter fields, we need to discuss the Yukawa interactions. The most general Yukawa Lagrangian reads

$$L_{y,L} = -f_{jk}\bar{L}_{j,L}\phi l_{k,R} + h.c. - \left(f_{jk}^{(d)}\bar{\Psi}_{L,j}\phi\, D_{k,R} + h.c\right) - \left(f_{jk}^{(u)}\bar{\Psi}_{L,j}\tilde{\phi}\, U_{k,R} + h.c\right) \qquad \tilde{\phi} = (i\tau_2\phi^*)$$

After the spontaneous symmetry breaking, we are left with terms that are quadratic in fermion fields, but where different "generations" of leptons and quarks mix

$$L_{\rm mass} = -\frac{v}{\sqrt{2}}\left(f_{jk}\bar{l}_{j,L}l_{k,R} + f_{j,k}^{*}\bar{l}_{k,R}l_{j,L}\right) - \frac{v}{\sqrt{2}}\left(\bar{D}_{L,j}f_{jk}^{(d)}D_{R,k} + h.c.\right) - \frac{v}{\sqrt{2}}\left(\bar{U}_{L,j}f_{jk}^{(u)}U_{R,k} + h.c.\right)$$

These terms have to be diagonalised. This requires different "inter-generational" rotations of left-handed U and D fields which, however, are part of the same left doublet. The gauge-interaction term that induces transitions from U to D and vice versa becomes affected by the 3 x 3 unitary mixing matrix, which describes the "mismatch" in rotations of up-type and down-types left-handed fields

$$V^{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 0.9743(1) & 0.2253(6) & 0.0035(1) \\ 0.2252(6) & 0.9734(1) & 0.041(1) \\ 0.0087(3) & 0.040(1) & 0.99915(3) \end{pmatrix}$$

Since for us neutrinos are massless, a similar matrix does not appear in the lepton sector.

It is convenient to re-write the Z-boson contribution to the covariant derivative through the third component of weak isospin T_3 and the electric charge. We find

$$-i Z_\mu \frac{g}{\cos\theta_W}\left(T_3 - Q\sin\theta_W^2\right)$$

The contribution to the Lagrangian is written as $$\mathcal{L}_Z = \frac{g}{\cos\theta_W} J_Z^\mu \, Z_\mu$$

The current reads $$J_Z^\mu = \frac{1}{2}\sum_{\psi\in l,q}\bar\psi\left[(T_L^3 - Q\sin^2\theta_W)\gamma_\mu - T_L^3\gamma^\mu\gamma_5\right]\psi$$

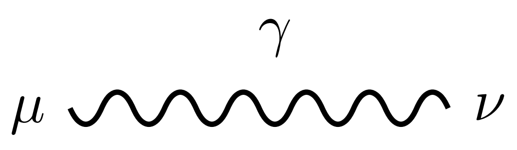
where we sum over all quarks and leptons.

The W-bosons contribution reads

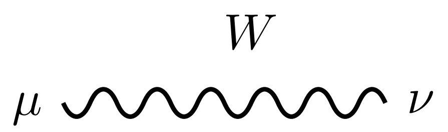
$$\mathcal{L}_W = \frac{g}{\sqrt{2}} W_\mu^+ J_W^{\mu,+} + h.c. \qquad J_W^{\mu,+} = \frac{1}{2}\sum_l \bar\nu_l\gamma^\mu(1-\gamma_5)l + \frac{1}{2}\sum_{i,j}\bar u_i\gamma^\mu(1-\gamma_5)V_{ij}d_j$$

Note the CKM matrix in the quark current.

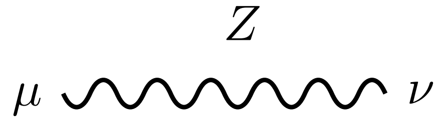
The Standard Model is a weakly-coupled theory; therefore, we can use the perturbative expansion to arrive at physical predictions. Here are the examples of the Feynman rules:



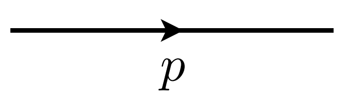
$$-i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}\right]$$



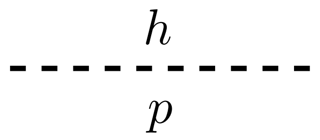
$$\frac{-i}{k^2-M_Z^2}\left[-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2-\xi m_Z^2}(1-\xi)\right]$$



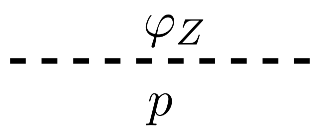
$$\frac{-i}{k^2-M_W^2}\left[-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2-\xi m_W^2}(1-\xi)\right]$$



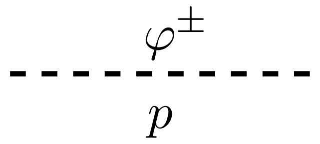
$$\frac{i(\not p+m_f)}{p^2-m_f^2+i\epsilon}$$



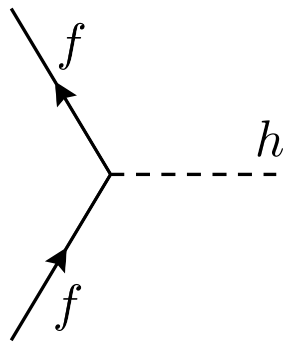
$$\frac{i}{p^2-M_h^2+i\epsilon}$$



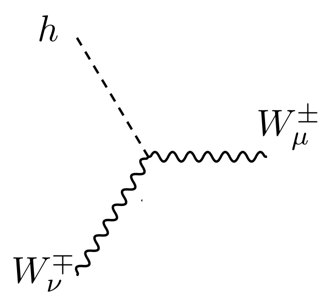
$$\frac{i}{p^2-\xi m_Z^2+i\epsilon}$$



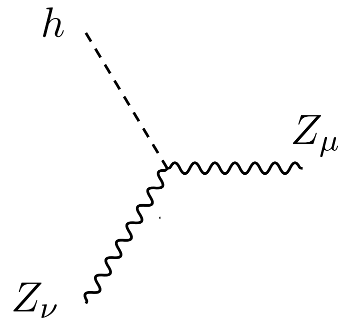
$$\frac{i}{p^2-\xi m_W^2+i\epsilon}$$



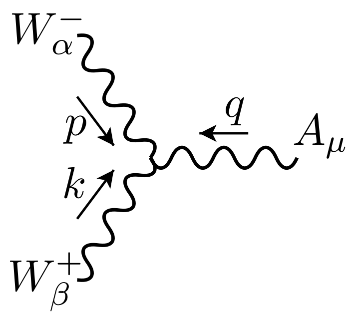
$$-i\,\frac{g}{2}\,\frac{m_f}{m_W}$$



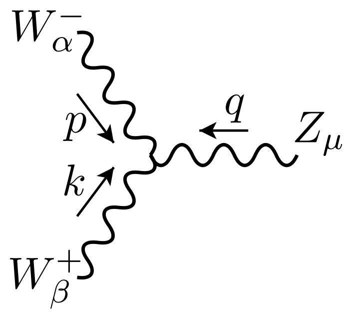
$$ig\,m_W\,g_{\mu\nu}$$



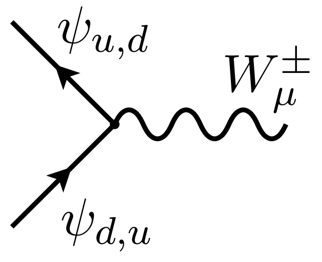
$$i\,\frac{g}{\cos\theta_W}\,m_Z\,g_{\mu\nu}$$



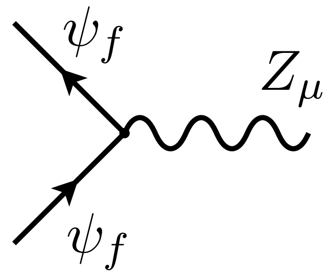
$$-ie\left[g_{\alpha\beta}(p-k)_\mu+g_{\beta\mu}(k-q)_\alpha+g_{\mu\alpha}(q-p)_\beta\right]$$



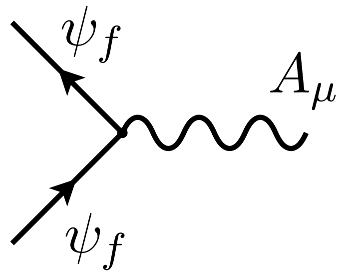
$$ig\cos\theta_W\left[g_{\alpha\beta}(p-k)_\mu+g_{\beta\mu}(k-q)_\alpha+g_{\mu\alpha}(q-p)_\beta\right]$$



$$i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1-\gamma_5}{2}$$

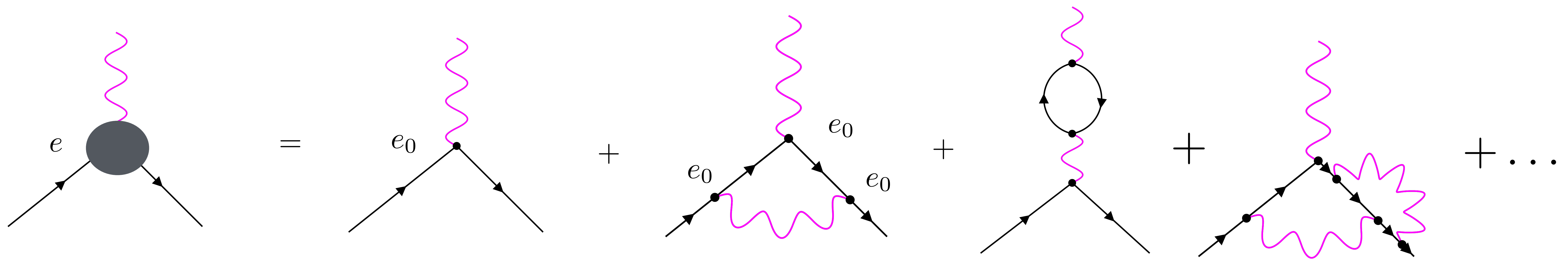


$$i\frac{g}{\cos\theta_W}\gamma_\mu\left(g_V^f-g_A^f\gamma_5\right)$$



$$-ieQ_f\gamma_\mu$$

In quantum field theory, one has to distinguish between parameters that appear in the Lagrangian and physical quantities observed in Nature since they may not be the same. Physical quantities, such as masses, couplings etc. have to be defined through physical observables and related to the Lagrangian parameters through (perturbative) computations.



$$e = F(e_0) \quad \Rightarrow \quad e_0 = F^{-1}(e)$$

Such computations may lead to poorly defined quantities that we refer to as divergent. These divergencies affect relations between physical and Lagrangian parameters, but this is a technical issue that, by itself, has nothing to do with the need for the renormalization.

In renormalisable theories, this problem of divergencies is taken care of by fixing a few physical parameters to their experimental values. If this is done and the results of the calculation are written in terms of physical parameters, all predictions become finite (i.e. independent of the “divergencies problem”).

The Standard Model was proven to be a renormalisable theory. This implies that by fixing the (finite) number of input SM parameters from experimental measurements, we get a theory with an absolute predictive power (provided that our computational prowess is sufficient).

The input parameters include masses of quarks and leptons, CKM matrix elements, the Higgs self-coupling constant, the Higgs field vacuum expectation value and the gauge couplings for SU(3), SU(2) and U(1) gauge groups.

A useful alternative to fixing two gauge couplings (SU(2) and U(1)), the Higgs self-coupling and the Higgs vacuum expectation value, is to fix masses of W and Z bosons, the Higgs boson mass and the value of the electromagnetic coupling constant.

Remarks on precision SM physics

Historically, after the SM was formulated as a theory of weak and electromagnetic interactions, and the first bunch of particles that one needed was discovered (charm quark, tau-neutrino, bottom quark, W and Z bosons), masses of the top quark and the Higgs boson remained unknown.

Because of this, the original goal of the SM precision physics was to determine the missing input parameters of the SM — the Higgs mass and the top quark mass — from their indirect effects on observables that can be precisely measured. However, with the discovery of the top quark mass and, later, the Higgs boson, and very precise measurements of their masses, the theory became fully determined.

For this reason, the current goal of the precision SM physics program is to systematically compare predictions of the SM with the results of measurements for many observables. We hope that by doing this, we will be able to establish a credible need for physics beyond the Standard Model.

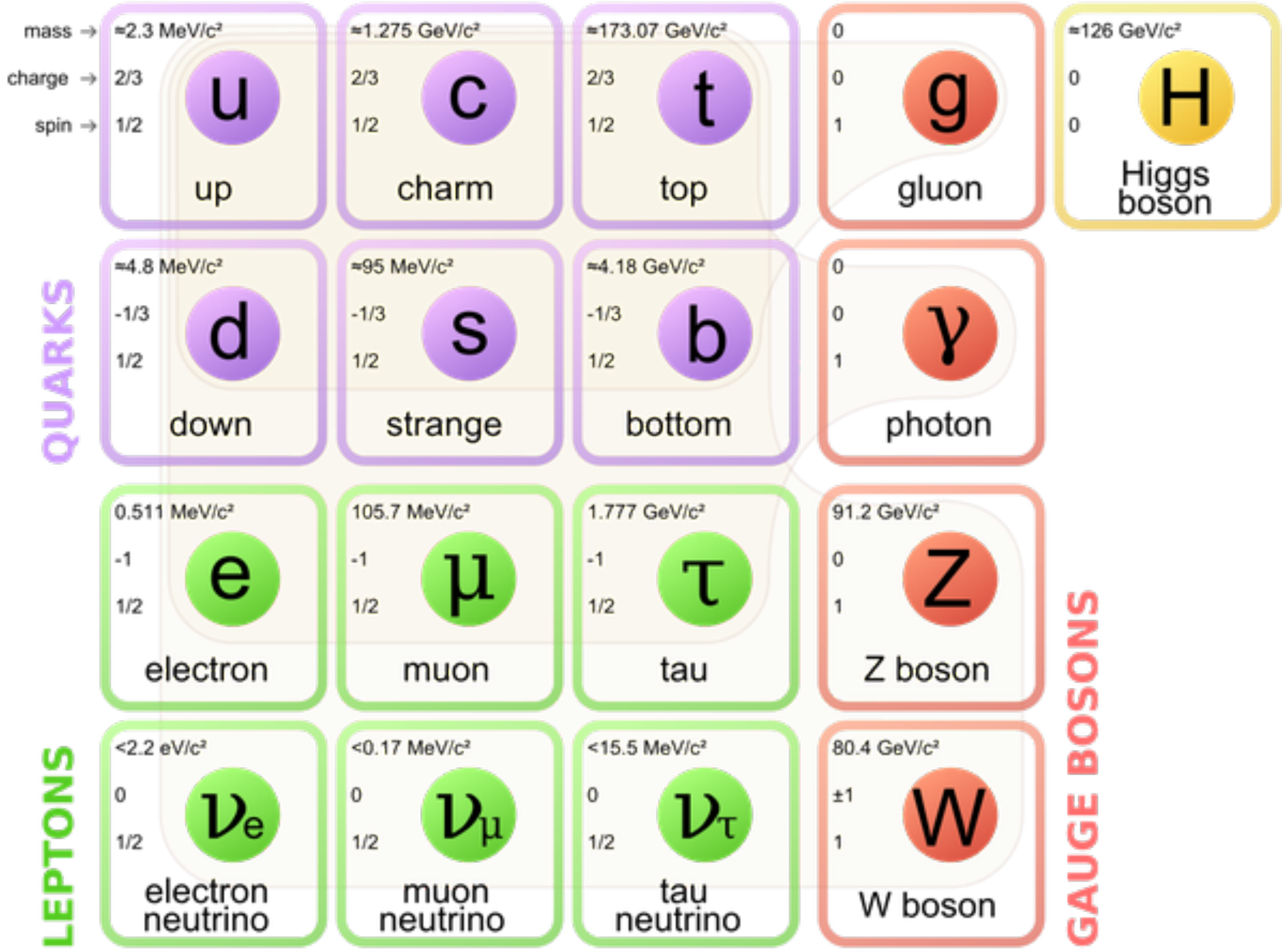
This simple idea is behind all high-precision low-energy experiments; it is also becoming the dominant philosophy behind the many LHC measurements. However, it has important limitations because getting to higher and higher precision forces us to dive deeper and deeper into complicated physics leading to uncertain outcomes.

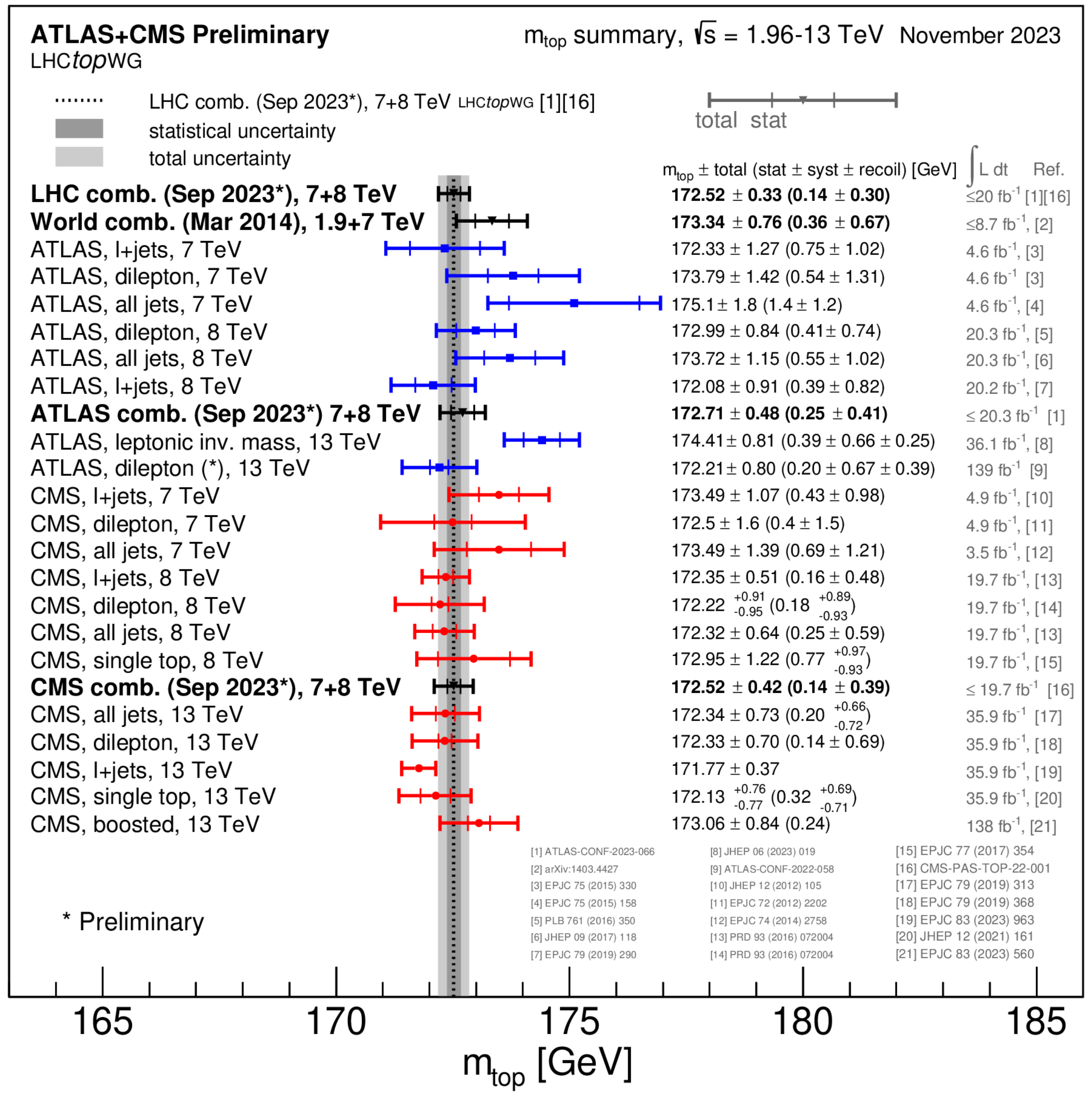
When discussing precision physics one has to remember that

- not all observables are equal; there are observables that are easier to understand theoretically than the other ones; simple observables must carry more weight in the comparison with the SM.
- SM physics is not the same as "perturbative SM physics"; in some cases, we start feeling "lack of perturbativity" by pushing to higher and higher precision.
- observables that can be studied at the highest energies are important because effects of heavy New Physics at high energies are more pronounced. Thus, lower relative precision of SM predictions at high energies is typically sufficient to probe for New Physics.

Masses of the Standard Model particles (and some other things)

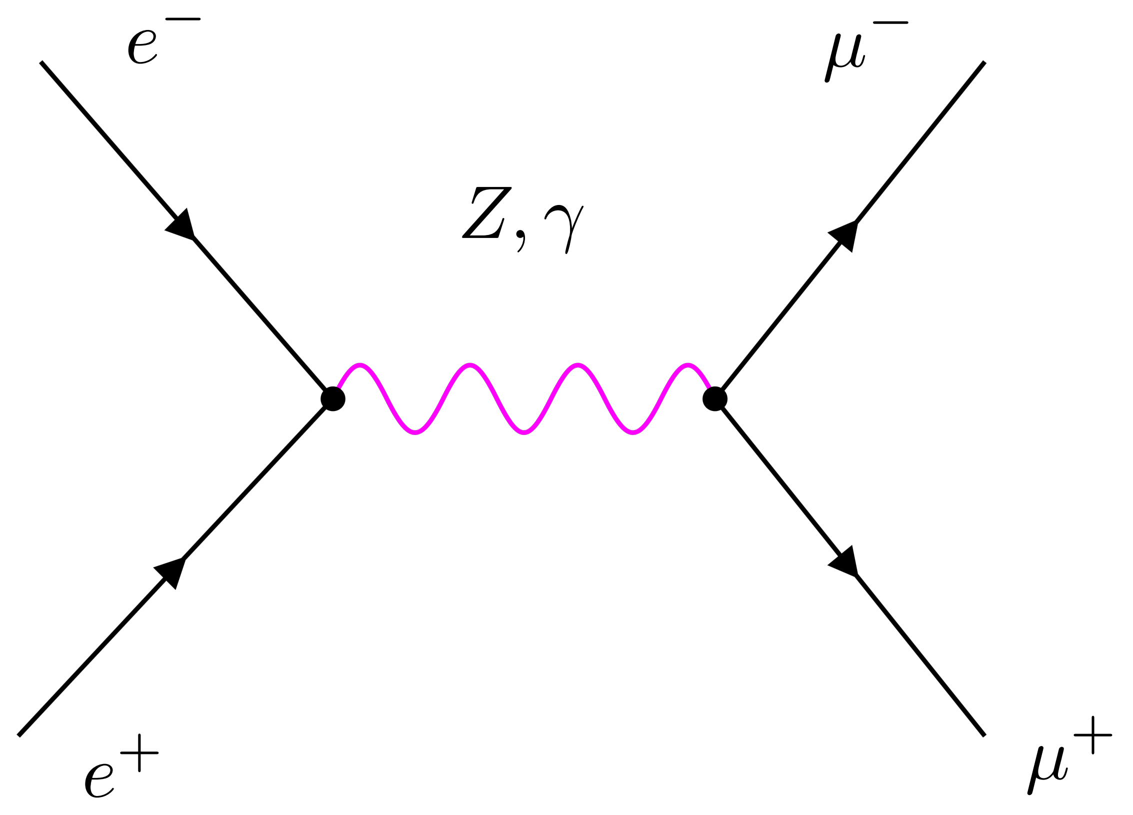
An important class of required SM inputs is comprised by the masses of elementary particles. Electron and muon masses are know from atomic physics, light-quark masses are known poorly but do not really matter, c-quark and b-quark masses are derived from D and B meson masses and the rest comes from collider physics measurements. Note that the (relative) precision of the top quark mass measurement is extraordinary (a few per mille).





$$m_t = 172.52 \pm 0.33 \text{ GeV}$$

The Z-boson mass was measured at LEP. Z-bosons were produced at LEP in collisions of electrons and positrons, and their decays into various final states were studied. An amplitude to describe the electron-positron annihilation into a pair of muons reads



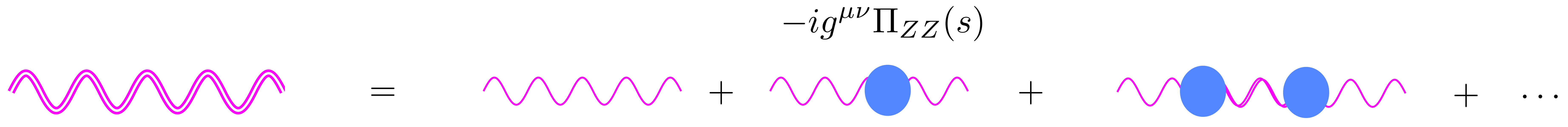
$$i\mathcal{M} = \tilde{J}^{\rho}_{Z,e} \frac{g_{\rho\sigma}}{s - m_Z^2} \tilde{J}^{\sigma}_{Z,\mu} + \tilde{J}^{\rho}_{\gamma,e} \frac{g_{\rho\sigma}}{s} \tilde{J}^{\sigma}_{\gamma,\mu}$$

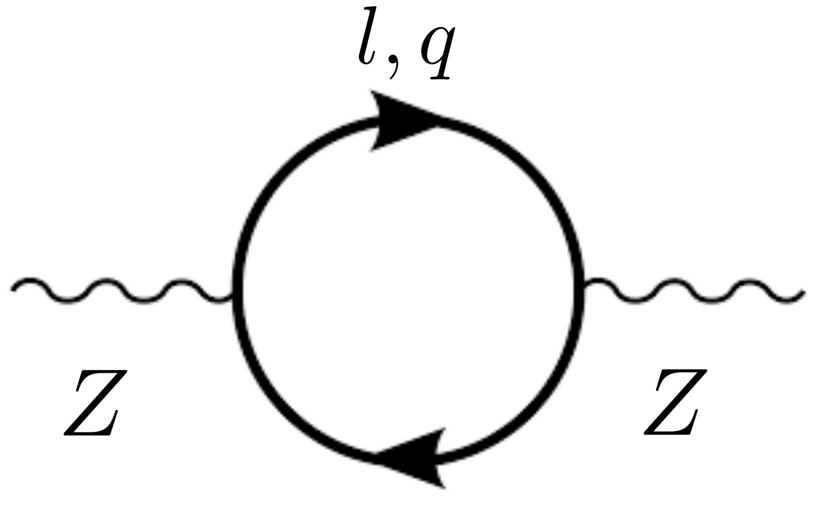
$$J_Z^\mu = \frac{1}{2} \sum_{\psi \in l,q} \bar{\psi} \left[(T_L^3 - Q \sin^2\theta_W)\gamma_\mu - T_L^3 \gamma^\mu \gamma_5 \right] \psi \qquad \mathcal{L}_Z = \frac{g}{\cos\theta_W} J_Z^\mu Z_\mu$$

At $s \approx m_Z^2$ the non-resonant (photon) term can be neglected.

$$i\mathcal{M} = \frac{R(m_Z^2)}{s - m_Z^2}$$

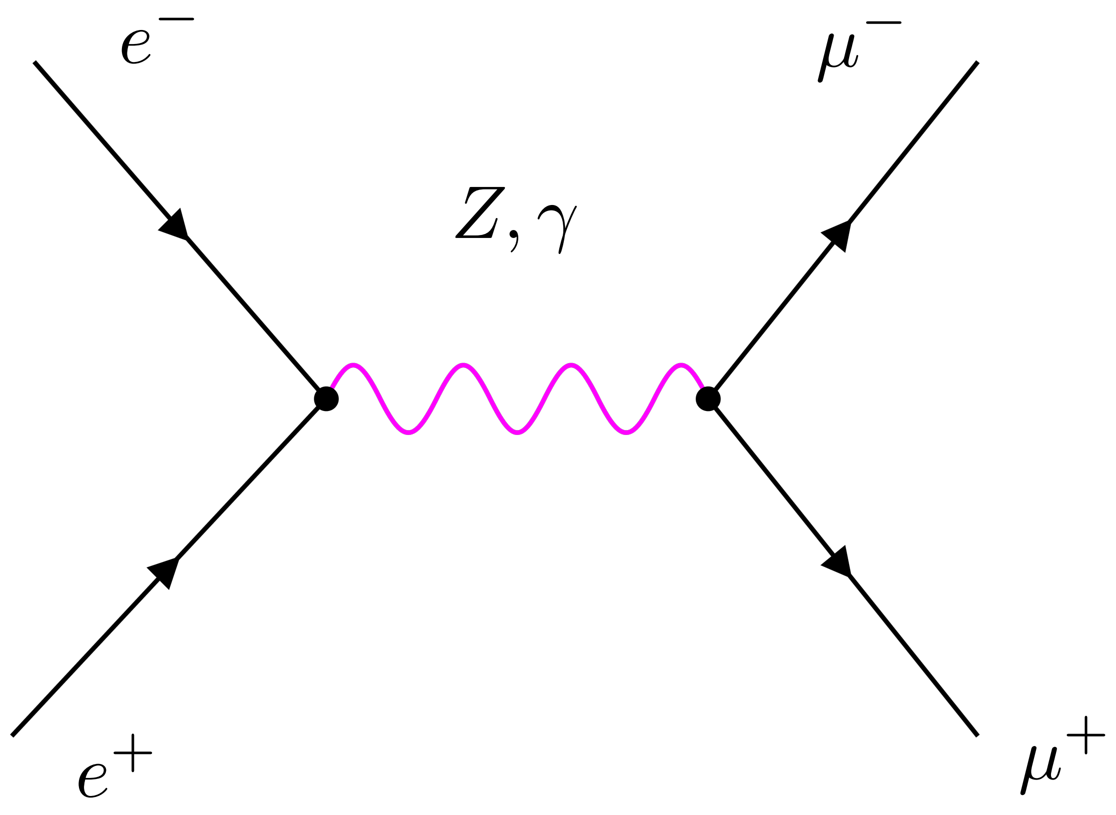
However, now there is an apparent problem since the above matrix element blows up at the most interesting kinematic point $s \approx m_Z^2$. What happens there?





$$\frac{-ig^{\mu\nu}}{s - m_Z^2} \Rightarrow \frac{-ig_{\mu\nu}}{s - m_Z^2 + \Pi_{ZZ}(s)}$$

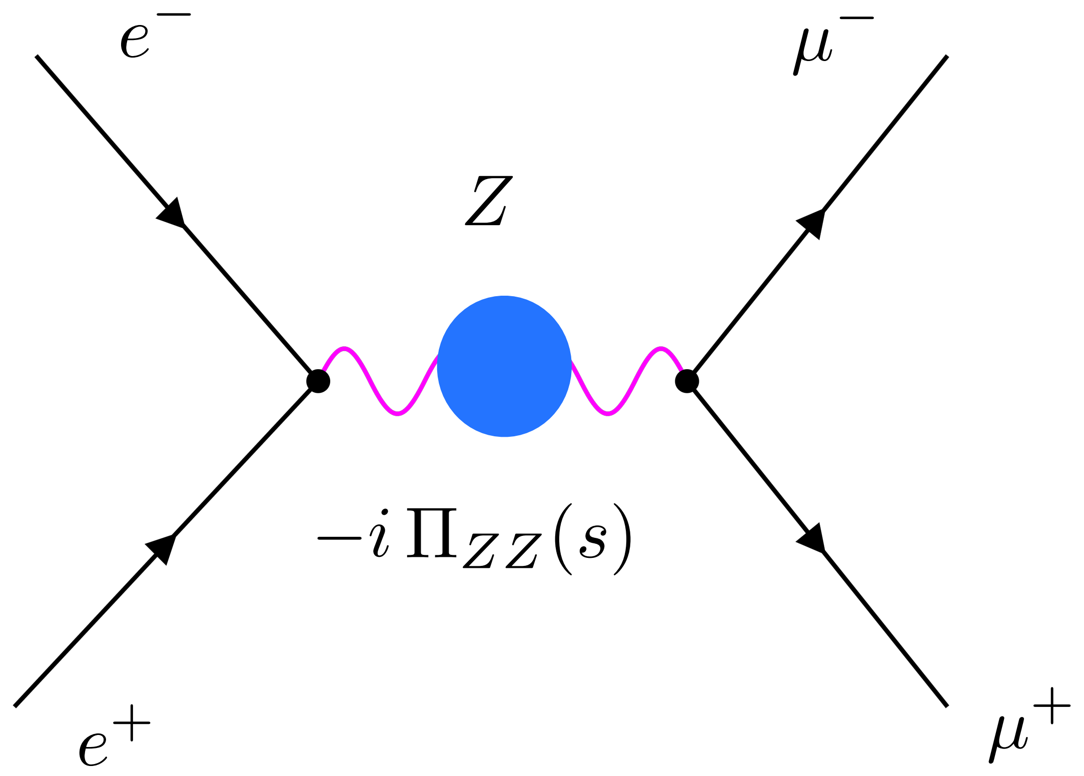
The Z-boson mass was measured at LEP. Z-bosons were produced at LEP in collisions of electrons and positrons, and their decays into various final states were studied. An amplitude to describe the electron-positron annihilation into a pair of muons reads



$$i\mathcal{M} = \tilde{J}^{\rho}_{Z,e} \frac{g_{\rho\sigma}}{s - m_Z^2} \tilde{J}^{\sigma}_{Z,\mu} + \tilde{J}^{\rho}_{\gamma,e} \frac{g_{\rho\sigma}}{s} \tilde{J}^{\sigma}_{\gamma,\mu}$$

At $s \approx m_Z^2$ the non-resonant (photon) term can be neglected.

$$i\mathcal{M} = \frac{R(m_Z^2)}{s - m_Z^2}$$



The residue R is computed from the product of the electron and the muon currents. The infinity at $s = m_Z^2$ is avoided by the re-summation of the vacuum polarization contributions in the vicinity of $s \approx m_Z^2$.

$$i\mathcal{M} = \frac{R}{s - m_Z^2 + \Pi_{zz}(s)}$$

$$m_Z^2 - \mathrm{Re}\left[\Pi_{ZZ}(m^2_{z,\mathrm{phys}})\right] = m^2_{z,\mathrm{phys}} \qquad \mathrm{Im}\left[\Pi_{ZZ}(m^2_{z,\mathrm{phys}})\right] = m_{z,\mathrm{phys}}\Gamma_Z$$

Once this is done the matrix element is defined for all values of s, and $m_{z,\mathrm{phys}} \to m_Z$ is the physical Z-boson mass.

$$i\mathcal{M} = \frac{R}{s - m_Z^2 + i\Gamma_z m_z} + \cdot$$

The cross section for mu-pair production in the electron-positron annihilation in the vicinity of the Z-pole is computed as follows. The starting point is the familiar expression that involves squared amplitude, the normalization factor and the phase space.

$$d\sigma_Z = \mathcal{N} \frac{L_{z,e}^{\rho\sigma} \; L_{z,\mu}^{\rho\sigma}}{(s-m_z)^2 + m_z^2\Gamma_z^2} d\Phi \qquad \mathcal{N} = \frac{1}{8s} \qquad L_{z,e(\mu)}^{\rho\sigma} = \sum_{\text{pol}} \tilde{J}_{Z,e(\mu)}^{\rho} \tilde{J}_{Z,e(\mu)}^{\sigma,*}$$

$$d\Phi = (2\pi)^4 \delta^{(4)}(p_{e^-} + p_{e^+} - p_{\mu^-} - p_{\mu^+}) \frac{d^3\vec{p}_{\mu^-}}{(2\pi)^3 2E_{\mu^-}} \frac{d^3\vec{p}_{\mu^+}}{(2\pi)^3 2E_{\mu^+}} \qquad \Phi = \frac{1}{8\pi}$$

We integrate over the phase space of the final-state muons.....

$$\int L_{z,\mu}^{\rho\sigma} \, d\Phi = A\left[g^{\rho\sigma} - \frac{q^\rho q^\sigma}{m_Z^2}\right] + B\frac{q^\rho q^\sigma}{m_Z^2}$$

$$q_\rho L_e^{\rho\sigma} = 0 \qquad A = -2m_Z\Gamma_{Z,\mu}$$

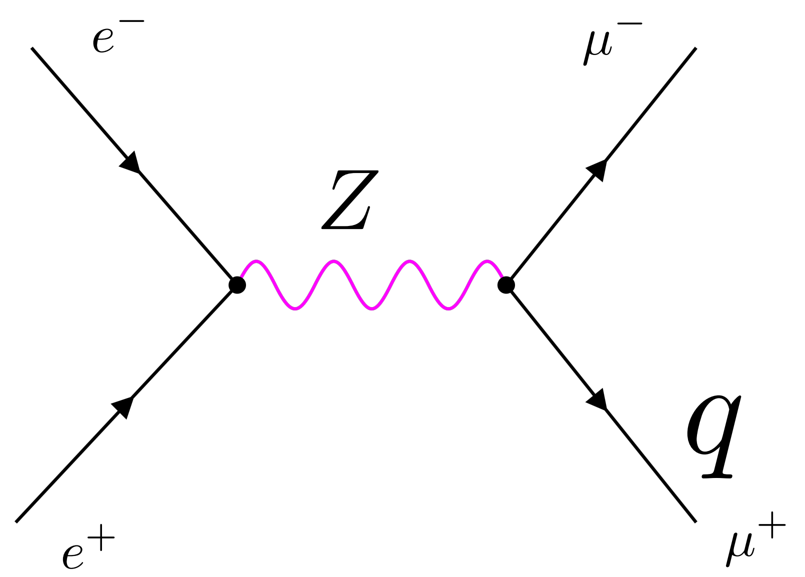
$$-g^{\rho\sigma} + \frac{q^\rho q^\sigma}{m_Z^2} = \sum_{\text{pol}} \epsilon_Z^\rho \epsilon_Z^{\sigma,*}$$

$$d\sigma = \frac{2m_Z}{8s} \frac{\Gamma_{Z,\mu}}{((s-m_Z)^2 + m_Z^2\Gamma_Z^2} \left(-g_{\rho\sigma} + \frac{q^\rho q^\sigma}{m_Z^2}\right) L_{z,e}^{\rho\sigma}$$

$$6m_z\Gamma_{z,e} = \left(-g_{\rho\sigma} + \frac{q^\rho q^\sigma}{m_z^2}\right) L_{z,e}^{\rho\sigma} \, \Phi$$

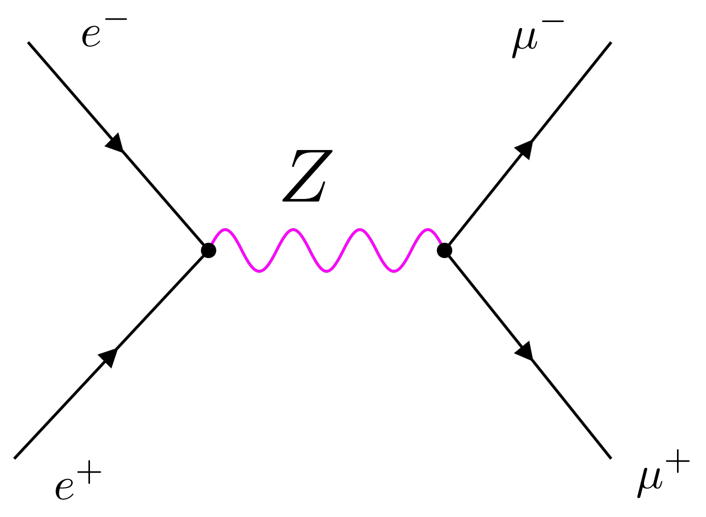
... and find the following (Breit-Wigner) result

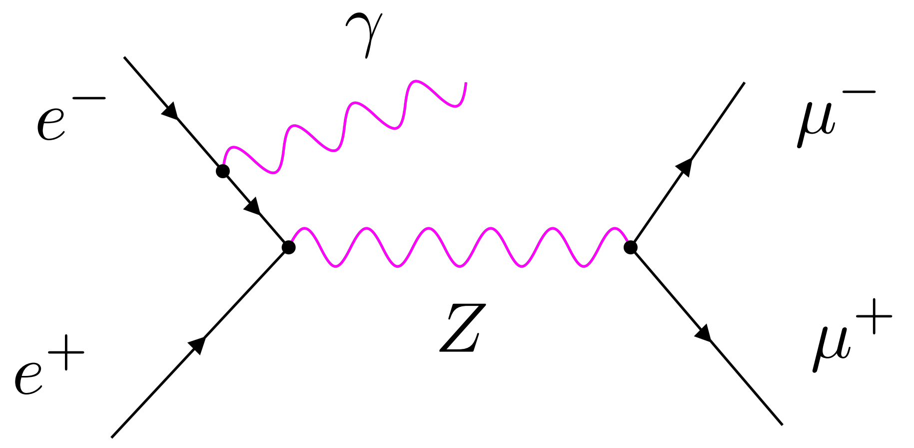
$$\sigma = \frac{12\pi \, \Gamma_{z,e}\Gamma_{z,f}}{(s-m_z)^2 + m_z^2\Gamma_z^2}$$



The measured cross section looks very different from a simple formula that we derived. The reason is the radiative corrections; chief among them is the initial state radiation as it distorts the shape of the Breit-Wigner distribution.

$$\sigma = \frac{12\pi\, \Gamma_{z,e}\Gamma_{z,f}}{(s - m_z)^2 + m_z^2\Gamma_z^2}$$





$$\lim_{\vec{k}||\vec{p}_{e^-}} |\mathcal{M}(p_{e^-}, p_{e^+}, k)|^2 \to \frac{e^2}{(p_{e^-}k)} P_{ee}(x)\, |\mathcal{M}(x p_{e^-}, p_{e^+})|^2$$

$$P_{ee}(x) = \frac{1+x^2}{1-x} \qquad x = \frac{E_{e^-} - \omega}{E_{e^-}}$$

$$\frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega} \frac{1}{(p_{e^-}k)} \approx \frac{1}{4\pi^2} \mathrm{d}x \frac{\theta_\gamma \mathrm{d}\theta_\gamma}{\frac{m_e^2}{E_e^2} + \theta_\gamma^2} \to \frac{1}{4\pi^2} \mathrm{d}x \, \ln \frac{m_Z^2}{m_e^2}$$

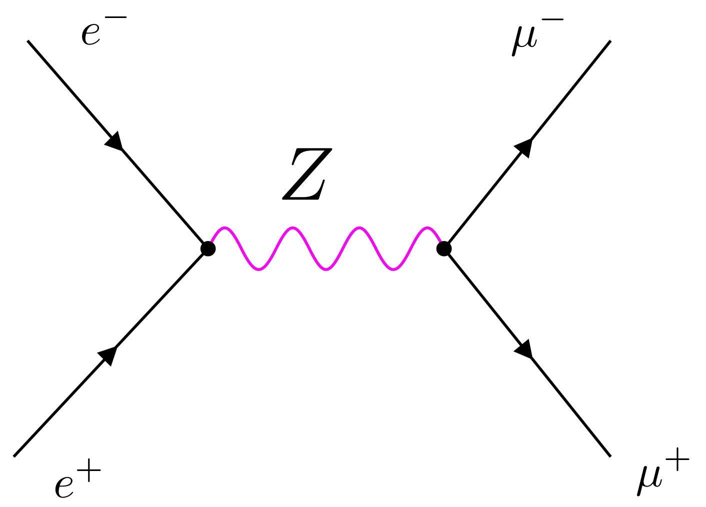
$$\frac{\delta\sigma}{\sigma} = \frac{\alpha}{\pi} \ln \frac{m_z^2}{m_e^2} \int_0^1 \mathrm{d}x \frac{1+x^2}{1-x} \left[\frac{\sigma(xs)}{\sigma(s)} - 1 \right]_{s=m_z^2} \Rightarrow \frac{\delta\sigma}{\sigma} = -\frac{\alpha}{\pi} \ln \frac{m_z^2}{m_e^2} \int_0^1 \mathrm{d}x \, (1+x^2) \frac{(1-x)}{(1-x)^2 + \gamma^2}$$

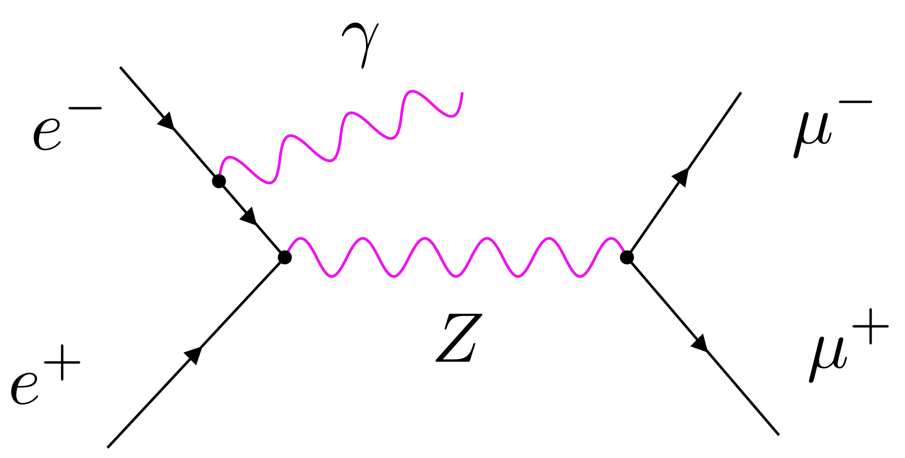
$$\gamma = \frac{\Gamma_Z}{m_Z}$$

Performing the integral, we find very large radiative corrections:

$$\frac{\delta\sigma}{\sigma} = -\frac{2\alpha}{\pi} \ln \frac{m_z^2}{m_e^2} \ln \frac{m_z}{\Gamma_z} \approx -0.4$$

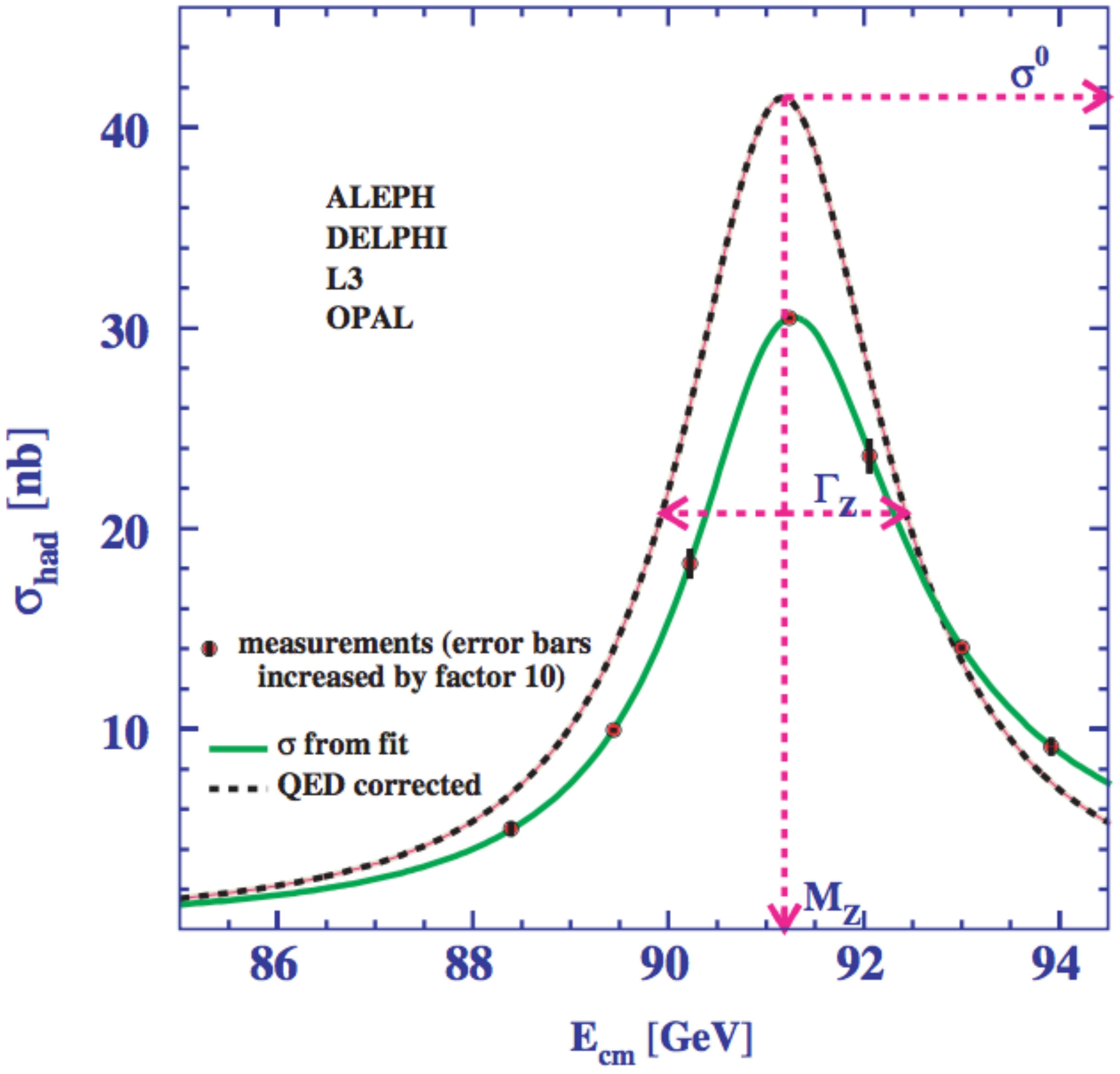
The mass of the Z boson is obtained from the peak position of the measured distribution; the total width — from its broadness, the hight at the peak gives access to partial decay widths. To properly extract all these quantities from the experimental measurements, radiative corrections are extremely important, as we have seen from the previous estimate.





$$\sigma = \frac{12\pi\,\Gamma_{z,e}\Gamma_{z,f}}{(s-m_z)^2 + m_z^2\Gamma_z^2}$$

$$\frac{\delta\sigma}{\sigma} = -\frac{2\alpha}{\pi}\ln\frac{m_z^2}{m_e^2}\ln\frac{m_z}{\Gamma_z} \approx -0.4$$

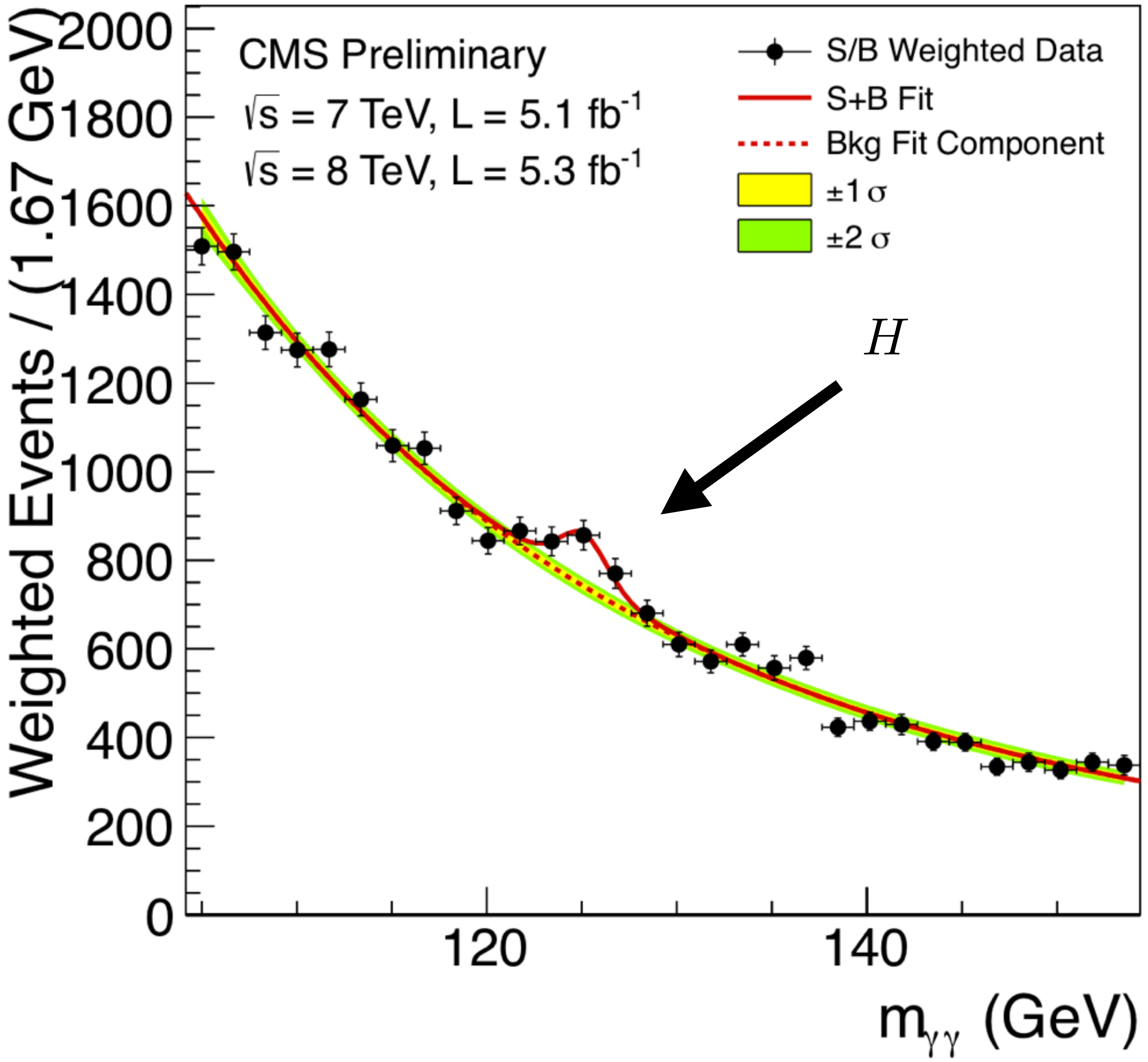


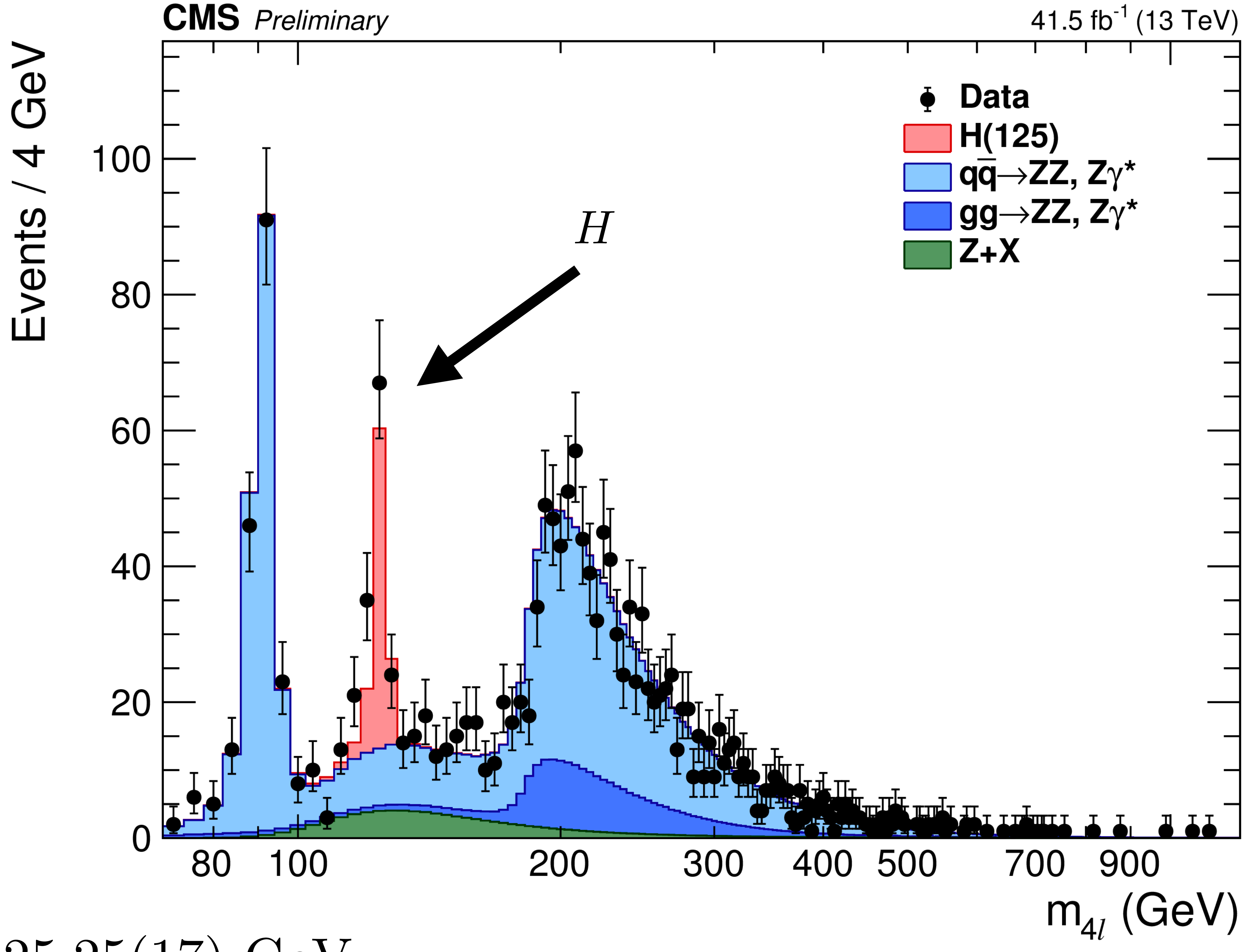
$$m_z = 91.1876 \text{ GeV}$$

The Higgs boson, discovered at the LHC, is produced in the gluon fusion and observed in two-photon and four-lepton final states. Higgs boson is a very narrow resonance; for this reason measuring its line shape at the LHC is impossible.

$$\sigma_{gg\to H} = \frac{4\pi^2}{9}\ \frac{\Gamma_{H\to gg}}{m_H}\delta(s-m_H^2)\ \frac{\Gamma_{H\to f}}{\Gamma_H}$$

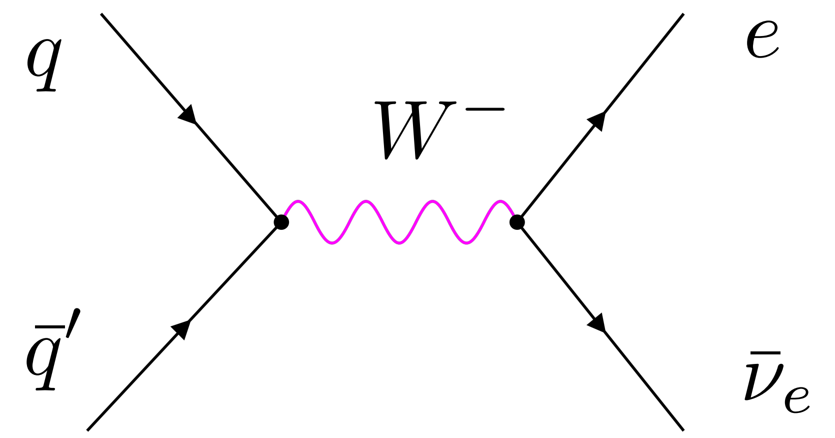
$$\sigma_{pp\to H} = \int \mathrm{d}x_1\mathrm{d}x_2\ g(x_1)g(x_2)\ \sigma_{gg\to H}(x_1x_2s)$$





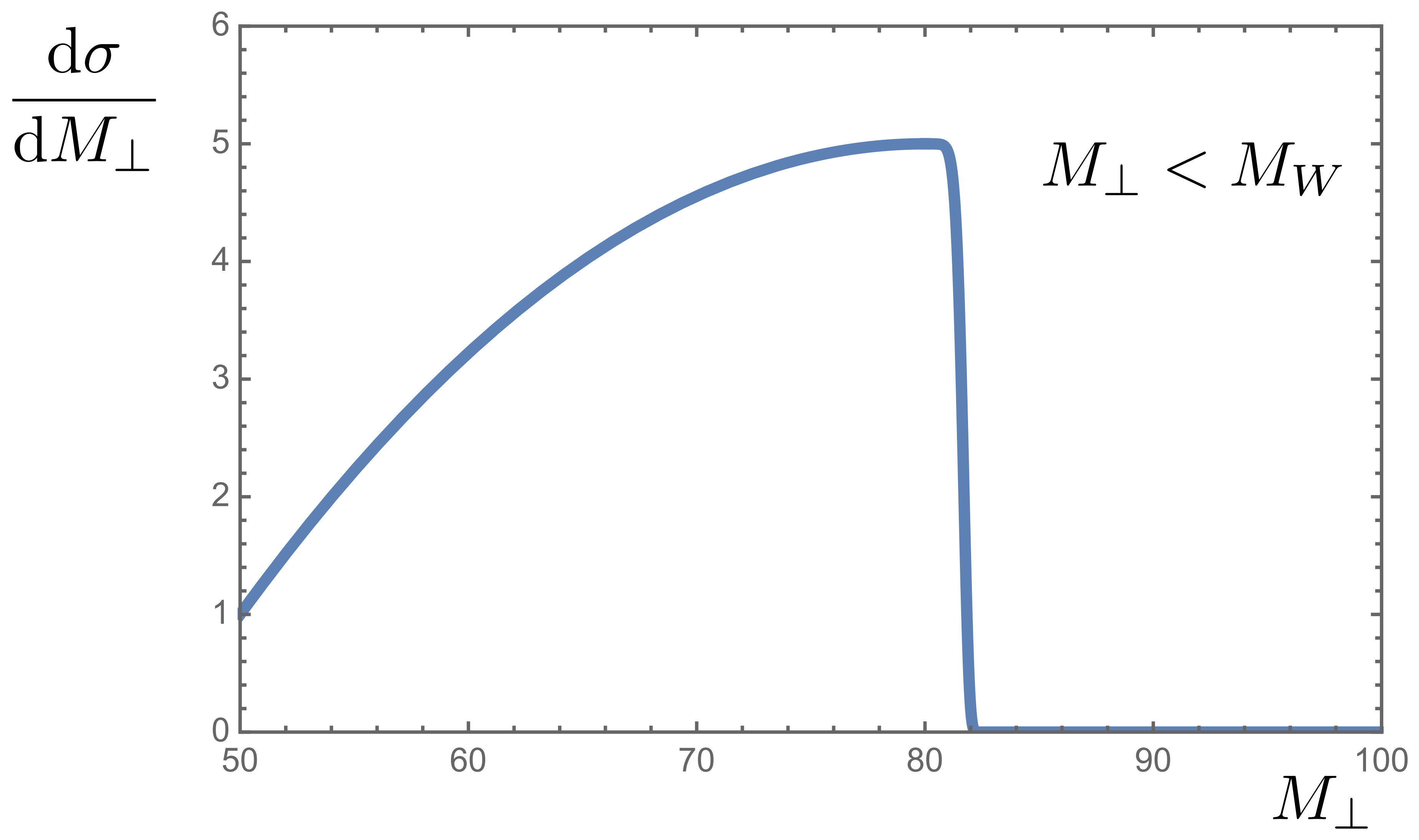
$$m_H = 125.25(17)\ \mathrm{GeV}$$

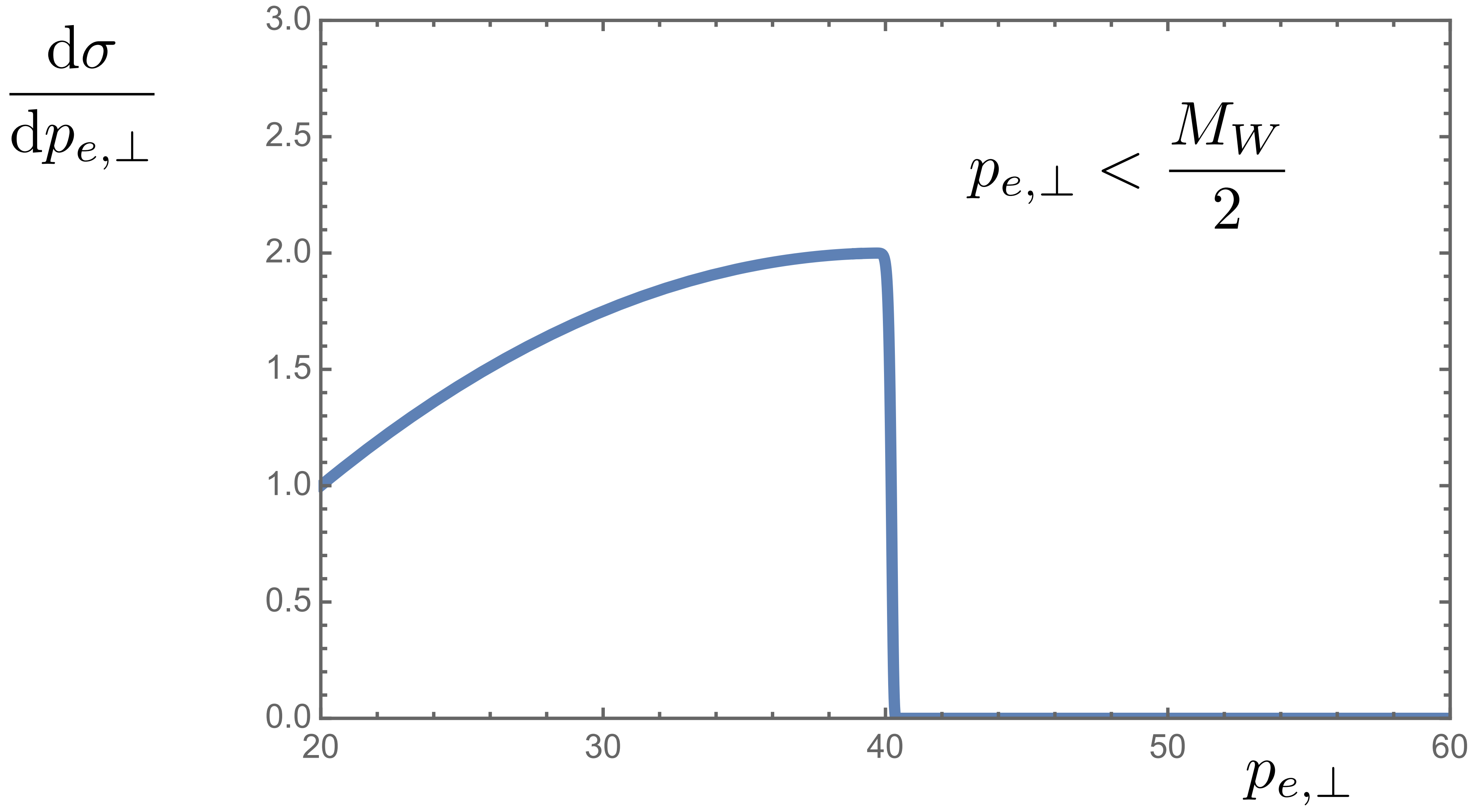
The W-boson mass measurement requires a somewhat different approach because hadronic decays of the W-boson are buried in backgrounds, and neutrino is not observable, so reconstructing the invariant mass is not an option.



It is possible to study two variables with pronounced kinematic boundaries — the transverse W-mass and the transverse momentum of the charged lepton.

$$M_\perp = \sqrt{2p_{e,\perp}p_{\nu,\perp} - 2\vec{p}_{e,\perp}\cdot\vec{p}_{\nu,\perp}}$$

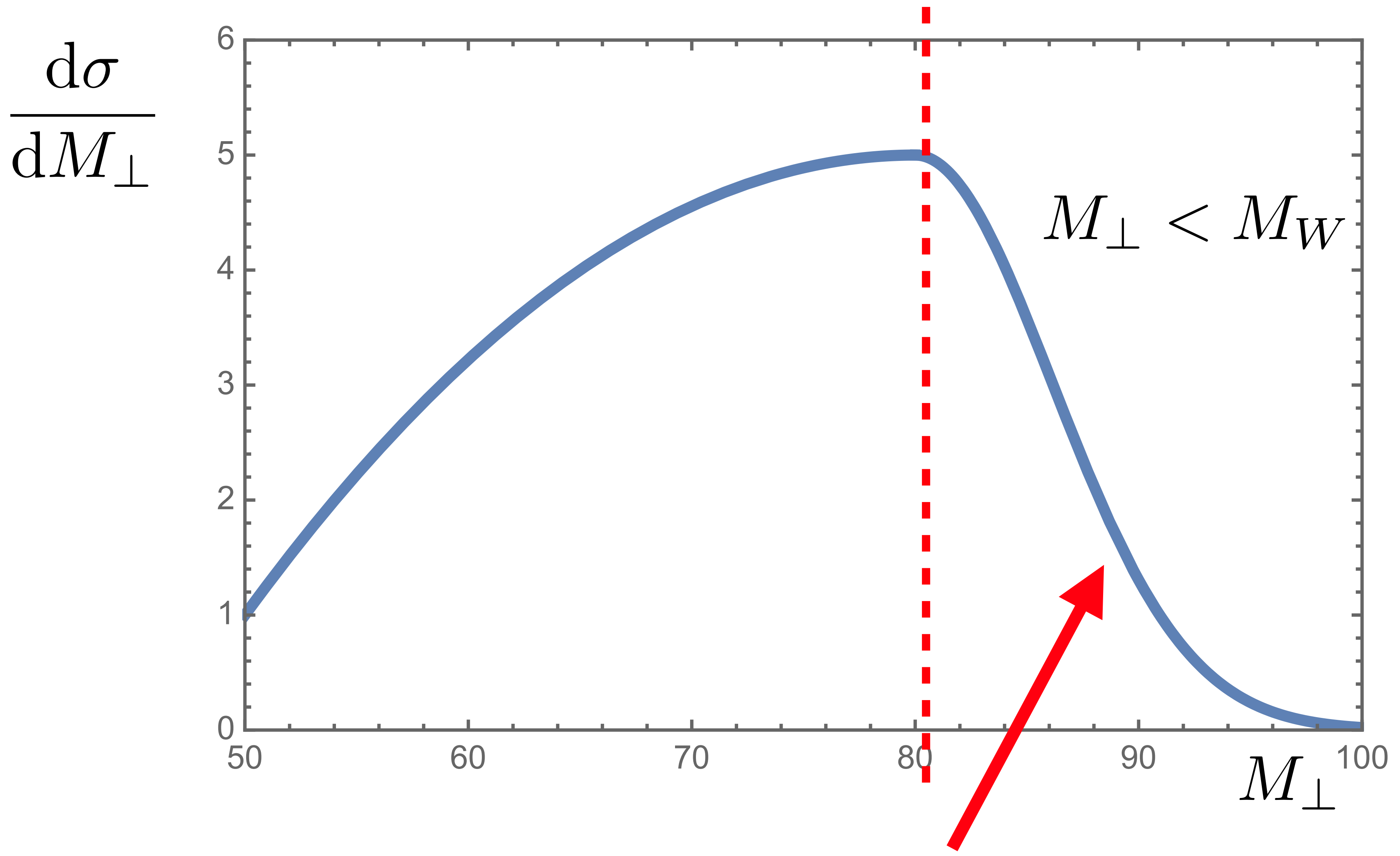




The W-boson mass measurement requires a somewhat different approach because W hadronic decays are buried in backgrounds, and neutrino is not observable. It is possible to study two variables with pronounced kinematic boundaries — the transverse W-mass and the transverse momentum of the charged lepton.

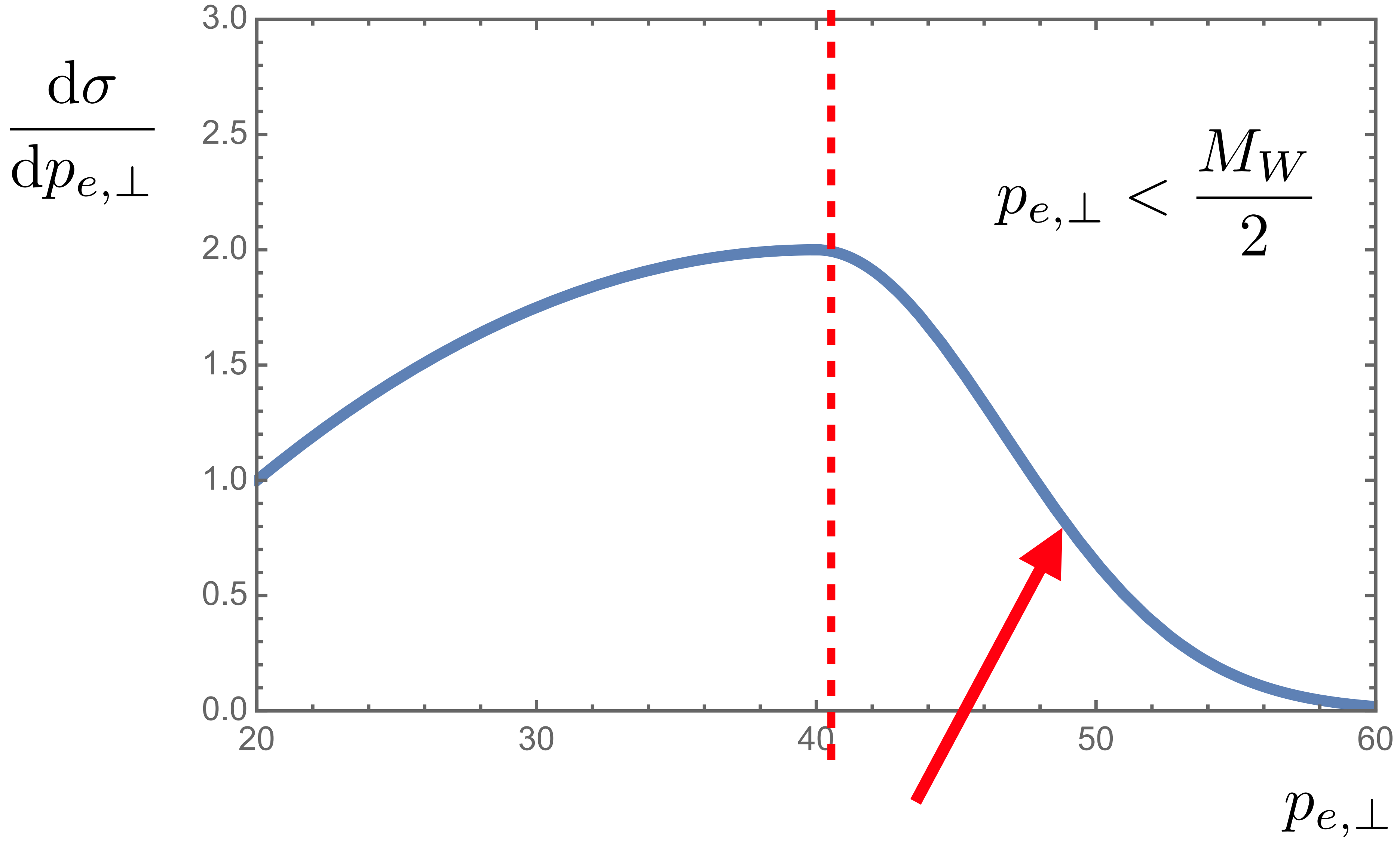
$$m_W = 80.377(12)\ \mathrm{GeV} \qquad \text{PDG}$$

The precision of the measurement is extraordinary given the fact that QCD radiative corrections to processes at the LHC are typically of the order of 10 percent.



Beyond the edge: mostly detector effects

$$M_\perp = \sqrt{2p_{e,\perp}p_{\nu,\perp} - 2\vec{p}_{e,\perp}\cdot\vec{p}_{\nu,\perp}}$$



Beyond the edge: the initial state radiation

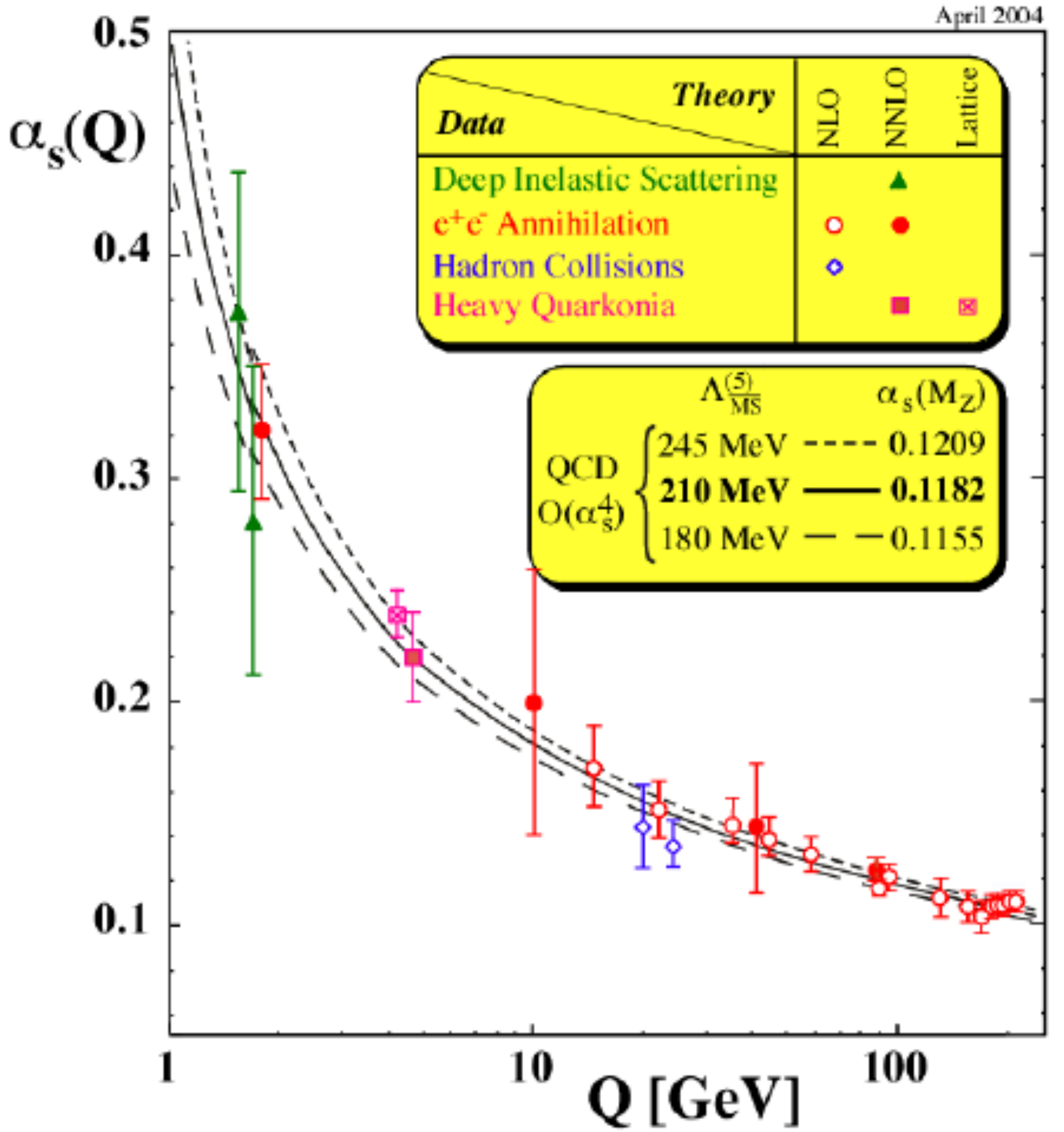
$$pp \to W + j$$

The strong coupling constant

The strong coupling constant is another input parameter of the Standard Model. We will discuss different ways used to determine it.

The main problem with the strong coupling constant determination is that strong interactions are indeed strong, in which case we do not know how to relate the α_s to observables.

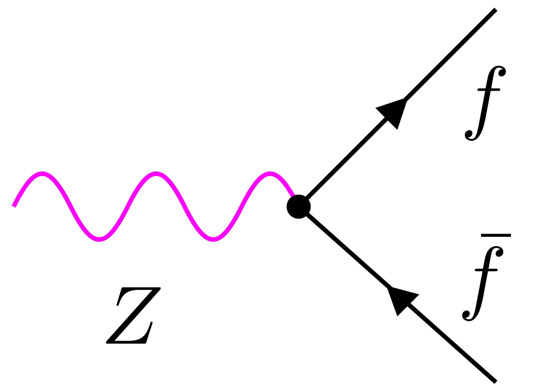
The idea is then to use the asymptotic freedom of QCD, which implies that the strong coupling constant becomes smallish at large energies/momenta (short distances), and to extract its value from short-distance processes that can be studied in QCD perturbation theory.



$$\alpha_s(\mu) = \frac{\pi}{\beta_0 \ln \frac{\mu}{\Lambda_{QCD}}} \qquad \beta_0 = \frac{11}{6} C_A - \frac{2}{3} n_f T_R$$

One option is to consider decays of Z-bosons to hadrons and to charged leptons. Z decays occur at distances that are about 100 times smaller than the Compton wave length of the proton (which is a distance of where strong force becomes strong). For this reason, we can use quarks and QCD perturbation theory.

$$\frac{\Gamma_{Z\to \text{hadr}}}{\Gamma_{Z\to \text{ch. lept}}} = \frac{1744(2)\ \text{MeV}}{3\times 83.984(86)\ \text{MeV}} = 6.92 \qquad \frac{\Gamma_{Z\to \text{hadr}}}{\Gamma_{Z\to \text{ch. lept}}} = \left(1+\frac{\alpha_s(m_Z)}{\pi}\right) N_c \frac{\sum_q (g_v^{(q)})^2 + (g_a^{(q)})^2}{\sum_l (g_v^{(l)})^2 + (g_a^{(l)})^2}$$



$$J_Z^\mu = \frac{1}{2}\sum_{\psi\in l,q} \bar\psi\left[(T_L^3 - Q\sin^2\theta_W)\gamma_\mu - T_L^3\gamma^\mu\gamma_5\right]\psi \quad = \frac{1}{4}\sum_{\psi\in l,q}\bar\psi\left[g_V^\psi\gamma_\mu + g_A^\psi\gamma_\mu\gamma_5\right]\psi$$

PDG $\quad s_w^2 = 0.231 \qquad {g_v^{(l)}}^2 + {g_a^{(l)}}^2 = (-1+4s_w^2)^2 + 1 \approx 1.0056$

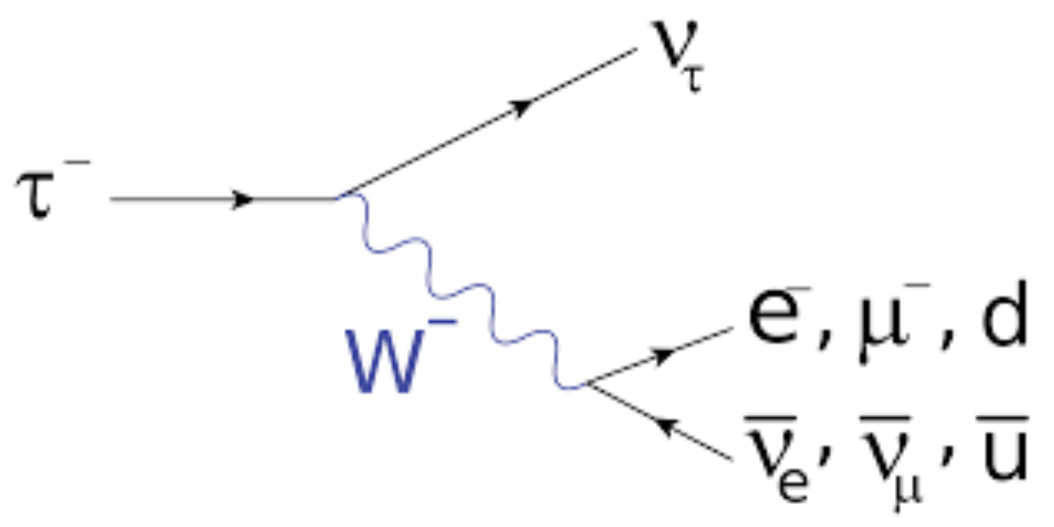
$${g_v^{(u)}}^2 + {g_a^{(u)}}^2 = (1-\frac{8}{3}s_W^2)^2 + 1 \approx 1.1475 \qquad {g_v^{(d)}}^2 + {g_a^{(d)}}^2 = (-1+\frac{4}{3}s_w^2)^2 + 1 \approx 1.4789$$

We obtain $\alpha_s(m_z) = 0.108$, but the result depends strongly on the assumed value of the weak mixing angle. The value of the strong coupling constant allows us to determine the energy scale where QCD becomes strongly-coupled (i.e. non-perturbative).

$$\alpha_s(\mu) = \frac{\pi}{\beta_0 \ln\frac{\mu}{\Lambda_{QCD}}} \qquad \beta_0 = \frac{11}{6}C_A - \frac{2}{3}n_f T_R \qquad \Lambda_{QCD} = m_Z e^{-\frac{\pi}{\beta_0\alpha_S(m_Z)}} \approx 140\ \text{MeV}$$

An alternative determination of the strong coupling constant involves decays of the tau lepton. The idea is identical to what we did with Z decays, but the energy scale (the tau mass) is smaller and the sensitivity to the strong coupling constant is (potentially) larger. The price is larger non-perturbative contributions that need to be analysed.

$$\tau \to \nu_\tau + l + \bar{\nu}_l, \quad l = e, \mu, \qquad \tau \to \nu_\tau + \text{hadrons}$$



$$\Gamma_{\tau,sl} \approx \Gamma_{\tau,0}\left(\text{pert} - \frac{c_q}{m_\tau^4}\langle \sum_{q\in\{u,d\}} m_q \bar{q} q\rangle + \frac{c_g}{m_\tau^4}\langle G^a_{\mu\nu} G^{a,\mu\nu}\rangle\right)$$

$$m_\tau = 1.77 \ \text{GeV}$$

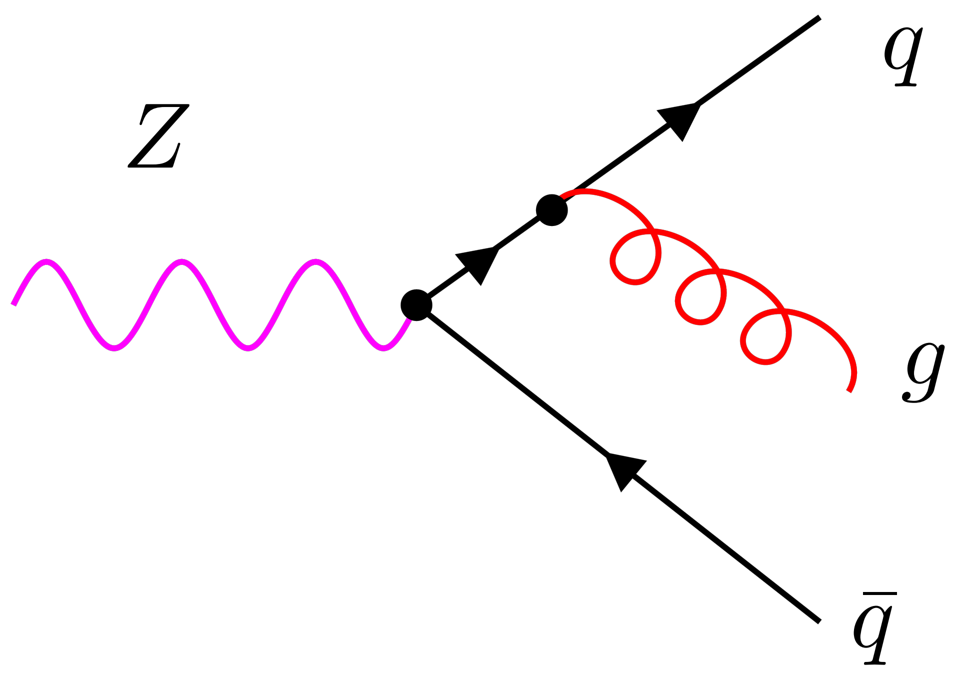
Non-perturbative contributions are quite small; they scale as $\Lambda_{\rm QCD}^4$ and, therefore, produce small corrections (fraction of a percent). The very important feature of the above formula is that there are no non-perturbative effects that are proportional to first, second or third power of $\Lambda_{\rm QCD}$, and this makes the non-perturbative corrections small.

$$\left.\frac{\Gamma_{\tau,sl}}{\Gamma_{\tau,l}}\right|_{\rm exp} = 1.8393 \qquad \frac{\Gamma_{\tau,sl}}{\Gamma_{\tau,l}} = \frac{N_c}{2}\left(1 + \frac{\alpha_s(m_\tau)}{\pi} + 5.2\left(\frac{\alpha_s(m_\tau)}{\pi}\right)^2 + 26.4\left(\frac{\alpha_s(m_\tau)}{\pi}\right)^3 + \cdots\right)$$

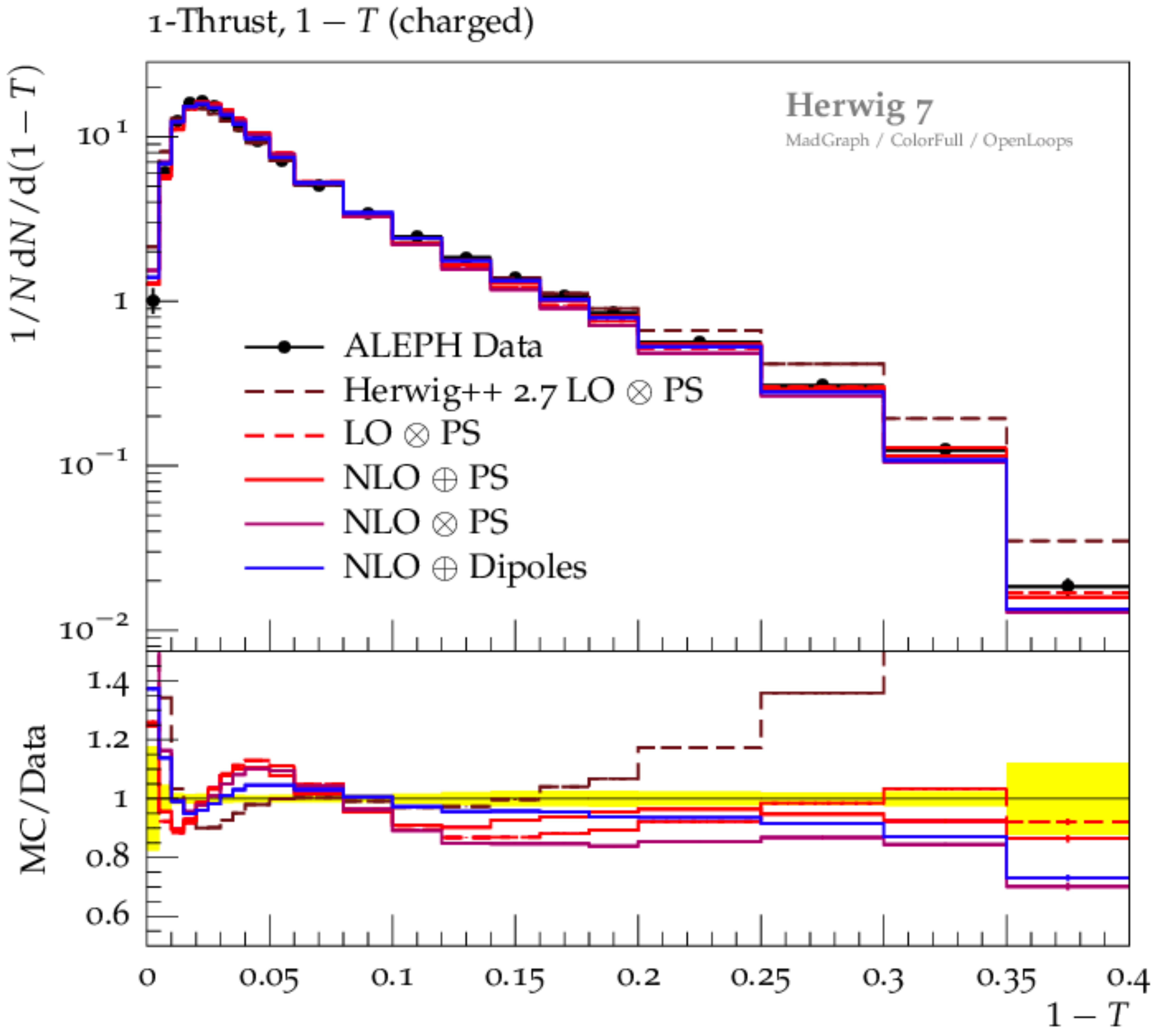
We solve the above equations for the strong coupling constant and use the RG equation to compute $\alpha_s(m_Z)$

$$\alpha_s(m_\tau) = 0.363(20) \qquad \alpha_s(m_z) = \frac{\alpha_s(m_\tau)}{1 + \frac{\alpha_s(m_\tau)}{\pi}\beta_0 \ln\frac{m_z}{m_\tau}} \approx 0.118$$

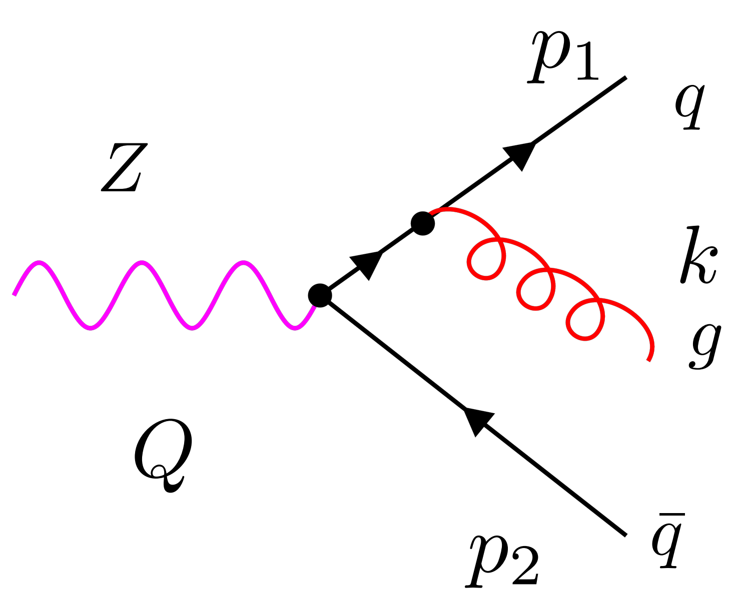
Rates and "shapes" of higher-multiplicity processes are sometimes proportional to the strong coupling constant. A good example is the thrust variable T that is often used for the determination of α_s.



$$T = \max_{\vec{n}} \sum_i \frac{|\vec{n} \cdot \vec{p}_i|}{Q}$$



To see how this works, we consider the case where the emitted gluon is soft $k \ll Q$



$$T = \max_{\vec{n}} \sum_i \frac{|\vec{n} \cdot \vec{p}_i|}{Q} \Bigg|_{k \ll Q} \Rightarrow \quad T = 1 - \min\left[\frac{2p_1 k}{Q}, \frac{2p_2 k}{Q}\right]$$

$$\tau = 1 - T$$

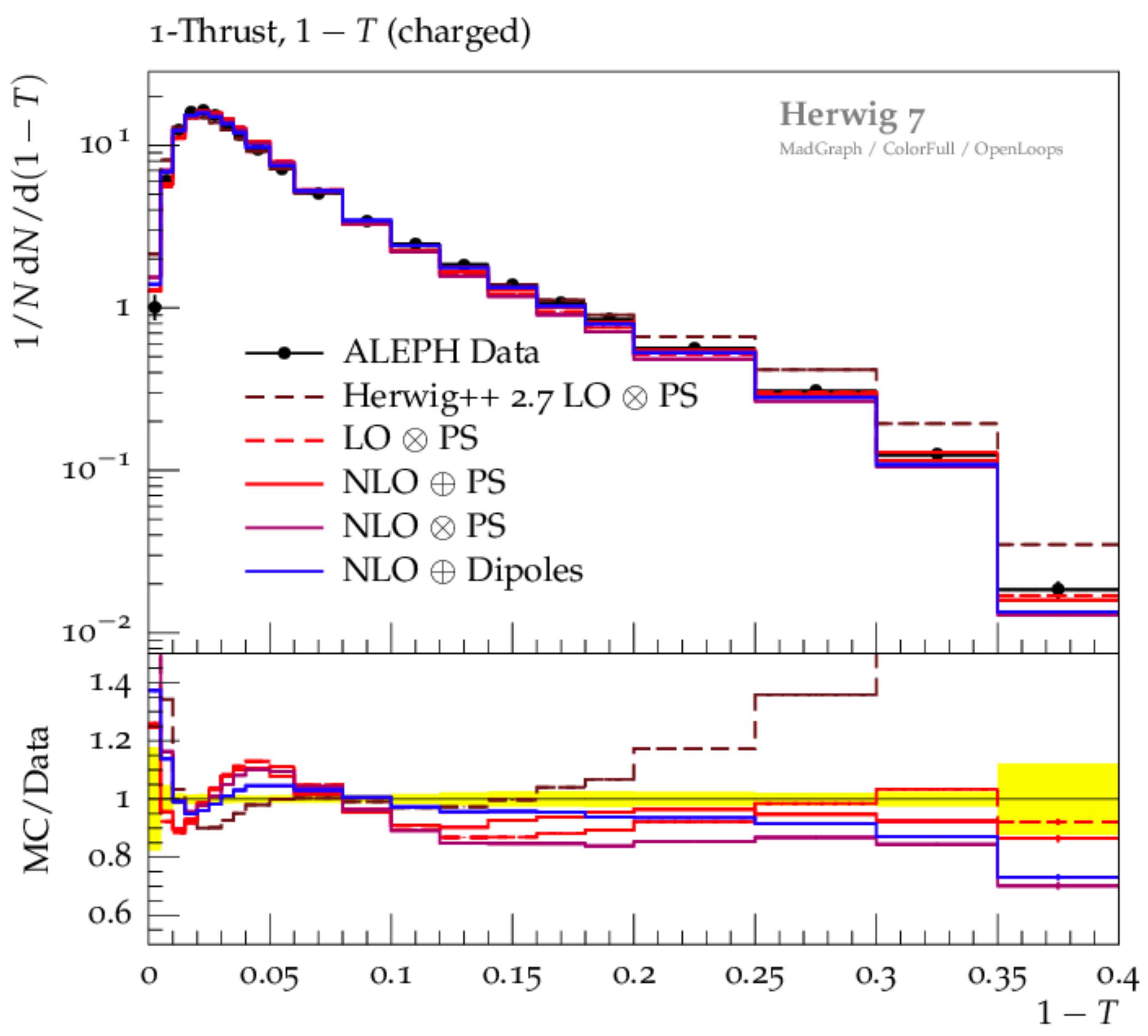
$$\frac{\mathrm{d}\sigma}{\sigma_0 \mathrm{d}\tau} = C_F g_s^2 \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3 \omega_k} \frac{2p_1 p_2}{(p_1 k)(p_2 k)} \delta\left(\tau - \min\left[\frac{2p_1 k}{Q^2}, \frac{2p_2 k}{Q^2}\right]\right)$$

$$k = \alpha p_1 + \beta p_2 + k_\perp \qquad k_\perp \cdot p_{1,2} = 0 \qquad \frac{2kp_1}{Q^2} = \beta \qquad \frac{2kp_2}{Q^2} = \alpha$$

$$\frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k} = \frac{\mathrm{d}^4 k}{(2\pi)^3}\delta(k^2) = \frac{\pi}{(2\pi)^3}\frac{Q^2}{2}\mathrm{d}\alpha\, \mathrm{d}\beta \mathrm{d}k_\perp^2\, \delta\left(Q^2\alpha\beta - k_\perp^2\right)$$

$$\frac{\mathrm{d}\sigma}{\sigma_0 \mathrm{d}\tau} = \frac{\alpha_s}{2\pi} 4C_F \frac{1}{2} \int \frac{\mathrm{d}\alpha\, \mathrm{d}\beta}{\alpha\beta}\, \delta\left(\tau - min(\alpha,\beta)\right)$$

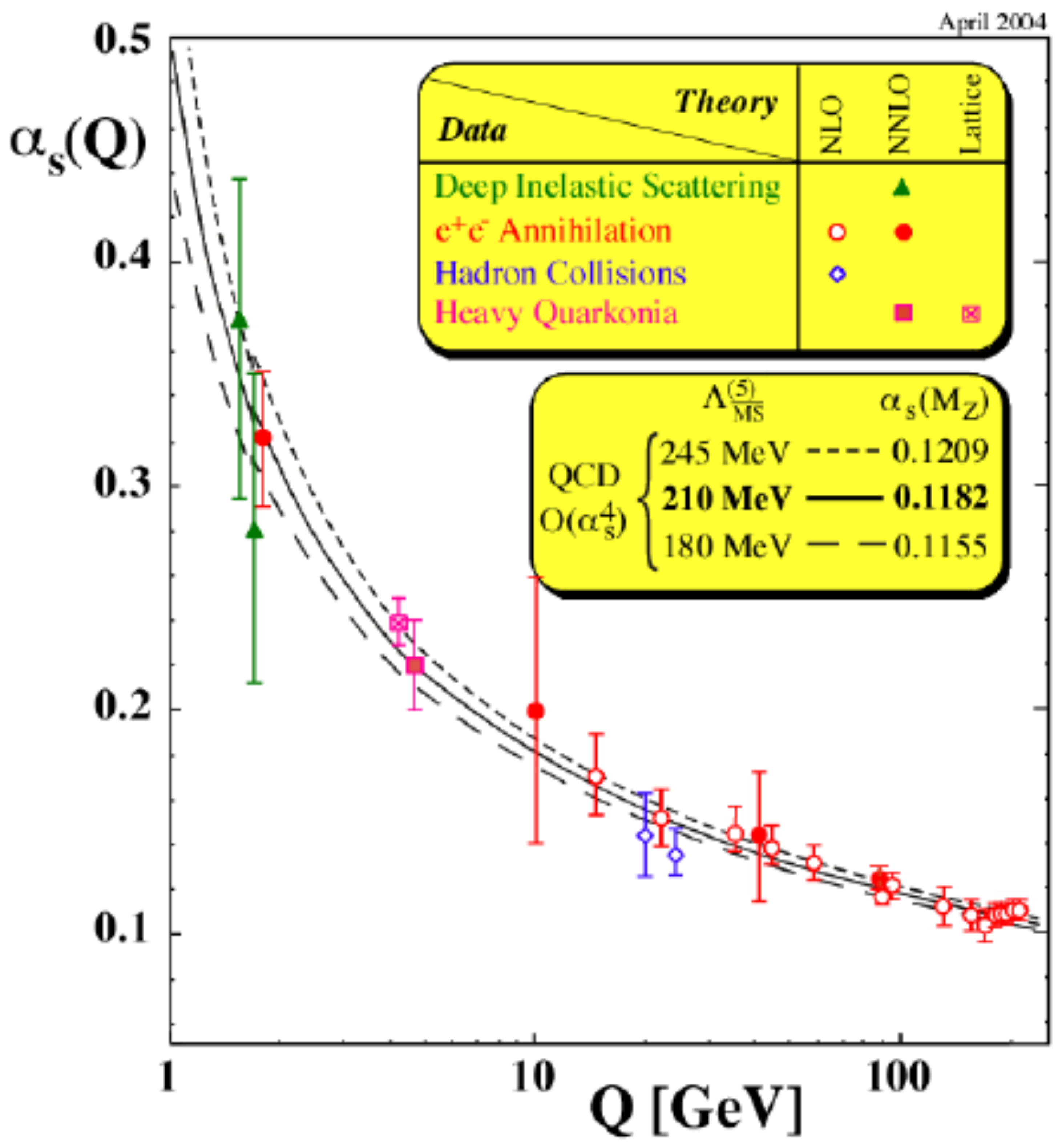
$$\int \frac{\mathrm{d}\alpha \mathrm{d}\beta}{\alpha\beta}\delta(\tau - \min(\alpha,\beta)) = 2\int_\tau^1 \frac{\mathrm{d}\alpha}{\alpha}\int_0^\alpha \frac{\mathrm{d}\beta}{\beta}\delta(\tau - \beta) = \frac{2}{\tau}\ln\frac{1}{\tau}$$

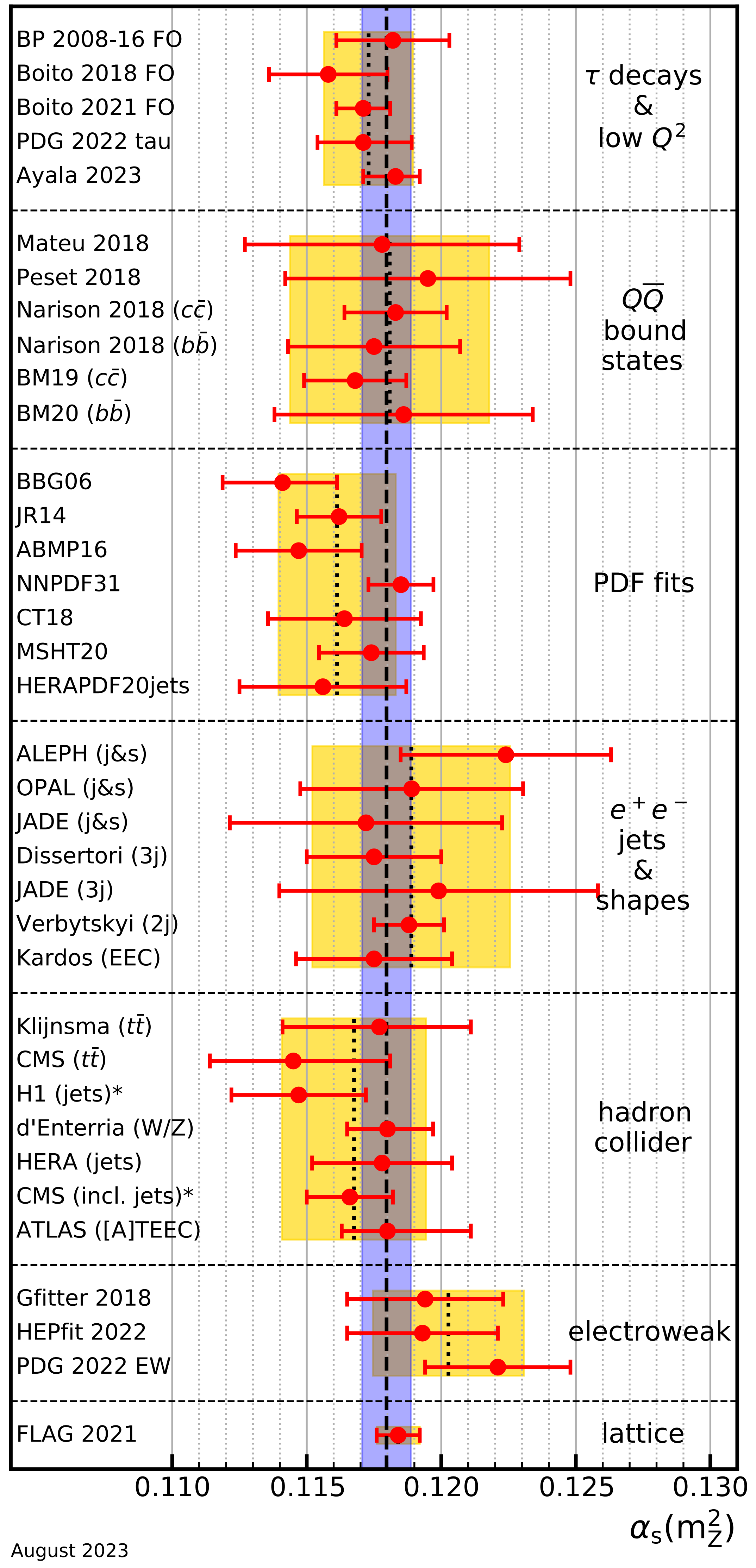


The result is proportional to the strong coupling constant;
it can be obtained from the comparison with data .

$$\boxed{\frac{\mathrm{d}\sigma}{\sigma_0 \mathrm{d}\tau} = \frac{\alpha_s}{2\pi} 4C_F \frac{1}{\tau} \ln\frac{1}{\tau}}$$

When many very different measurements and analyses are combined, a rather consistent value of the strong coupling constant is obtained: $\alpha_s(m_Z) = 0.118 \pm 0.001.$





Note that lattice computations play a very important role in this, providing one of the most precise determinations of the strong coupling constant. In fact, we will see in what follows that lattice calculations start playing decisive role in several precision measurements.

The electromagnetic coupling constant

The electromagnetic coupling constant determines the strength of two charges separated by a distance r. In Quantum Electrodynamics, the “constant” becomes a function of the distance. The distance scale where r-dependence becomes strong, is the Compton wave length of an electron.

$$V(r) = \frac{Q_1 Q_2 \alpha}{r} \qquad \lambda_e = 1/m \sim 10^{-12}\ \mathrm{m} \qquad r \gg \lambda_e \to \alpha(r) \to \alpha.$$

$$\alpha(q) = \frac{\alpha}{1 + \Pi_{\gamma\gamma}(q^2)} \qquad \Pi_{\gamma\gamma}(q^2 = 0) = 0 \qquad \Longrightarrow \qquad \alpha(0) = \alpha.$$

In practice, measuring the Coulomb force isn’t the best option; instead, one finds a quantity that can be measured and computed with very high precision. A suitable option is the electron’s anomalous magnetic moment.

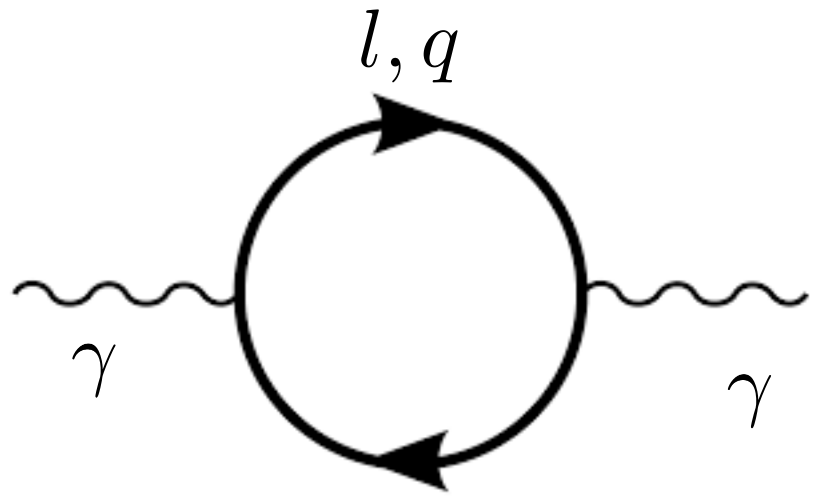
$$H_e = -\vec{\mu} \cdot \vec{B} \qquad \vec{\mu} = g \mu_0 \vec{s} \qquad \mu_0 = \frac{e\hbar}{2mc} \qquad g = 2(1 + a_e)$$

$$a_e = 1.15965218073(28) \times 10^{-3} \qquad a_e = \frac{\alpha}{2\pi} + ..$$

$$\boxed{\alpha^{-1} = 137.035999174(35)}$$

Contributions to precision electroweak observables must be expressed through fine structure constant at the scale $q = m_Z$. At low values of q, one cannot use quarks to compute hadronic contributions to the vacuum polarisation function.

$$\alpha(q) = \frac{\alpha}{1+\Pi_{\gamma\gamma}(q^2)} \qquad \Pi_{\gamma\gamma}(q^2) = \frac{q^2}{\pi}\int_{s_0}^{\infty} \frac{ds}{s(s-q^2-i0)}\,\mathrm{Im}\left[\Pi_{\gamma\gamma}(s)\right]$$

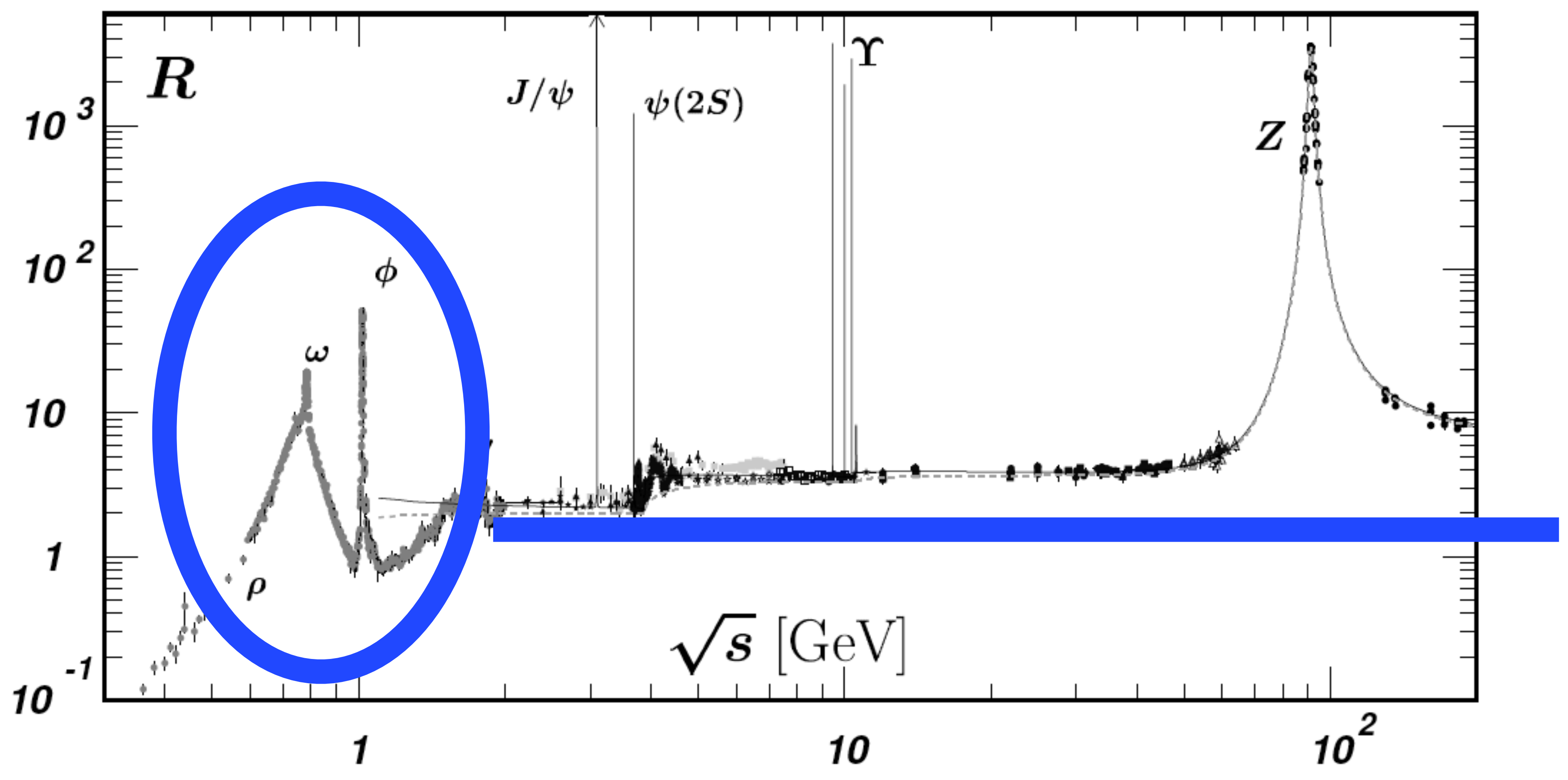


$$\mathrm{Im}\Pi_{\gamma\gamma}(s) = \frac{e^2}{12}\left(R^{\mathrm{lept}}(s) + R^{\mathrm{hadr}}(s) + R^{\mathrm{rest}}(s)\right)$$

$$R^{\mathrm{lept}}(s) = \frac{\sigma_{\mathrm{lept}}(s)}{\sigma_{\mathrm{point}}(s)}, \qquad R^{\mathrm{hadr}}(s) = \frac{\sigma_{\mathrm{hadr}}(s)}{\sigma_{\mathrm{point}}(s)} \qquad \sigma_{\mathrm{point}} = \frac{4\pi\alpha^2}{3s}$$

$$\sigma_{\mathrm{lept}} = \sigma_{\mathrm{point}}\sqrt{1-\frac{4m_l^2}{s}}\left(1+\frac{2m_{\mathrm{lept}^2}}{s}\right)$$

$$\Pi^{\mathrm{lept}}_{\gamma\gamma}(m_z) = -\frac{\alpha}{3\pi}\sum_{l\in\{e,\mu,\tau\}} \ln\left(\frac{m_Z^2}{m_l^2}\right) - \frac{5}{3} \approx -0.032$$



$$\sigma_{e^+e^-\to V} = \frac{12\pi^2\Gamma_{V\to e^+e}}{m_V}\delta(s-m_V^2)$$

$$\Pi^{\mathrm{hadr}}_{\gamma\gamma}(m_z) = -0.0045|_{\rho,\omega,\phi} - 0.022|_{\mathrm{cont}} = -0.026$$

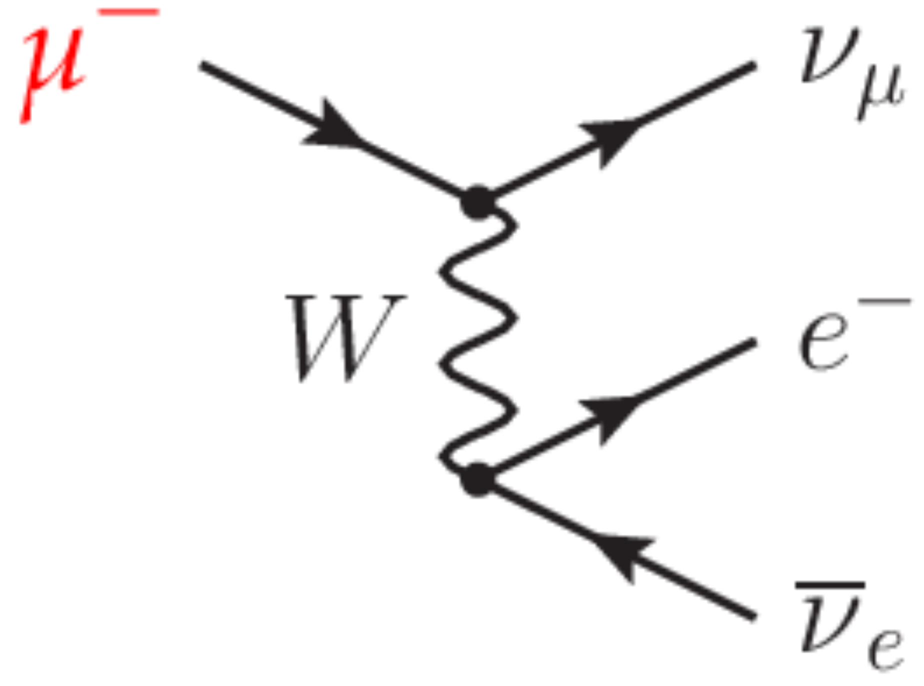
From the above estimate: $\Rightarrow \quad \alpha^{-1}(m_z) = 129.05$

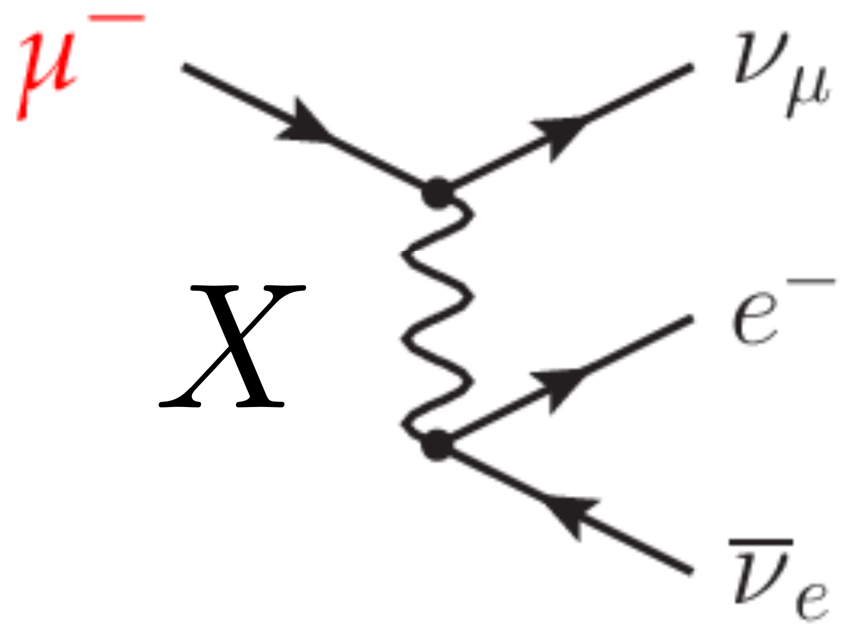
A careful data-based computation gives: $\alpha^{-1}(m_z) = 128.89 \pm 0.09$

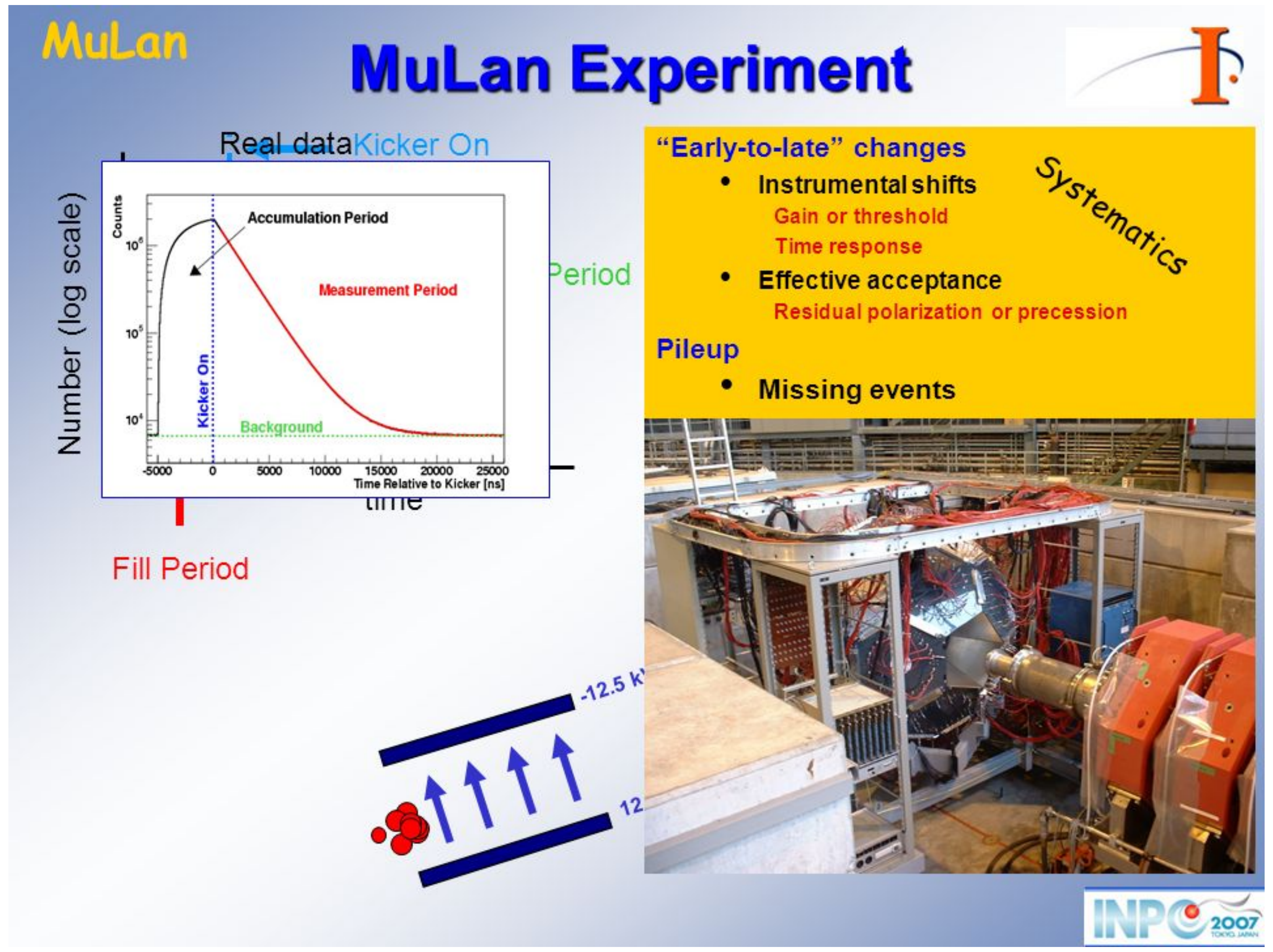
A simple test of the Standard Model

The muon decay in the Standard Model proceeds through the exchange of a W-boson. As we discussed, the mass of the W-boson has been precisely measured. Hence, we can compute the decay rate with high precision and compare the prediction with the result of the measurement.

We do this to test the consistency of the Standard Model and check for possible contributions of physics beyond the Standard Model to the muon decay.

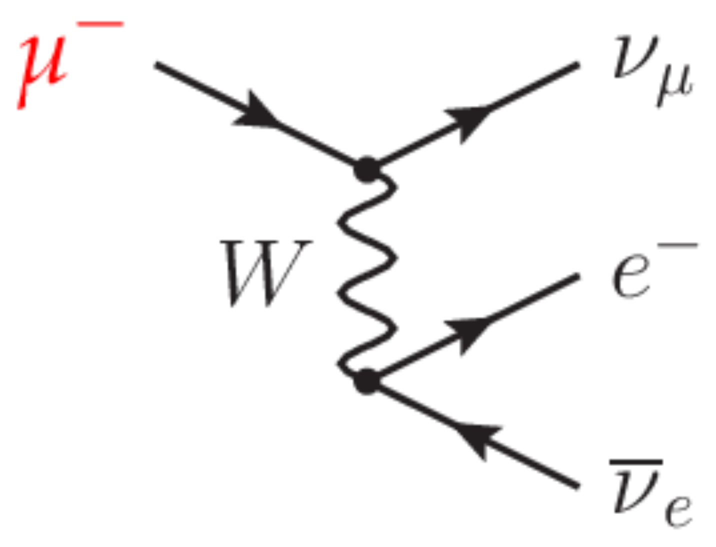




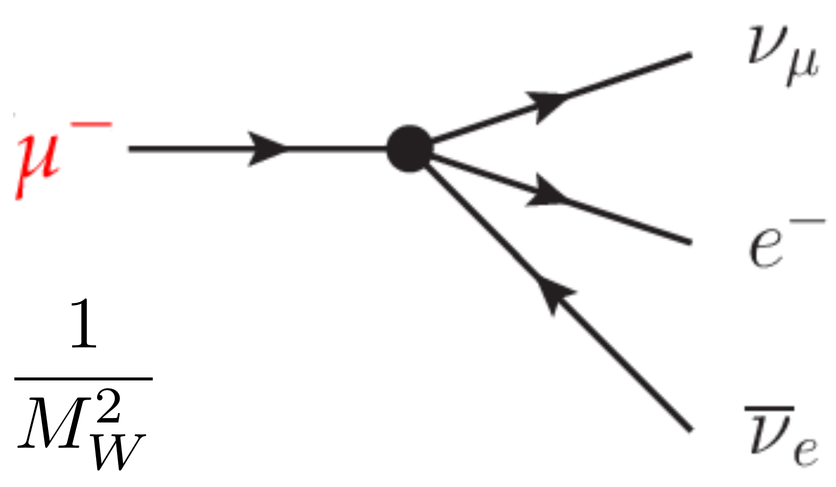


MuLan experiment at PSI

The muon decay rate is usually computed in the context of the Fermi theory. Then, effects of heavy electroweak physics are hidden in the Fermi constant.



$$\Rightarrow \qquad \frac{1}{q^2 - M_W^2} \to -\frac{1}{M_W^2}$$



$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}} \bar{e}\gamma_\alpha (1-\gamma_5)\nu_e \, \bar{\nu}_\mu (1-\gamma_5)\mu + h.c$$

$$G_F = \frac{g^2}{4\sqrt{2} m_W^2}$$

$$e = g \sin\theta_W \qquad G_F = \frac{\pi\alpha}{\sqrt{2} m_W^2 \left(1 - \frac{m_W^2}{m_Z^2}\right)}$$

$$G_F = 1.1245 \times 10^{-5} \text{ GeV}^{-2}$$

Why there is a 3.5 percent difference?

$$\frac{1}{\tau_\mu} = \Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e^2}{m_\mu^2}\right)$$

$$f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x$$

$$\tau = 2.1969811(22) \times 10^{-6} \text{ sec} \qquad \text{PDG}$$

$$G_F = \sqrt{\frac{192\pi^3 \hbar}{\tau m_\mu^5 f(m_e^2/m_\mu^2)}}$$

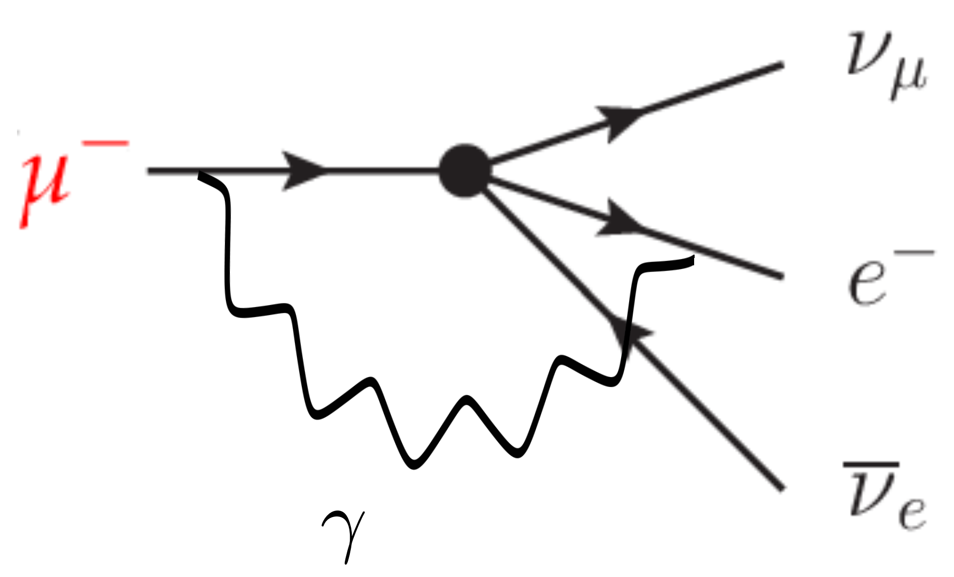
$$\hbar = 6.58211928 \times 10^{-22} \text{ MeV sec}$$

$$m_\mu = 105.6583715 \text{ MeV}, \quad m_e = 0.510998928 \text{ MeV}$$

$$G_F = 1.16394 \times 10^{-5} \text{ GeV}^{-2}$$

A percent-level comparison requires us to consider radiative corrections. Traditionally, one distinguishes between QED corrections in the Fermi theory and weak corrections in the SM, although at the end of the day, all of them are just electroweak corrections.

In the modern language we will say that we compute the effective Lagrangian, including short-distance electroweak corrections, and then use it to calculated the matrix element for the muon decay including long-distance QED corrections.

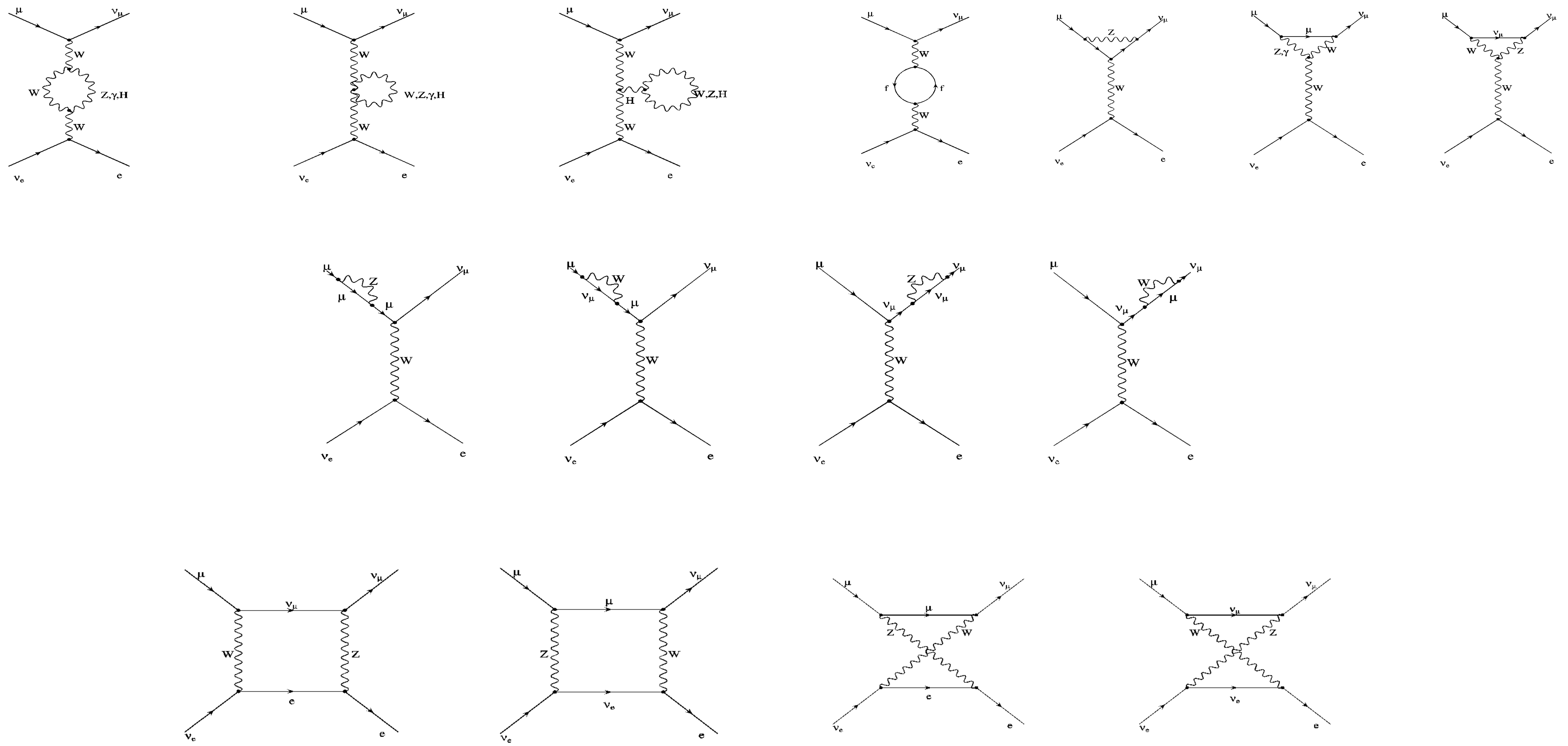


$$\frac{1}{\tau_\mu} = \Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e^2}{m_\mu^2}\right) F(\alpha)$$

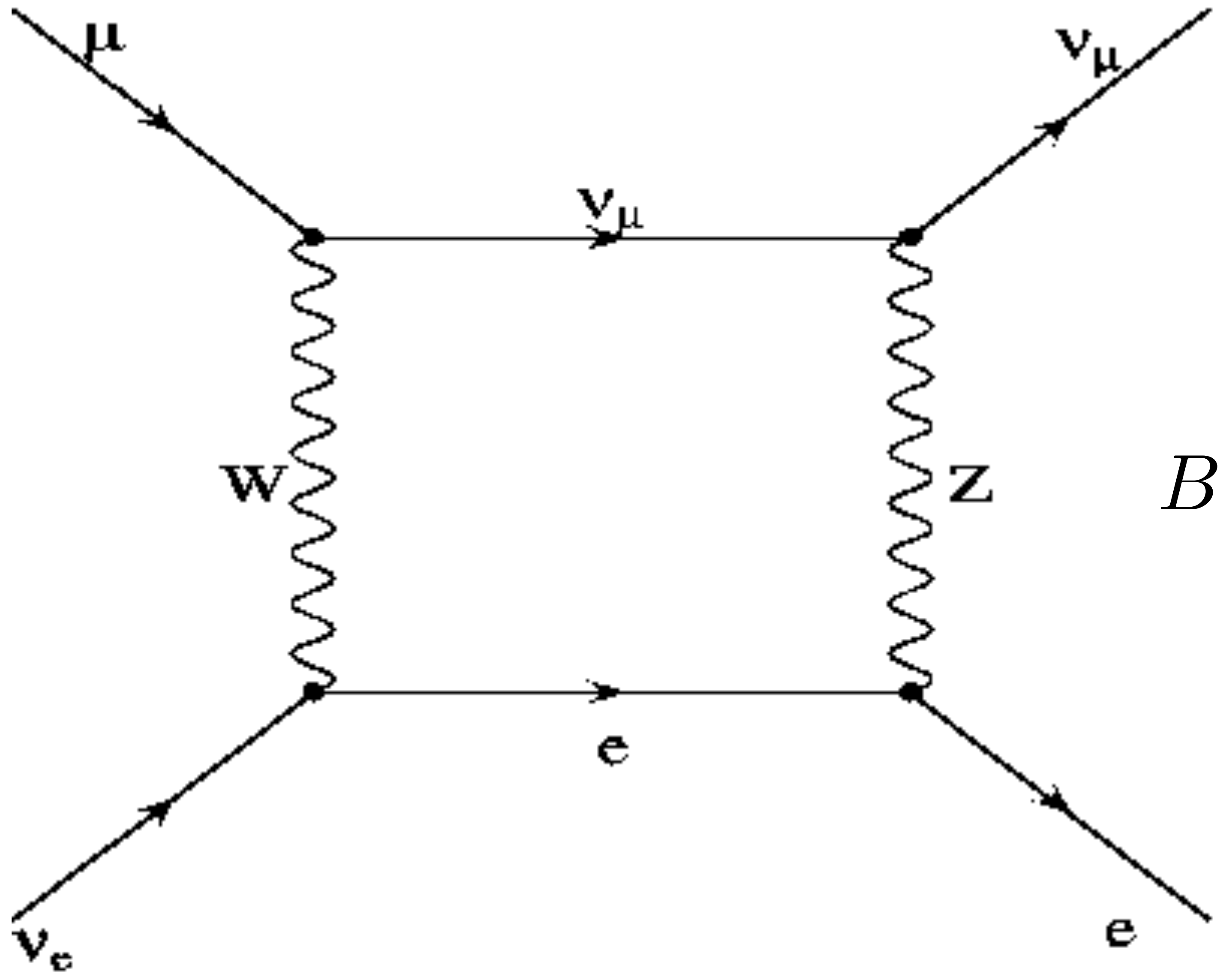
$$F = 1 + \frac{\alpha}{\pi}\left(\frac{25}{4} - \pi^2\right) + \mathcal{O}(\alpha^2) \approx 1 - 8 \times 10^{-3}$$

Hence, it follows that the QED radiative correction is way too small to reconcile two values of the Fermi constant (and, furthermore, it works in the wrong direction increasing the difference between them).

However, not all is lost since, leaving the QED corrections aside, we find many electroweak one-loop diagrams that provide corrections to the muon decay that need to be analysed...



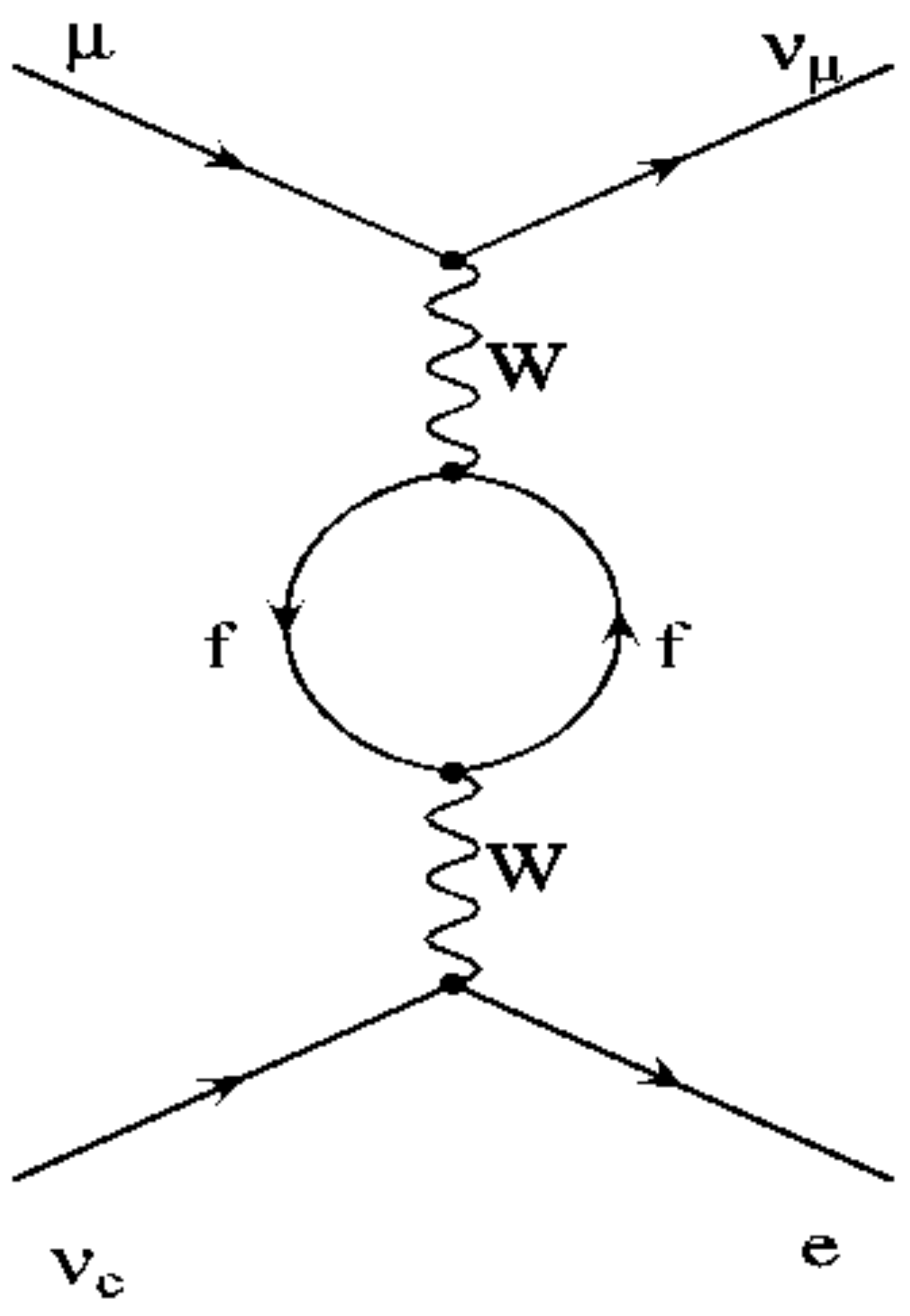
We will try to estimate the magnitude of the expected corrections by considering one of the box diagrams. Such diagrams renormalise the Fermi Lagrangian but the expected corrections are tiny; it is hard to argue that such corrections can reconcile the (measured and computed) Fermi constants unless an enhancement factor is found.

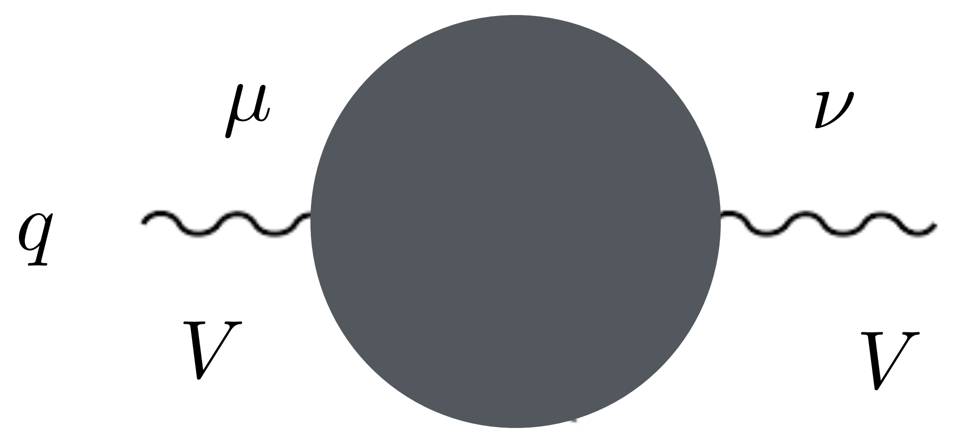


$$B \sim \frac{g^4}{c_W^2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\gamma_\mu (1-\gamma_5) \frac{1}{\hat{k}} \gamma_\nu (1-\gamma_5)\right] \otimes \left[\gamma_\mu \gamma_5 \frac{1}{\hat{k}} \gamma_\nu (1-\gamma_5)\right] \frac{1}{k^2 - m_W^2} \frac{1}{k^2 - m_Z^2}$$

$$B \approx \frac{g^2}{16\pi^2 c_W^2} \mathcal{L}_F \approx \frac{\alpha}{4\pi s_W^2 c_W^2} \mathcal{L}_F \approx 3 \times 10^{-3} \mathcal{L}_F$$

It turns out that such enhanced corrections are hiding in the vacuum polarization functions of electroweak gauge bosons. We will now investigate how this happens and what is the enhancement factor. We will start with the discussion of the impact of the vacuum polarization contributions on the Fermi constant.





$$-i\Pi_{VV}(q^2)g^{\mu\nu} + \mathcal{O}(q^\mu q^\nu)$$

$$\frac{-ig_{\mu\nu}}{q^2 - m^2_{V,0}} \rightarrow \frac{-ig_{\mu\nu}}{q^2 - m^2_{V,0} + \Pi_{VV}(q^2)}$$

$$m^2_{V,0} = m^2_V + \delta m^2_V$$

$$\frac{-ig_{\mu\nu}}{q^2 - m^2_V + \left[\Pi_{VV}(q^2) - \Pi_{VV}(m^2_V)\right]}$$

$$\delta m^2_V = \Pi_{VV}(m^2_V)$$

Note that g_0 is still a bare (i.e. Lagrangian, unphysical) weak coupling.

$$G_F = \frac{g_0^2}{\sqrt{2}\,4}\frac{1}{m^2_{W,0} - \Pi_{WW}(0)} = \frac{g_0^2}{\sqrt{2}\,4m^2_{W,0}}\left[1 + \frac{\Pi_{WW}(0)}{m^2_w}\right]$$

In general since the SM is renormalisable theory, we have to express all quantities through physical parameters. In case of the Fermi constant, the relevant ones are the Z-mass, the W-mass and the fine structure constant.

$$m_{z,0}^2 = m_z^2 + \delta m_z^2, \quad m_{w,0}^2 = m_w^2 + \delta m_w^2, \quad \alpha_0 = \alpha(m_z) + \delta\alpha$$

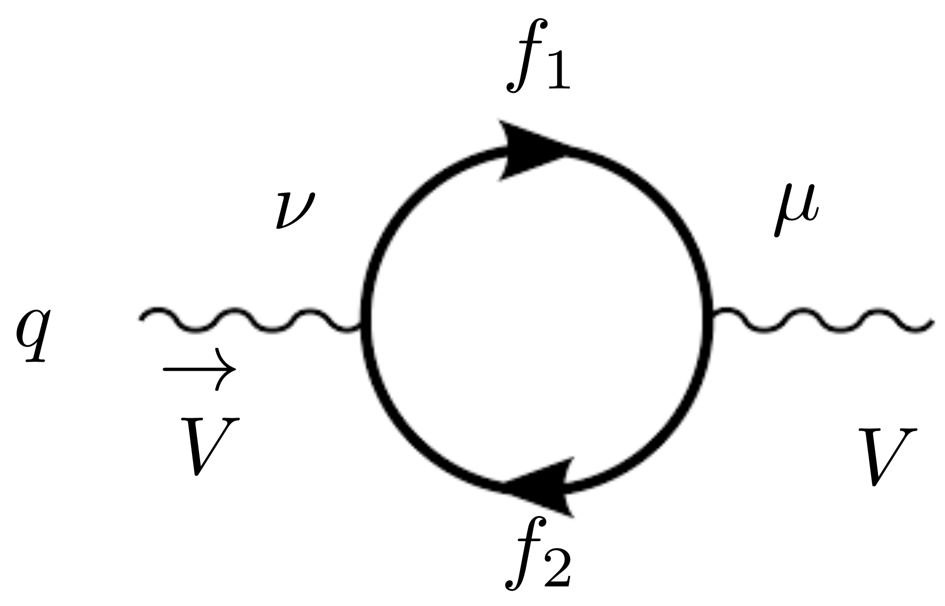
$$g_0^2 = \frac{4\pi\alpha_0}{s_{W,0}^2} \qquad\qquad s_{W,0}^2 = 1 - \frac{m_{w,0}^2}{m_{z,0}^2}$$

$$\frac{\delta G_F}{G_F} = \frac{\delta\alpha}{\alpha} - \frac{\delta s_W^2}{s_W^2} - \frac{\delta m_W^2}{m_W^2} + \frac{\Pi_{WW}(0)}{m_w^2} \qquad\qquad \frac{\delta s_W^2}{s_W^2} = -\frac{c_W^2}{s_W^2}\left(\frac{\delta m_w^2}{m_w^2} - \frac{\delta m_z^2}{m_z^2}\right)$$

$$\frac{\delta G_F}{G_F} = \frac{\delta\alpha}{\alpha} - \frac{c_W^2}{s_W^2}\left(\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(m_W^2)}{m_W^2}\right) + \frac{\Pi_{WW}(0) - \Pi_{WW}(m_W^2)}{m_W^2}$$

We will continue with the study of the the self-energy corrections to the propagators of Z and W bosons.

We will focus on the fermionic contributions to vacuum polarisation functions of the gauge bosons — they are the important quantities; they involve vector and axial-vector currents.



$$T^{\mu\nu}_{VV,AA} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu(\gamma_5)(\hat{k}+m_1)\gamma^\nu(\gamma_5)(\hat{k}-\hat{q}+m_2)\right]}{(k^2-m_1^2)((k-q)^2-m_2^2)}$$

$$d = 4-2\epsilon$$

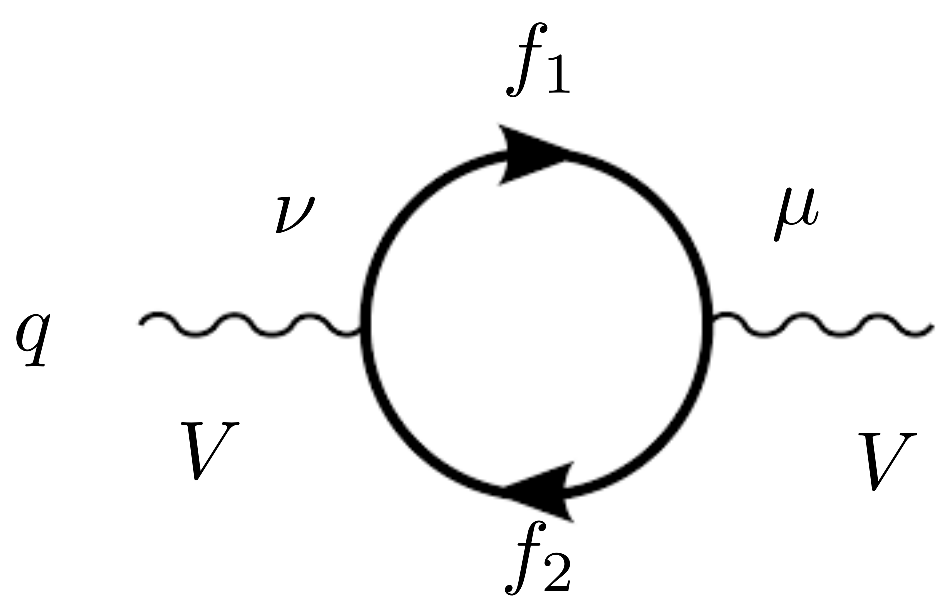
$$T^{\mu\nu}_{VV} = T^{\mu\nu}_1 + T^{\mu\nu}_2 \qquad T^{\mu\nu}_{AA} = T^{\mu\nu}_1 - T^{\mu\nu}_2$$

$$T^{\mu\nu}_1 = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu \hat{k}\gamma^\nu(\hat{k}-\hat{q})\right]}{(k^2-m_1^2)((k-q)^2-m_2^2)} \qquad T^{\mu\nu}_2 = m_1 m_2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]}{(k^2-m_1^2)((k-q)^2-m_2^2)}$$

Combining propagators using the Feynman parameters and shifting the loop momentum, we obtain to obtain the unshifted momentum in the denominator, we find

$$T^{\mu\nu}_1 = 4\int [\mathrm{dx}]_{12} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\frac{(2-d)}{d} g^{\mu\nu}k^2 - (2q^\mu q^\nu - q^2 g^{\mu\nu})x_2(1-x_2)}{(k^2-\Delta+i0)^2} \qquad T^{\mu\nu}_2 = 4m_1 m_2 g^{\mu\nu}\int [\mathrm{dx}]_{12} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta+i0)^2}$$

$$[\mathrm{dx}]_{12} = \mathrm{d}x_1\, \mathrm{d}x_2 \delta(1-x_1-x_2) \qquad \Delta = m_1^2 x_1 + m_2^2 x_2 - q^2 x_2(1-x_2)$$



$$T^{\mu\nu}_{VV,AA} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu(\gamma_5)(\hat{k}+m_1)\gamma^\nu(\gamma_5)(\hat{k}-\hat{q}+m_2)\right]}{(k^2-m_1^2)((k-q)^2-m_2^2)}$$

$$T^{\mu\nu}_{VV} = T^{\mu\nu}_1 + T^{\mu\nu}_2 \qquad T^{\mu\nu}_{AA} = T^{\mu\nu}_1 - T^{\mu\nu}_2$$

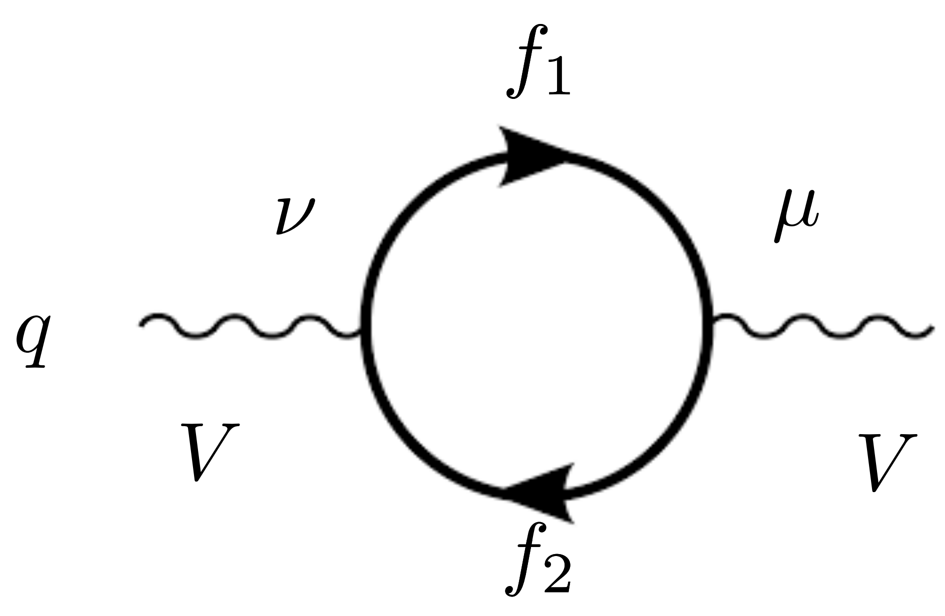
$$T^{\mu\nu}_1 = 4\int [\mathrm{d}x]_{12} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\frac{(2-d)}{d} g^{\mu\nu} k^2 - (2q^\mu q^\nu - q^2 g^{\mu\nu}) x_2(1-x_2)}{(k^2-\Delta+i0)^2} \qquad T^{\mu\nu}_2 = 4m_1 m_2 g^{\mu\nu} \int [\mathrm{d}x]_{12} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta+i0)^2}$$

$$[\mathrm{d}x]_{12} = \mathrm{d}x_1\, \mathrm{d}x_2 \delta(1-x_1-x_2) \qquad \Delta = m_1^2 x_1 + m_2^2 x_2 - q^2 x_2(1-x_2)$$

Integration over the loop momentum is performed using the following (standard) formula

$$I_n(m^2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-m^2+i0)^n} = \frac{i(-1)^n\ (m^2-i0)^{d/2-n}}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

We will drop terms proportional to $q^\mu q^\nu$ because they lead to the mass-suppressed terms.



$$T^{\mu\nu}_{VV} = T^{\mu\nu}_1 + T^{\mu\nu}_2 \qquad T^{\mu\nu}_{AA} = T^{\mu\nu}_1 - T^{\mu\nu}_2 \qquad T^{\mu\nu}_{1,2} = iT_{1,2}g^{\mu\nu}$$

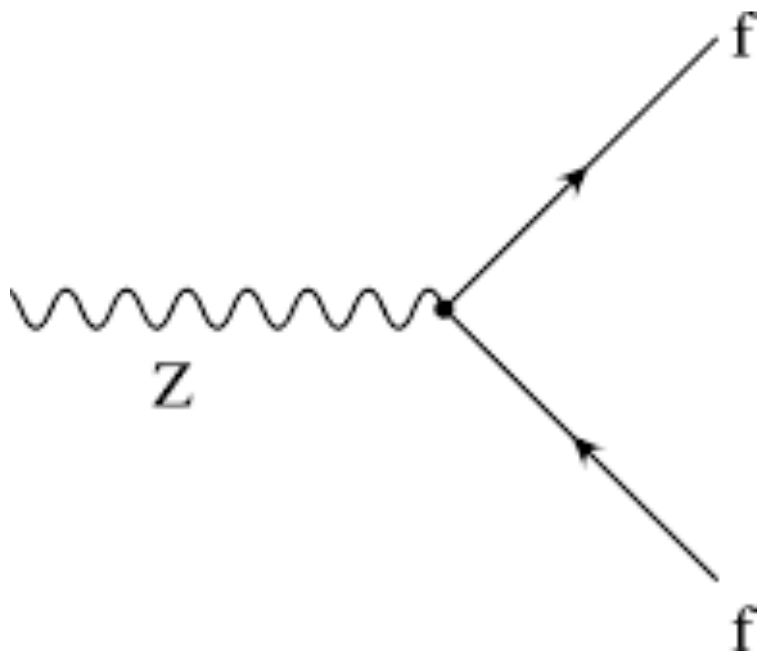
$$T_1 = \frac{4N_\epsilon}{\epsilon}\int [\mathrm{d}x]_{12} \left(\frac{\Delta}{\mu^2}\right)^{-\epsilon} \left[-\Delta(x_1,x_2) + q^2 x_2(1-x_2)\right] \qquad T_2 = \frac{4N_\epsilon}{\epsilon} m_1 m_2 \int [\mathrm{d}x]_{12} \left(\frac{\Delta}{\mu^2}\right)^{-\epsilon} \qquad N_\epsilon = \frac{\Gamma(1+\epsilon)\mu^{-2\epsilon}}{(4\pi)^{d/2}}$$

We can assemble the vacuum polarization contributions for different vector bosons. For the photon, where only vector current contributes, we find ($\eta_f = N_c$ for a quark and $\eta_f = 1$ for a lepton).

$$\tilde{\Pi}^{(f)}_{\gamma\gamma}(q^2) = q^2 \Pi^{(f)}_{\gamma}\gamma(q^2) = e^2 Q_f^2 \eta_f (T_1 + T_2)|_{m_1=m_2=m} = q^2 e^2 Q_f^2 \eta_f N_\epsilon \left(\frac{4}{3\epsilon} - 8\int_0^1 \mathrm{d}x\, x(1-x) \ln\frac{\Delta}{\mu^2} \right)$$

We note that the result is proportional to q², which implies that the photon remains massless (as it should be, of course).

Next, we compute the Z-boson vacuum polarization. The interaction of Z-bosons and fermions involves vector and vector-axial currents.

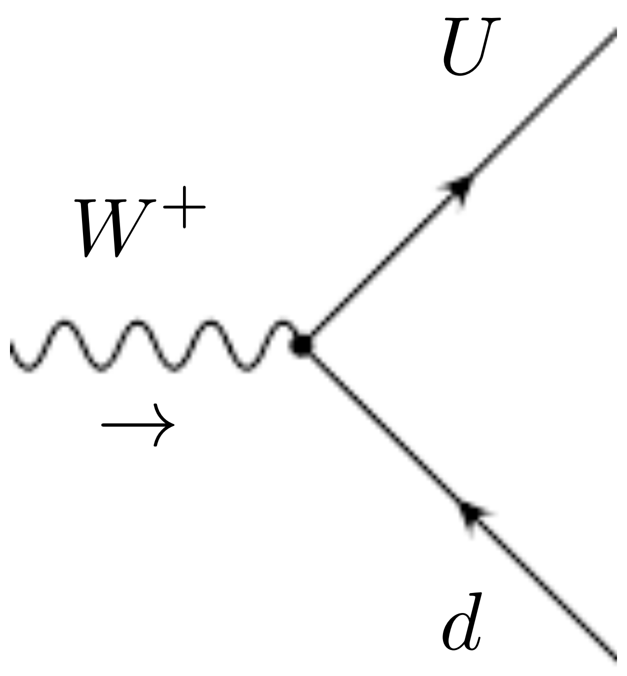


$$i\frac{g}{4c_W}\left(g_V^{(f)}\gamma^\mu + g_A^{(f)}\gamma^\mu\gamma_5\right)$$

$$\Pi_{ZZ}^{(f)}(q^2) = \frac{g^2}{16c_W^2}\eta_f\left(\left(g_{V,f}^2 + g_{A,f}^2\right)(T_1+T_2) - 2g_{A,f}^2 T_2\right)\Big|_{m_1=m_2=m_f}$$

Note that there is a contribution that involves only T_2, and that T_2 is proportional to the mass of the fermion in the loop squared. If the fermion is heavier that the Z-boson , this gives a significant enhancement. The same applies to the vacuum polarisation of the W-boson where T_1 appears.

$$\Pi_{WW} = \frac{g^2}{8}V_{UD}V_{UD}^*\,\eta_f\,(T_1+T_2+T_1-T_2)_{m_1\to m_U, m_2\to m_D}$$
$$= \frac{g^2}{4}V_{UD}V_{UD}^*\,\eta_f T_1\Big|_{m_1\to m_U, m_2\to m_D}$$



$$i\frac{g}{2\sqrt{2}}V_{UD}\gamma^\mu(1-\gamma_5)$$

We will now use these results to compute the vacuum polarisation corrections to the Fermi constant making use of the fact that leading contributions to ZZ and WW vacuum polarisation functions are independent of q (since they are proportional to the heaviest (top) quark mass).

$$\frac{\delta G_F}{G_F} = \frac{\delta\alpha}{\alpha} - \frac{c_W^2}{s_W^2}\left(\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(m_W^2)}{m_W^2}\right) + \frac{\Pi_{WW}(0) - \Pi_{WW}(m_W^2)}{m_W^2}$$

$$\frac{\delta G_F}{G_F} \approx \frac{\delta\alpha}{\alpha} - \frac{c_W^2}{s_W^2}\left(\frac{\Pi^{tt}_{ZZ}(0)}{m_Z^2} - \frac{\Pi^{tb}_{WW}(0)}{m_W^2}\right)$$

$$\frac{\Pi_{ZZ}(0)}{m_Z^2} = -\frac{g^2}{2c_W^2 m_Z^2}\frac{N_\epsilon}{\epsilon}\sum_f g_{A,f}^2 \eta_f\, m_f^2\left(1 - \epsilon\ln\frac{m_f^2}{\mu^2}\right) \qquad g_{A,f} = 2T_L^{(3)} = \pm 1$$

$$\frac{\Pi_{WW}(0)}{m_W^2} = \frac{g^2}{m_W^2}\frac{N_\epsilon}{\epsilon}\sum_{f_u,f_d} V_{f_u,f_d}V^*_{f_u,f_d}\,\eta_f\,\Big(-\frac{m_{f,u}^2 + m_{f,d}^2}{2} + \epsilon\int_0^1 \mathrm{d}x\ \left(m_{f,u}^2\, x + m_{f,d}^2(1-x)\right)\,\ln\frac{m_{f,u}^2\, x + m_{f,d}^2(1-x)}{\mu^2}\Big),$$

$$\frac{\Pi^{tt}_{ZZ}(0)}{m_Z^2} - \frac{\Pi^{tb}_{WW}(0)}{m_W^2} = \frac{g^2 m_t^2 N_c}{64\pi^2 m_W^2} \approx 0.02.$$

Corrections enhanced by the square of the top quark mass and fairly large correction related to the change in the fine structure constant help us reconcile measured and predicted values of the Fermi constant.

$$G_F = \frac{4\pi\alpha}{\sqrt{2}4s_w^2 m_W^2}\left(1+\delta\alpha - \frac{c_w^2}{s_w^2}\left(\frac{\Pi_{zz}^{tt}(0)}{m_z^2} - \frac{\Pi_{WW}^{tb}(0)}{m_W^2}\right)\right) \approx 1.16 \times 10^{-5}\ \mathrm{GeV}^2$$

Another interesting quantity is the relative strength of charged and neutral current interactions, at low energies. It is characterised by the so-called rho-parameter.

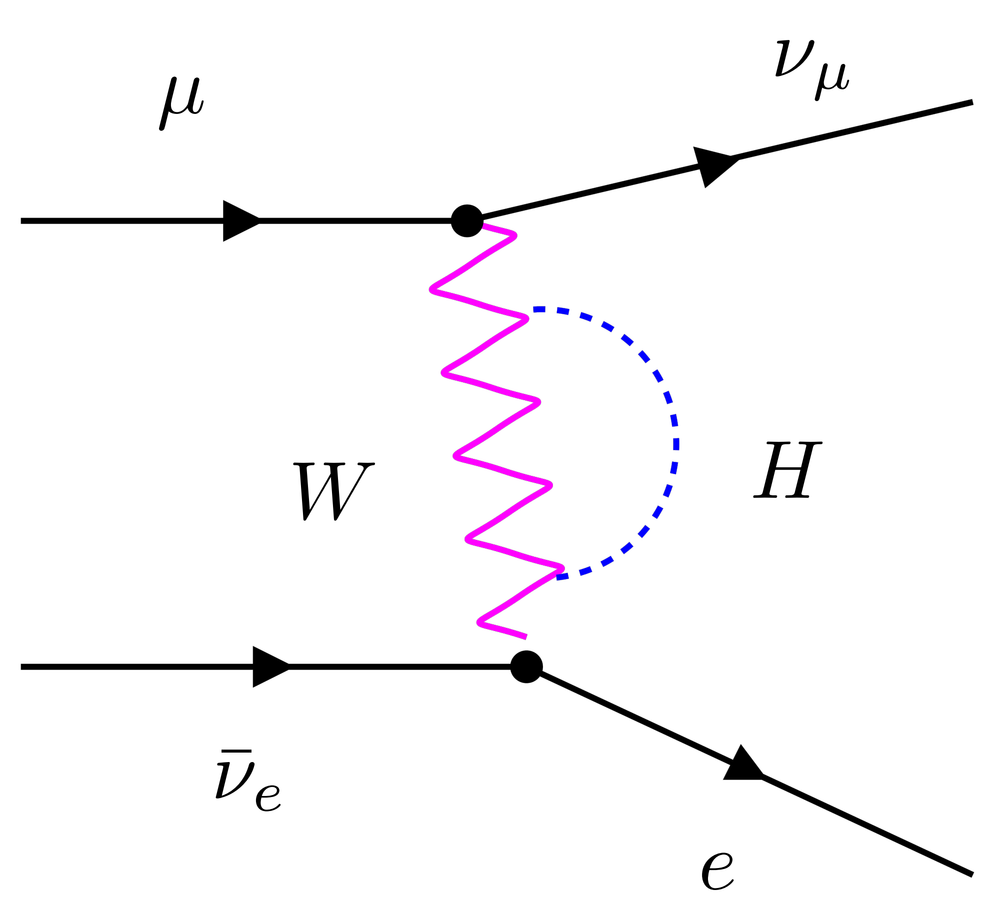
$$\mathcal{L}_W = \frac{g}{\sqrt{2}} W_\mu^+ J_W^{\mu,+} + h.c. \qquad \mathcal{L} = \frac{g^2}{2m_W^2} J_W^{\mu,+} J_{W,\mu}^- + \frac{g^2}{2c_W^2 m_Z^2} J_Z^\mu J_{z,\mu} = \frac{g^2}{2m_W^2}\left[J_W^{\mu,+} J_{W,\mu}^- + \rho\, J_Z^\mu J_{z,\mu}\right]$$

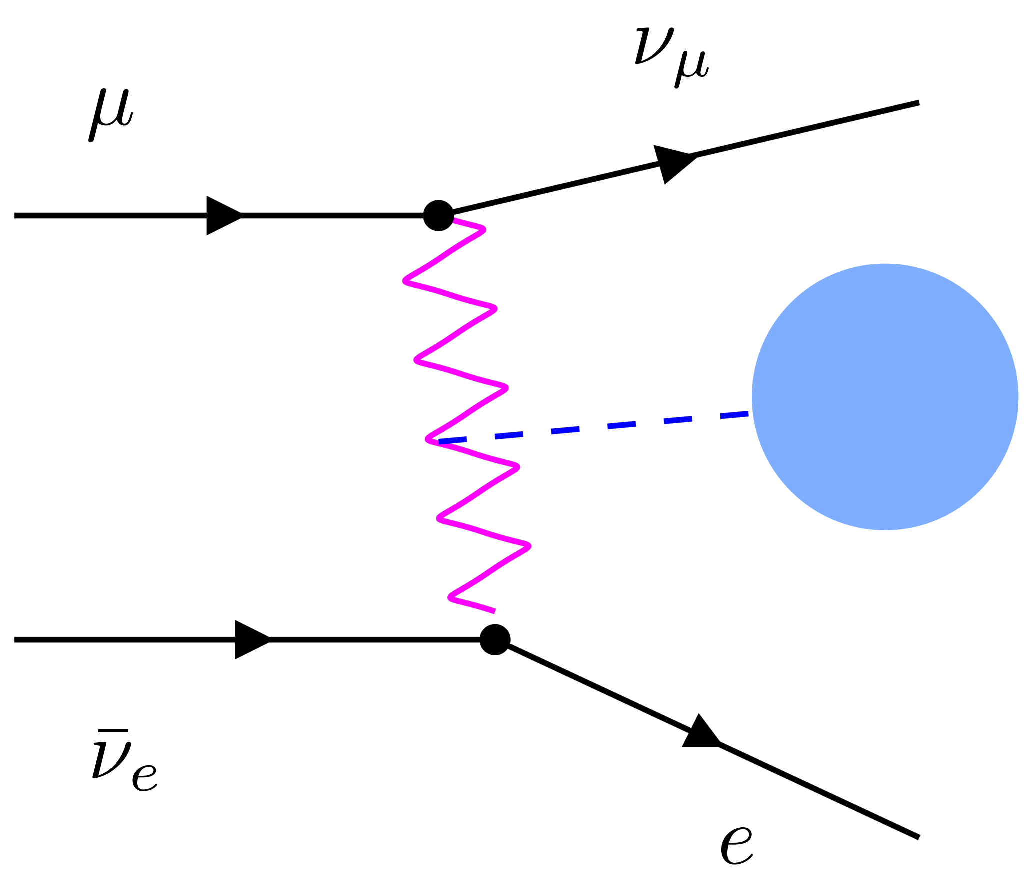
$$\mathcal{L}_Z = \frac{g}{\cos\theta_W} J_Z^\mu \qquad \rho = \frac{g_0^2}{2(m_{Z,0}^2 - \Pi_{ZZ}(0))c_{w,0}^2}\,\frac{2(m_{W,0}^2 - \Pi_{WW}(0))}{g_0^2} \qquad c_{W,0}^2 = \frac{m_{W,0}^2}{m_{Z,0}^2}$$

$$\rho = 1 + \frac{\Pi_{ZZ}(0)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2} \approx 1 + \frac{g^2 N_c m_t^2}{64\pi^2 m_W^2} \approx 1.02$$

For historical reasons, it was interesting to know the impact of the Higgs boson on the radiative corrections in the Standard Model.

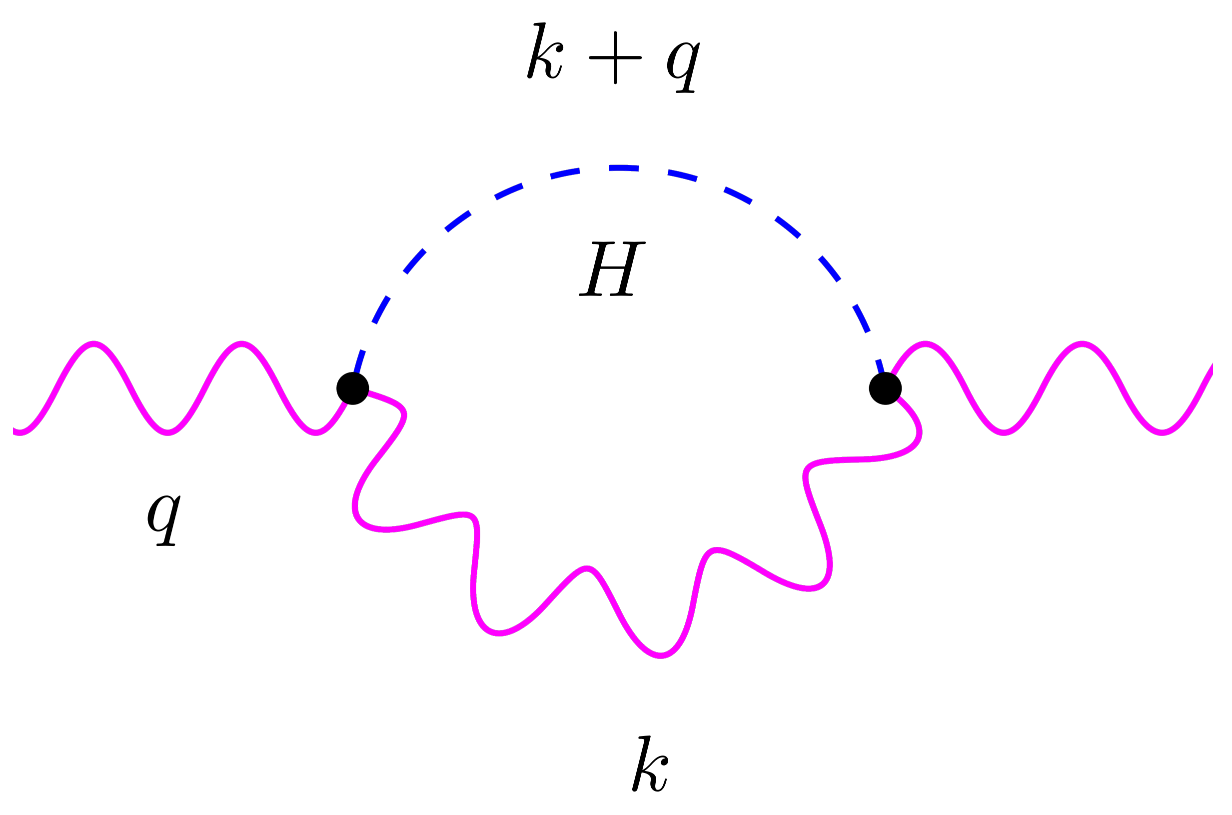
The Higgs boson couples to fermions with the strength proportional to their masses; for currents that appear in the Fermi Lagrangian, all such masses are tiny. Hence, the only diagrams to study in this case are the ones where the Higgs boson couples to the propagator of the virtual W boson.





We will again compute a generic vacuum-polarisation contribution and then use it to derive corrections to various relevant quantities. We work in the unitary gauge where unphysical parts of the Higgs doublet are not present.

$$T_3^{\mu\nu} = m^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}}{k^2 - m^2} \frac{1}{(k+q)^2 - m_H^2}$$



Combining propagators using Feynman parameters, shifting the loop momentum and neglecting all terms that are proportional to the momentum q in the numerator, we arrive at

$$T_3^{\mu\nu} = m^2 \int \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}}{(k^2 - \Delta_H)^2}$$

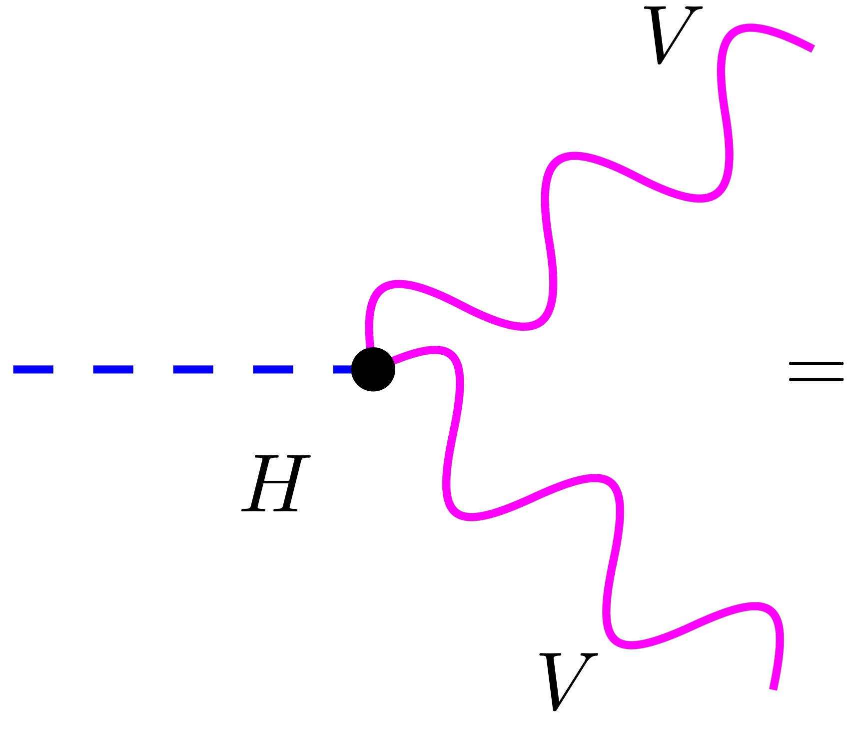
$$\Delta_H = m_H^2 x + m^2(1-x) - q^2 x(1-x).$$

$$T_3^{\mu\nu} = i g^{\mu\nu} T_3,$$

$$T_3 = -i m^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1 - \frac{k^2}{d m^2}}{(k^2 - \Delta_H)^2}$$

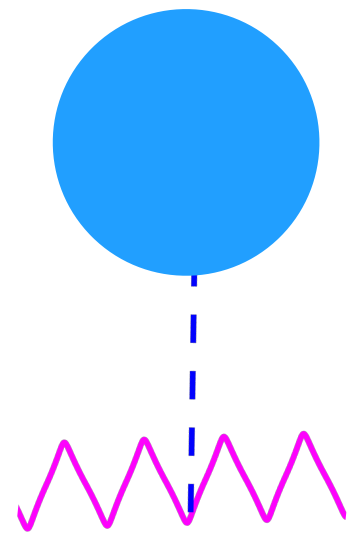
Computing the contribution of the Higgs boson to the rho-parameter, we find a logarithmic senstitivity in the limit $m_H \gg m_{Z,W}$

$$T_3 = \frac{N_\epsilon}{\epsilon} \int_0^1 dx \left(\frac{\Delta_H}{\mu^2}\right)^{-\epsilon} \left(m^2 - \frac{\Delta_H}{(d-2)}\right) \quad \Rightarrow \quad T_3 = \frac{N_\epsilon}{4\epsilon}\left[\left(3m^2 - m_H^2 + \frac{q^2}{3}\right) - \epsilon\left(m^2 + m_H^2 - \frac{q^2}{3}\right) - 2\epsilon \int_0^1 dx \left(m^2(1+x) - m_H^2 x + q^2 x(1-x)\right) \ln\left(\frac{\Delta_H}{\mu^2}\right)\right]$$



$$V = W \qquad V = Z$$

$$= i g_{\mu\nu}\, g m_W, \quad i g_{\mu\nu} \frac{g}{c_W} m_Z$$



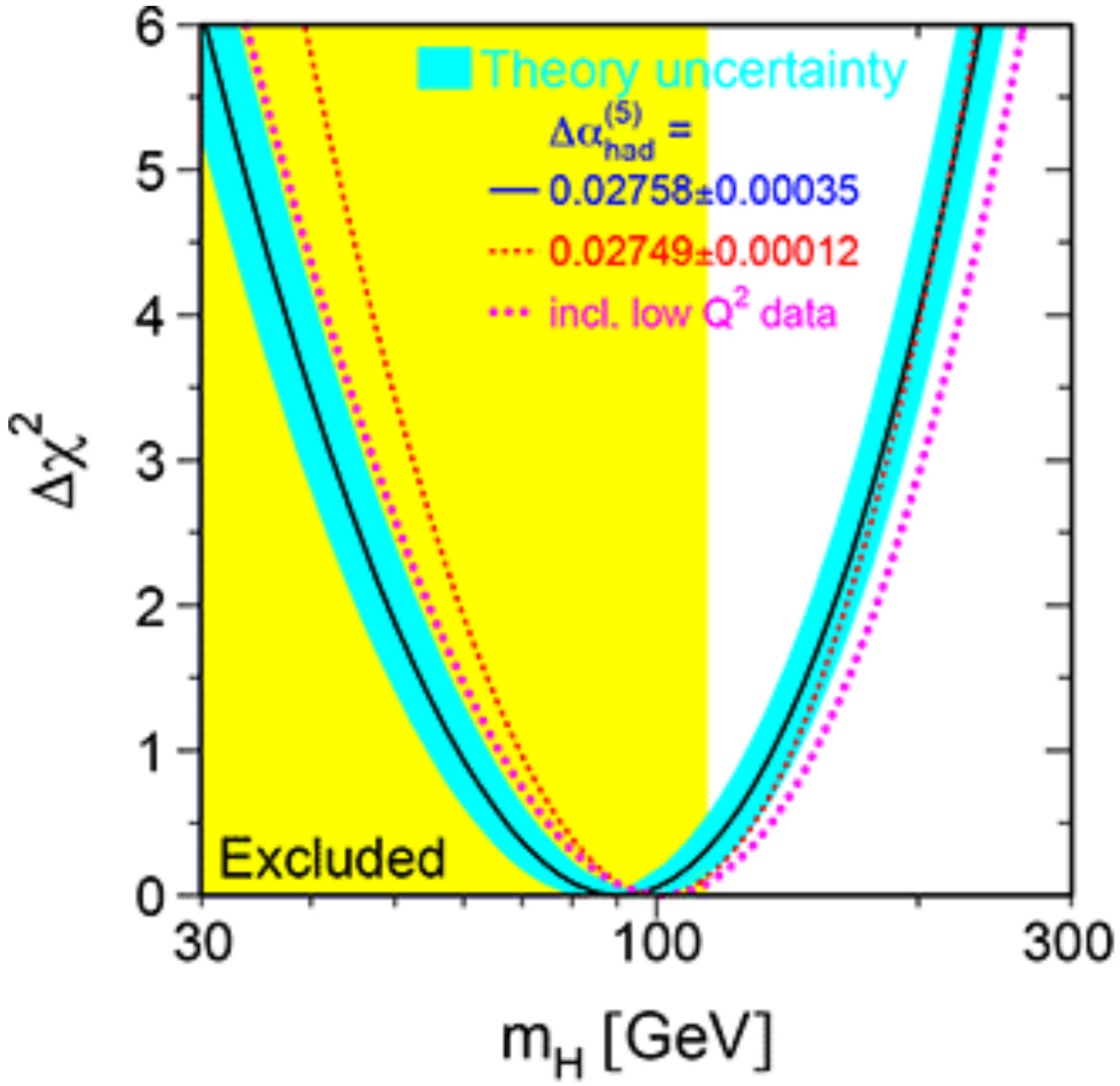
Tadpole diagrams fully cancel in the contribution to the rho parameter

$$\rho = 1 + \frac{\Pi_{ZZ}(0)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2}$$

$$\Delta\rho_H = \left.\frac{g^2 T_3(m_Z^2)}{\cos\theta_W^2 m_Z^2}\right|_{q^2\to 0} - \left.\frac{g^2 T_3(m_w^2)}{m_W^2}\right|_{q^2\to 0}$$

$$\Delta\rho_H = \frac{g^2}{(4\pi)^2 m_W^2}\frac{3}{4}\left(m_W^2 - m_Z^2\right)\ln\frac{m_H^2}{m_Z^2} = -\frac{3\alpha}{16\pi}\frac{m_z^2}{m_w^2}\ln\frac{m_H^2}{m_z^2}$$

Because of the weak sensitivity to the Higgs mass, it was difficult to pinpoint its value from the precision electroweak fits, although there were clear indications that the mass is relatively low.



The muon anomalous magnetic moment: a two-decade saga

Nothing epitomises challenges faced by the precision SM physics program better than the muon anomalous magnetic moment. This quantity was measured in the dedicated experiment at BNL where circa a 3-sigma deviation between theoretical and experimental results was observed. To clarify the origin of this discrepancy, the storage ring was moved to FNAL and the new experiment there was started.

$$a_\mu^{\rm exp} = 116\ 592\ 082(55) \times 10^{-11}$$

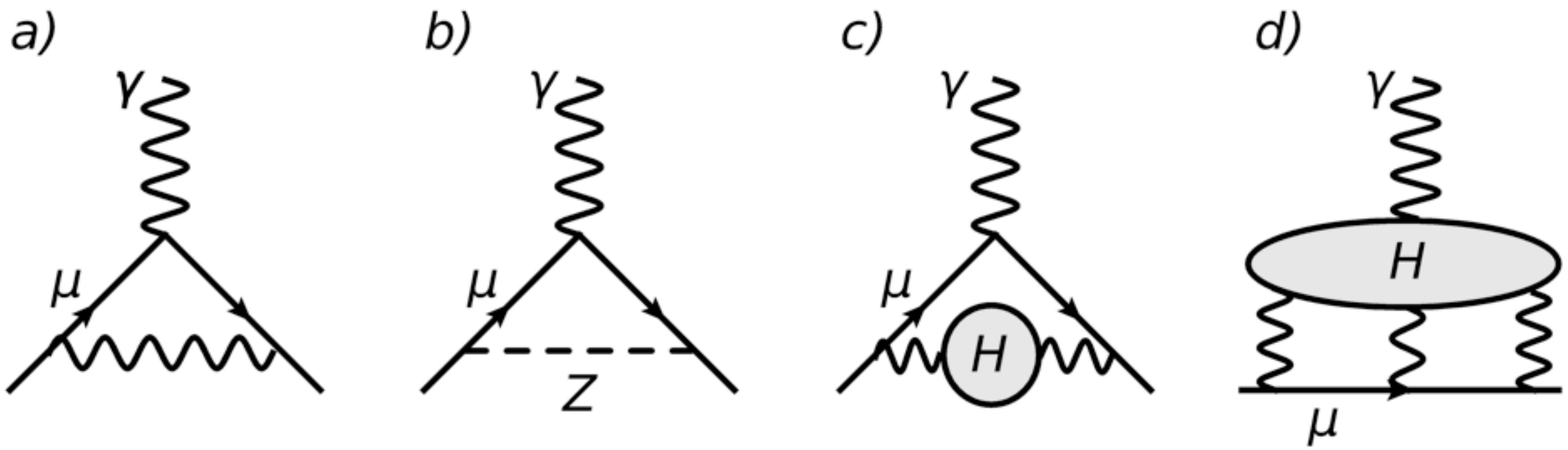
$$a_\mu^{\rm th} = 116\ 591\ 846(63) \times 10^{-11}$$

$$\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm th} = (237 \pm 80) \times 10^{-11}$$

Note that we like to study the muon and not the electron magnetic anomaly because New Physics contributions affect the muon anomalous magnetic moment 40.000 times stronger than the electron one. In fact, we have used the electron magnetic anomaly to fix the value of the fine structure constant....

$$a_l^{\rm BSM} \sim \frac{\alpha}{\pi}\frac{m_l^2}{M^2} \sim 60 \times 10^{-11} \qquad M \sim 200\ {\rm GeV}$$

Several contributions need to be computed to arrive at the muon anomalous magnetic moment at the shown level of precision.

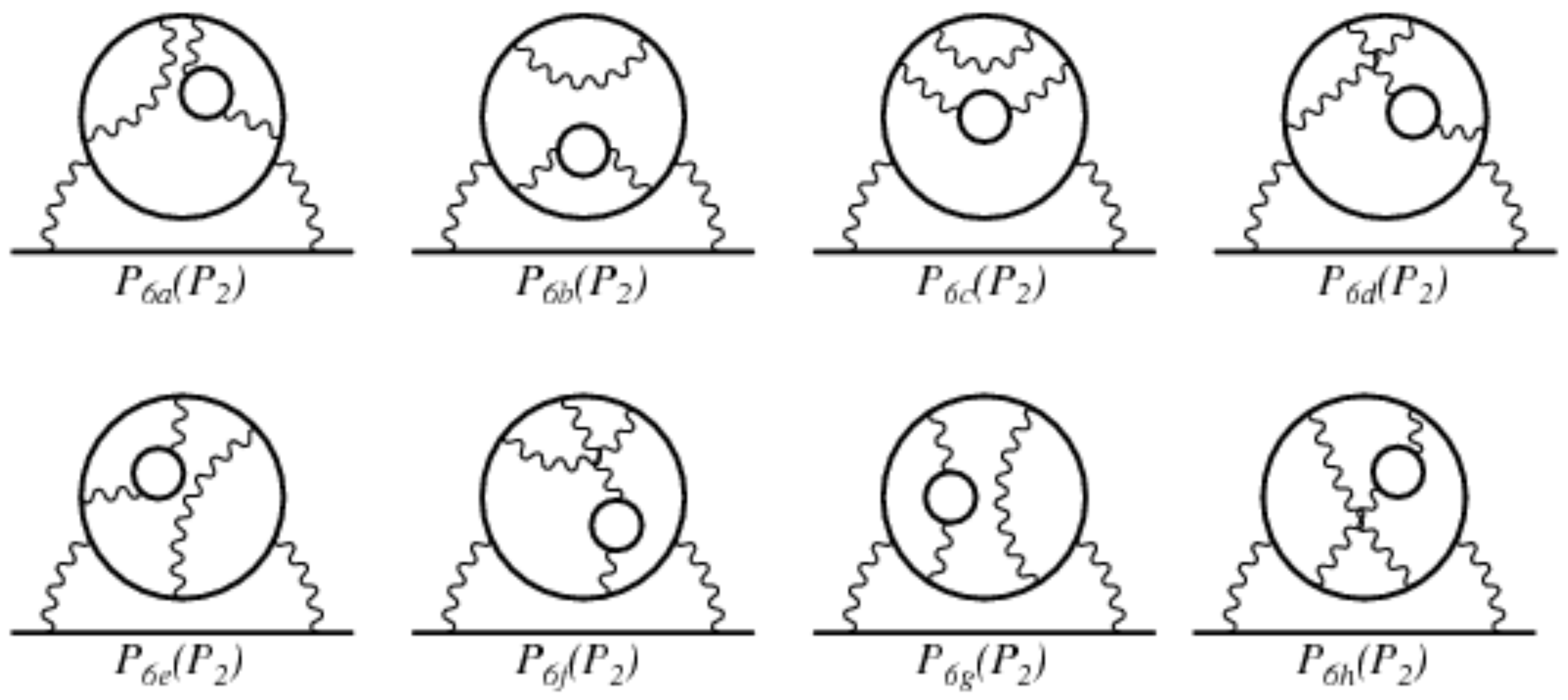


QED	116584718.95(8)
Electroweak	154 ± 2
Hadronic vacuum polarization, LO	$6949 \pm 37 \pm 21$
Hadronic vacuum polarization, NLO	-98.4
Hadronic light-by-light	105 ± 26

Deviation: 240

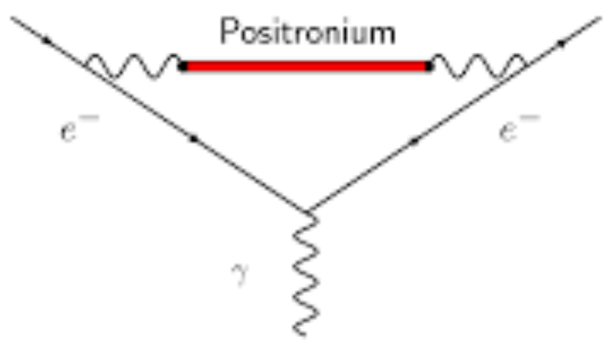
The magnitude of various contributions, in units of 10^{-11}.

QED provides the largest contribution, by far. The current frontier is the five-loop QED; its calculation requires an extremely high degree of specialisation, outsiders cannot judge the correctness of these results.



Nevertheless, one can argue that QED cannot be the reason for the large g-2 discrepancy since in this case either the three-loop contribution or the enhanced four-loop contributions must be wrong. But all these contributions have been checked multiple times, so we are confident that QED is not a reason for the current discrepancy.

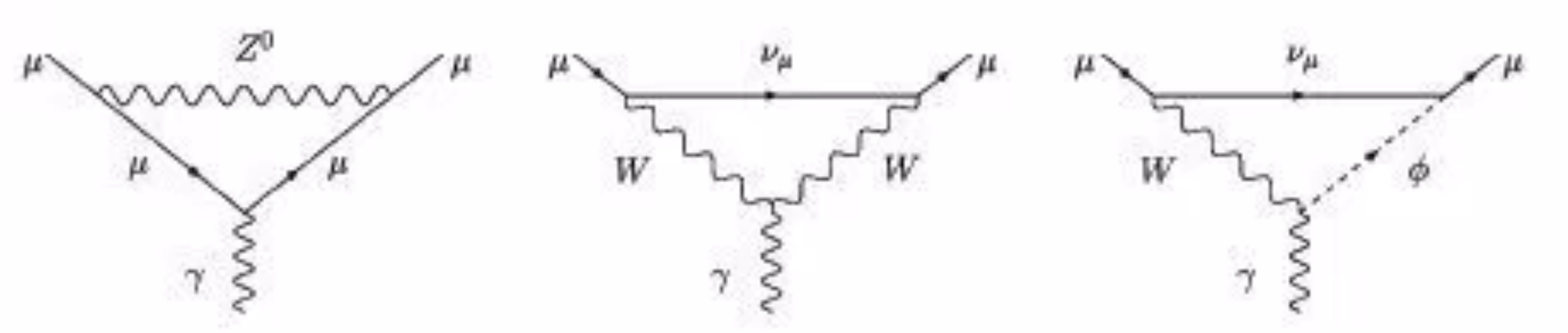
N.B. Since very high precision is required in this case, sometimes interesting side questions are being discussed that people have not thought about before (actually, people did think about it well before and forgot). For example, can QED bound states (positronium) contribute to g-2 and whether or not such contributions are "outside" of the conventional perturbative computations?



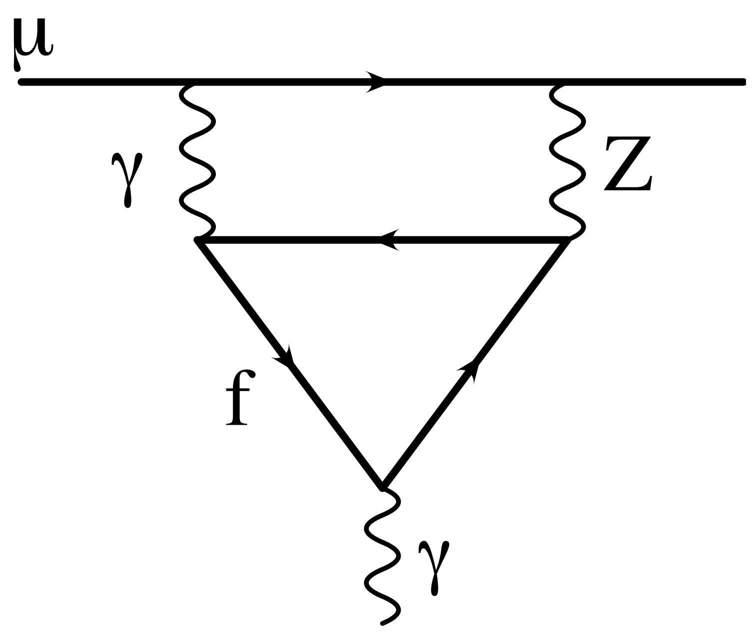
$$a_\mu^{\rm PS} = \frac{\alpha^5}{8\pi}\zeta(3)$$

$$a_\mu^{\rm cont} = -a_\mu^{\rm PS} + \text{ regular PT}$$

Electroweak contributions start at one loop; the two-loop contributions are known and are significant. You would think that it is straightforward (i.e. difficult but straightforward) to compute them, but this is not the case.

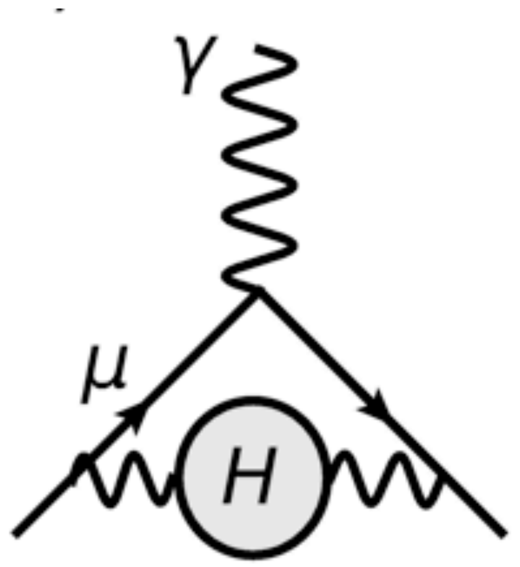


The reason is the "anomalous" contributions which exhibit strong sensitivity to infra-red physics where using quarks in the loops becomes invalid and hadrons are needed. Note that the result below contains hadron masses which is not something that is seen often when loop diagrams are calculated.



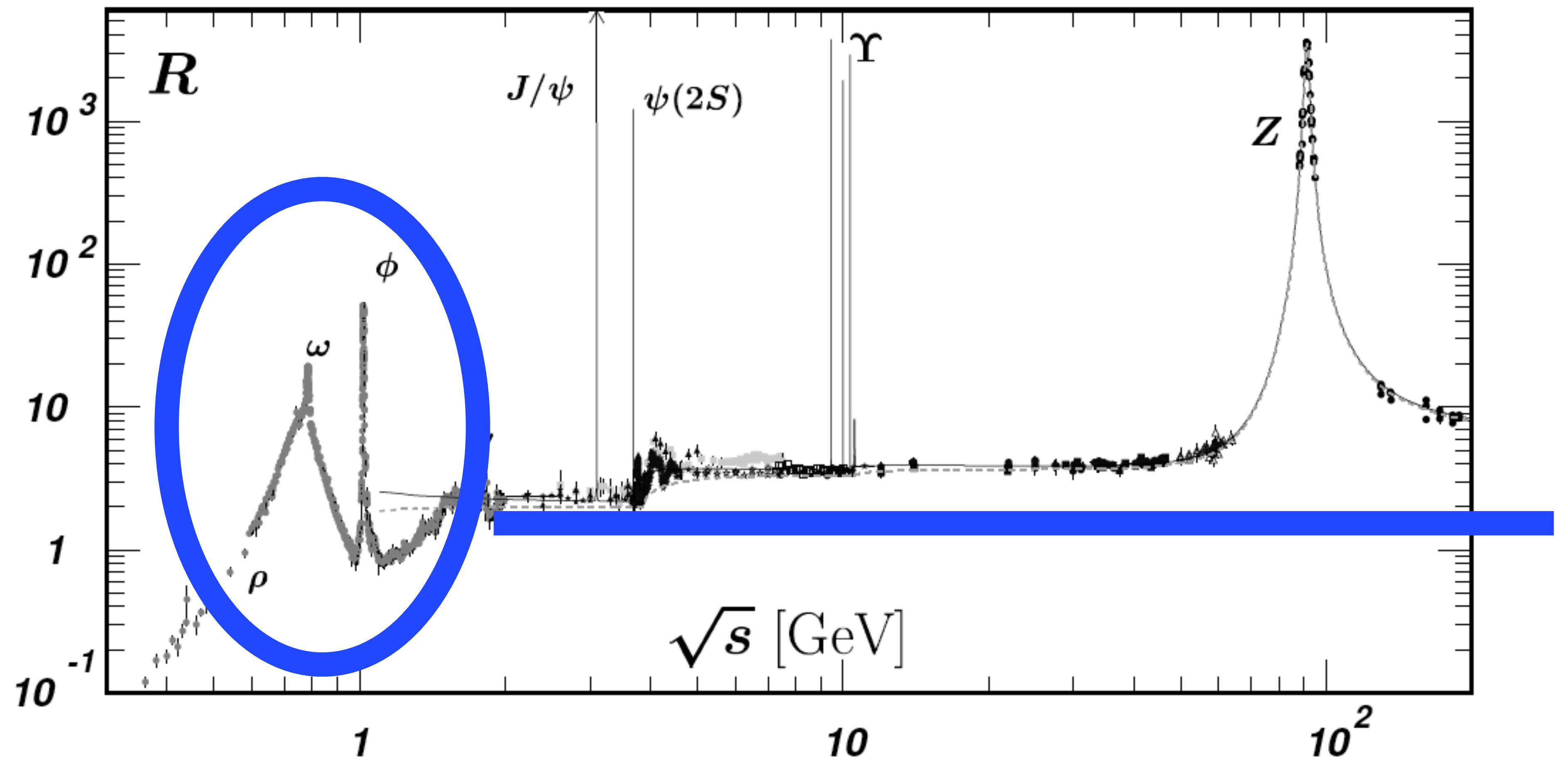
$$\Delta a_\mu(e,u,d) = -\frac{\alpha}{\pi}\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}\left(2\ln\frac{m_\pi^2}{m_\mu^2} + 1 + \ln\frac{m_\rho^2}{m_\mu^2} - \frac{m_\rho^2}{m_{a_1}^2 - m_\rho^2}\ln\frac{m_{a_1}^2}{m_\rho^2} + \frac{3}{2}\right)$$

The next contribution we discuss is the hadronic vacuum polarization. It is a large contribution and it needs to be know to a precision of about 1 percent. The enhancement of the low-s region makes estimating it difficult (this is a big difference with respect to a similar contribution to the electromagnetic coupling constant).



$$a_\mu^{\text{hvp}} = \frac{\alpha}{3\pi}\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}R^{\text{had}}(s)\; a^{(1)}(s)$$

$$a^{(1)}(s) \sim \frac{m_\mu^2}{s}.$$



- the chirally-enhanced two-pion threshold contribution $a_\mu^{\pi\pi}$ defined with an upper energy cut-off at $\sqrt{s} = m_\rho/2$, computed using scalar QED;
- the contribution of ρ, ω, ϕ vector mesons $a_\mu^{\rho,\omega,\phi}$;
- the "continuum" contribution a_μ^{cont} that starts above $\sqrt{s} \sim m_\phi$;

$a_\mu^{\pi\pi} = 400 \times 10^{-11}$ $\quad a_\mu^{\rho,\omega,\phi} = 5514 \times 10^{-11}$ $\quad a_\mu^{\text{cont}} = 1240 \times 10^{-11}$

$$a_\mu^{\text{hvp,th}} = a_\mu^{\pi\pi} + a_\mu^{\rho,\omega,\phi} + a_\mu^{\text{cont}} \approx 7160 \times 10^{-11}$$

A careful data-based evaluation gives the following result for the hadronic vacuum polarisation (SND, CMD-2, BABAR, KLOE)

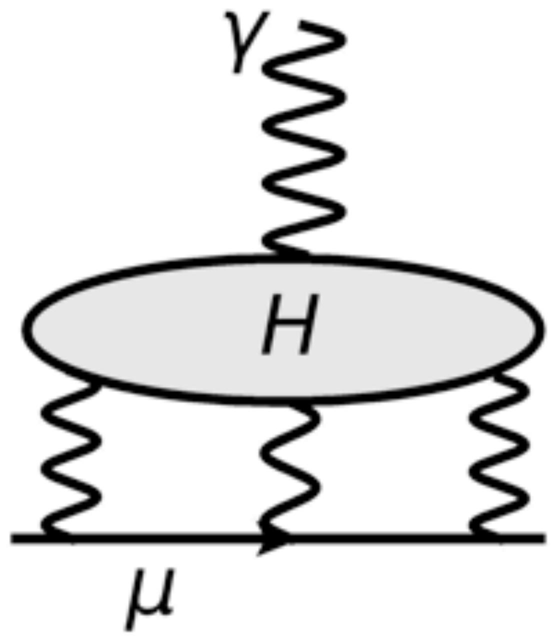
$$a_\mu^{\text{hvp}} = (6949 \pm 37.2 \pm 21.0) \times 10^{-11}$$

The recent result of the CMD-3 collaboration increased the rho-meson contribution by about 3 percent. This moves the hadronic vacuum polarisation contribution up by about 150 x 10^{-11} reducing the discrepancy with the experimental result by more than 2 sigma.

By itself, the CMD-3 result would have remained controversial because other measurements consistently arrive at lower result for a hadronic vacuum polarization. However, a new lattice calculation by the BMW collaboration also arrived at the result that is close to that of CMD-3 and (maybe !) solves the g-2 problem fully.

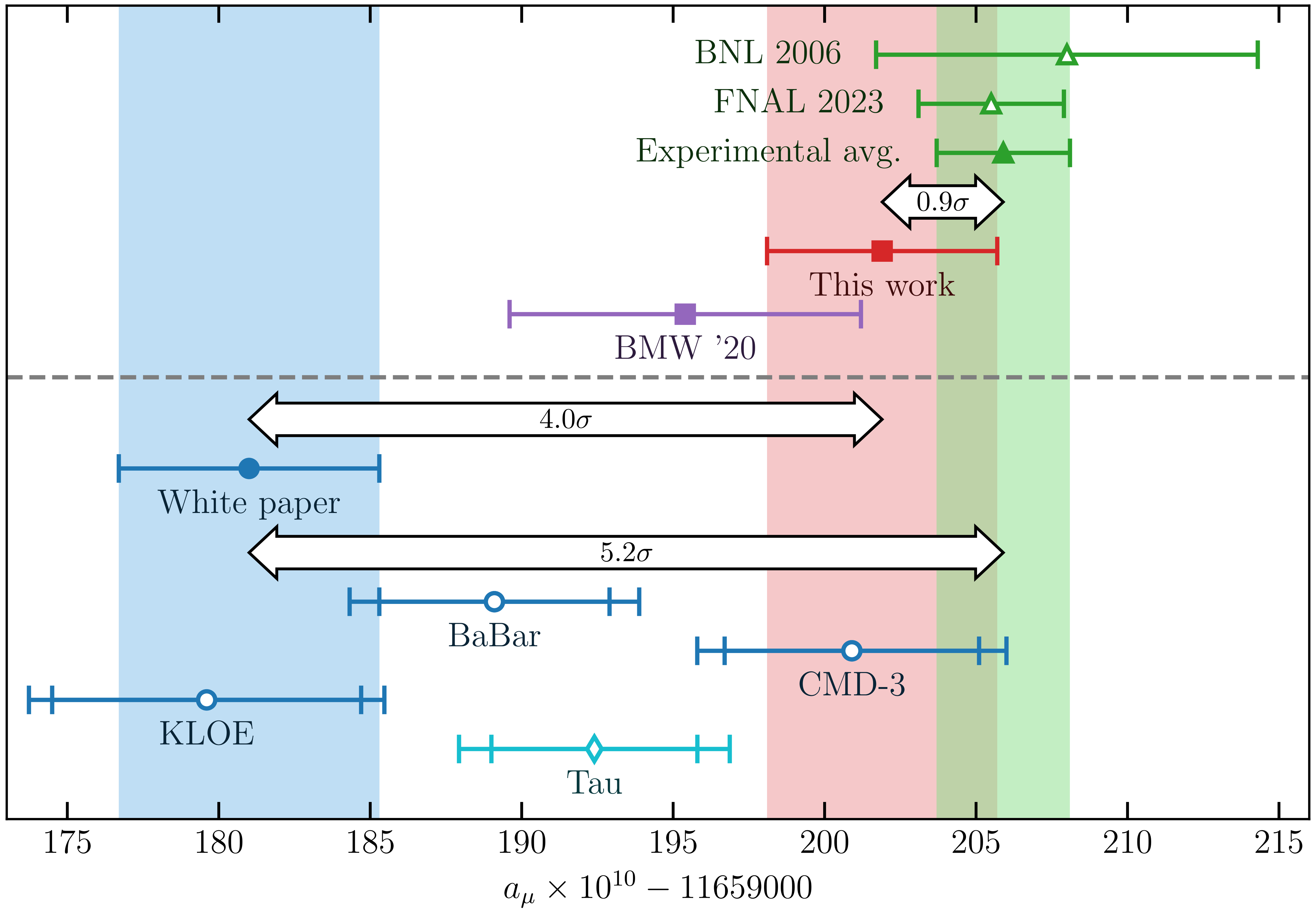
$$a_\mu^{\rm hvp} = 7141(45) \times 10^{-11}$$ BMW lattice collaboration

For completeness: the last contribution is the hadronic light-by-light scattering one. It is small but very troubled contribution, that was often considered as a leading candidate for being wrong (primarily because there is no data that can be used to evaluate it and because it used to be a negative of its current value for a while). However, theorists seem to have gotten it right (within a fairly large error bars), as is also confirmed by the dedicated lattice calculations.



$$a_\mu^{\rm hlbl} = (105 \pm 26) \times 10^{-11}$$

Hence, after more than 20 years, thanks to the new result of the BWM collaboration, the new measurement of CMD-3 and work of many theorists, it seems that the muon magnetic anomaly crises is starting to ease. Of course, cross checks of these results will continue and, hopefully, we will start seeing the convergence of the theoretical results soon.



Z. Fodor, BMW collaboration, ICHEP 2024

The unitarity of the CKM matrix

As we have seen, the CKM matrix arises when generic Yukawa interactions are diagonalised by independent unitary transformations of left up-type and down-type quarks. The CKM matrix is unitary. How this can be checked?

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} W_\mu^+ J_W^{\mu,+} + h.c.$$

$$V^{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$J_W^{\mu,+} = \frac{1}{2}\sum_l \bar{\nu}_l \gamma^\mu (1-\gamma_5) l + \frac{1}{2}\sum_{i,j} \bar{u}_i \gamma^\mu (1-\gamma_5) V_{ij} d_j$$

We will discuss tests of unitarity relations that involve light-quark CKM elements

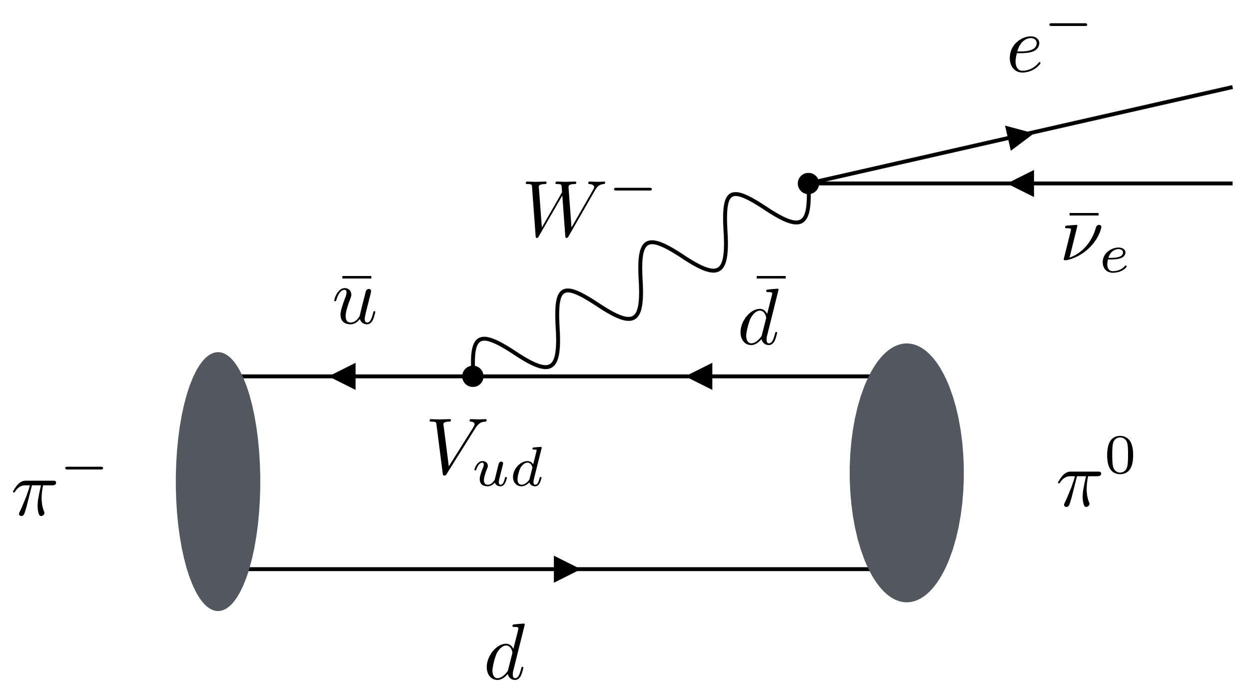
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Since $|V_{ub}| \sim 10^{-5}$ we will drop it from the above equation. Hence, we will be checking the relation between cosine and sine of the Cabibbo angle.

$$|V_{ud}|^2 + |V_{us}|^2 = 1$$

To get access to these CKM matrix elements, we need to understand weak decays of light hadrons. Hadrons are always difficult and, if we aim at testing the above relation with high precision, we should find a way to avoid the hadronic uncertainties impacting our predictions.

Consider the decay $\pi^- \to \pi^0 + e + \bar{\nu}_e$ $\quad m_\pi^- = 139.57\ \text{MeV}$ $\quad m_\pi^0 = 134.98\ \text{MeV}$



$$\mathcal{L} = \frac{G_F}{\sqrt{2}} V_{ud} J^q_\alpha(x) J^\alpha_{\text{lep}}(x)$$

$$J_q^\alpha = \bar{u}\gamma^\alpha(1-\gamma_5)d \qquad J^\alpha_{\text{lept}} = \bar{e}\gamma^\alpha(1-\gamma_5)\nu_e$$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ud} \langle \pi^0(p_{\pi^0}) | J_q^\alpha | \pi^-(p_{\pi^-}) \rangle \times \bar{u}_e(p_3)\gamma_\alpha(1-\gamma_5)v_\nu(p_4)$$

$$\langle \pi^0(p_{\pi^0}) | J_q^\alpha(0) | \pi^-(p_{\pi^-}) \rangle = \langle \pi^0(p_{\pi^0}) | \bar{u}\gamma^\alpha d | \pi^-(p_{\pi^-}) \rangle = f_1(q^2)\,(p^\alpha_{\pi^0} + p^\alpha_{\pi^-}) + f_2(q^2)\, q^\alpha \qquad q = p_{\pi^0} - p_{\pi^-}$$

The isospin symmetry implies $f_2(q^2) = 0$ because the current $\bar{u}\gamma^\mu d$ is conserved, similar to the electromagnetic current. The isospin symmetry does not restrict the form factor $f_1(q^2)$.

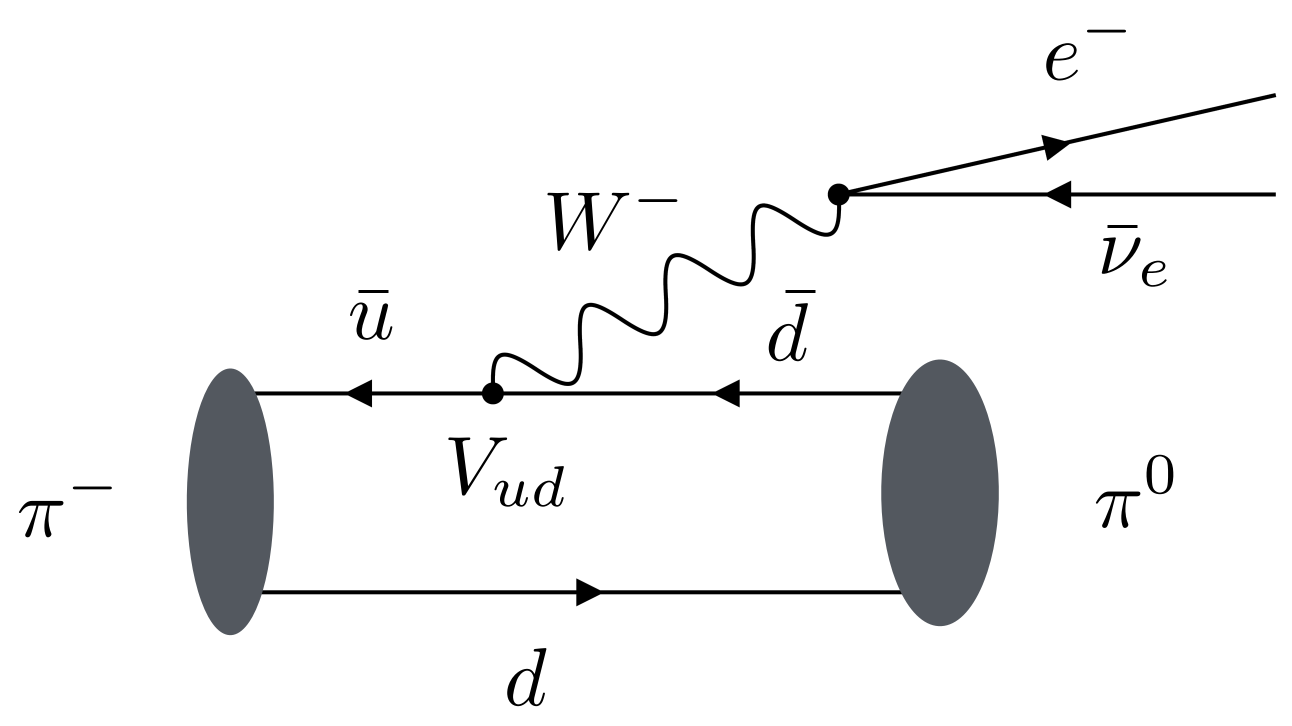
The important simplification arises because in the decay $\pi^- \to \pi^0 + e + \bar{\nu}_e$ the momentum transfer q is very small.

$$q^2 = (p_{\pi^-} - p_{\pi^0})^2 = m^2_{\pi^-} + m^2_{\pi^0} - 2m_{\pi^-}E_{\pi^0}$$

$$q^2|_{\text{max}} = (m_{\pi^-} - m_{\pi^0})^2 \approx (5\ \text{MeV})^2 \qquad \sqrt{q^2_{\text{max}}} \ll \Lambda_{\text{QCD}}$$

The decay $\pi^- \to \pi^0 + e + \bar{\nu}_e$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ud} \langle \pi^0(p_{\pi^0}) | J_q^\alpha | \pi^-(p_{\pi^-}) \rangle \times \bar{u}_e(p_3)\gamma_\alpha(1-\gamma_5)v_\nu(p_4)$$



$$\langle \pi^0(p_{\pi^0}) | J_q^\alpha(0) | \pi^-(p_{\pi^-}) \rangle = \langle \pi^0(p_{\pi^0}) | \bar{u}\gamma^\alpha d | \pi^-(p_{\pi^-}) \rangle = f_1(q^2)\,(p_{\pi^0} + p_{\pi^-})\,.$$

$$f_1(q^2) \approx f_1(0)\left(1 + \frac{q^2}{\Lambda^2_{\mathrm{QCD}}}\right) \to f_1(0) \qquad q = p_{\pi^0} - p_{\pi^-}$$

Conservation of the isospin current $\bar{u}\gamma^\mu d$ implies that $f_1(0) = \sqrt{2}$, i.e. the value of the form factor at q = 0 is completely fixed! Hence, the only unknown parameter in the decay amplitude is the CKM matrix element V_{ud}.

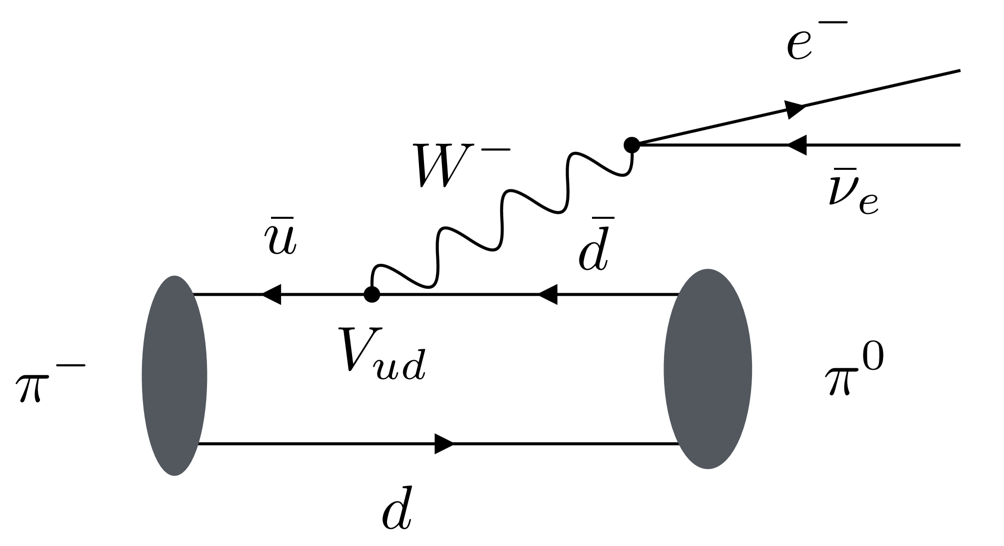
$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{ud}\, f_1(0)(p^\mu_{\pi^-} + p^\mu_{\pi^0})\, \bar{u}(p_l)\gamma_\mu(1+\gamma_5)u(p_\nu)$$

$$\Gamma = \frac{G_F |V_{ud}|^2 \Delta^5}{30\pi^3}\left(1 - \frac{5m_e^2}{\Delta^2} - \frac{3}{2}\frac{\Delta}{m_\pi}\right) \qquad \Delta = m_{\pi^-} - m_{\pi^0}$$

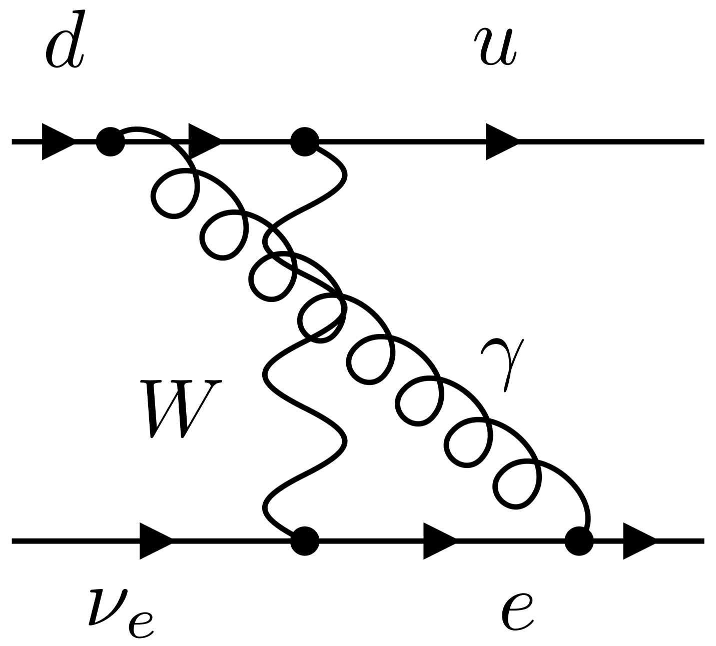
We can immediately use this formula to extract V_{ud} from the decay of a charged pion.
However, if we do so, we will miss an important effect, related to the radiative corrections.

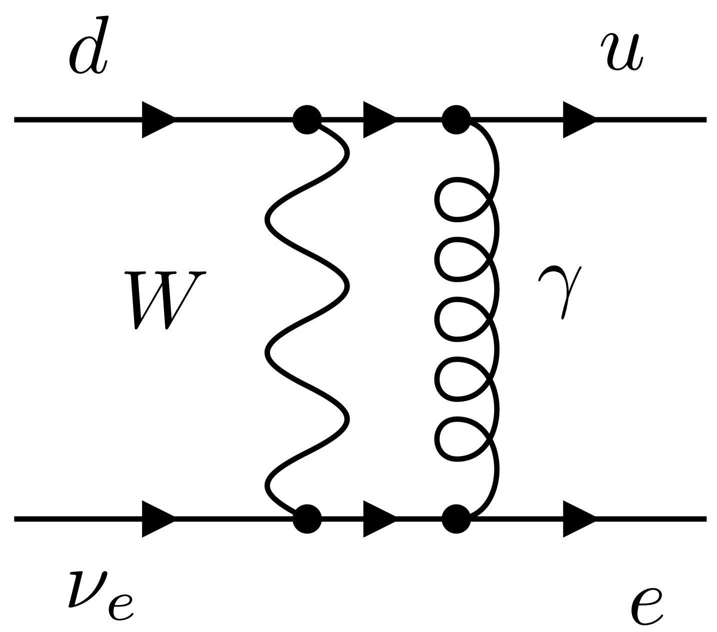
This effect is a logarithmically-enhanced short-distance renormalisation of the Fermi Lagrangian that is relevant for describing decay of the charged pion. Our goal is to find diagrams that exhibit logarithmic sensitivity to the mass of the W boson which is much larger than the mass of the pion.

$$\pi^- \to \pi^0 + e + \bar{\nu}_e$$



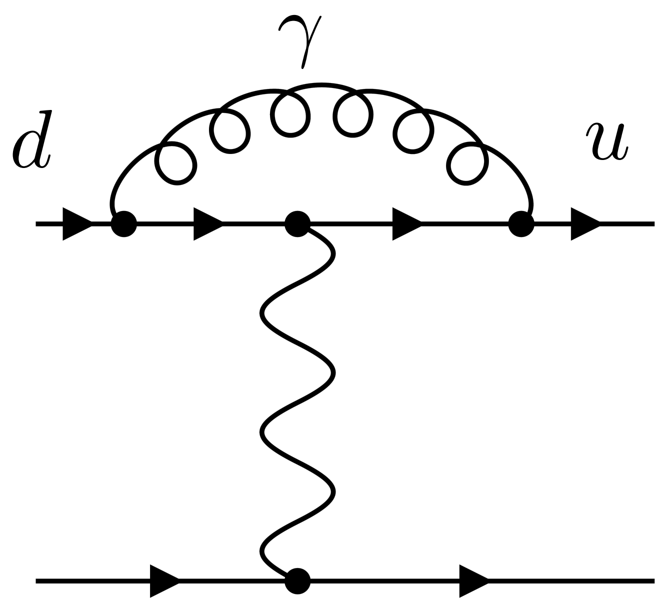
$$\mathcal{L} = \frac{G_F}{\sqrt{2}} J_q^\alpha(x) J_{\mathrm{lep},\alpha}(x) \qquad J_q^\alpha = \bar{u}\gamma^\alpha(1-\gamma_5)d \qquad J_{\mathrm{lept}}^\alpha = \bar{e}\gamma^\alpha(1-\gamma_5)\nu_e$$





To simplify the calculation, it is useful to employ the Landau gauge for the photon propagator since in this gauge the vertex corrections are ultraviolet finite.

$$D_{\alpha\beta}(k) = \frac{-i}{k^2}\left(-g_{\alpha\beta} + \frac{k_\alpha k_\beta}{k^2}\right)$$

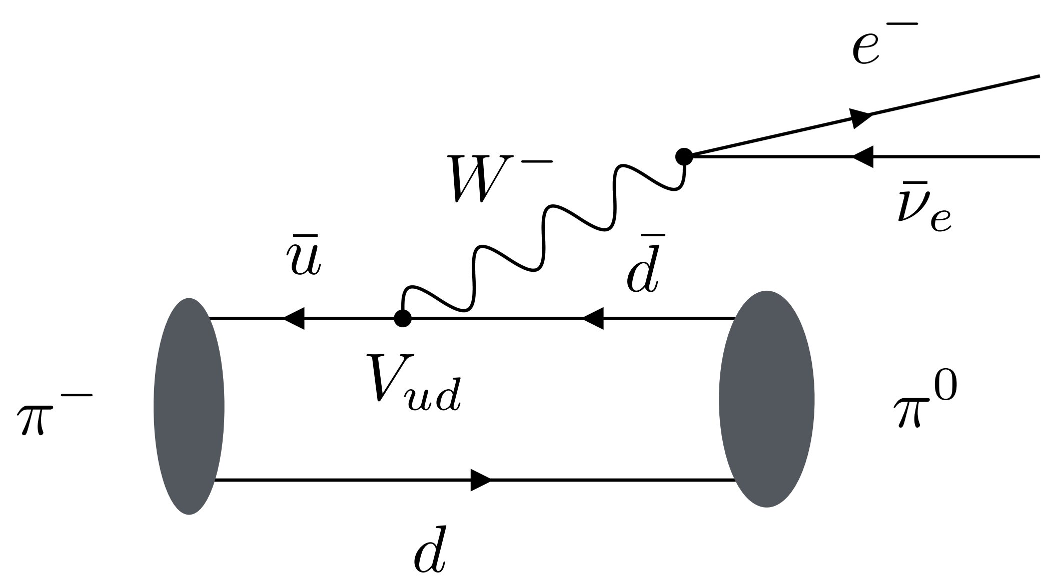


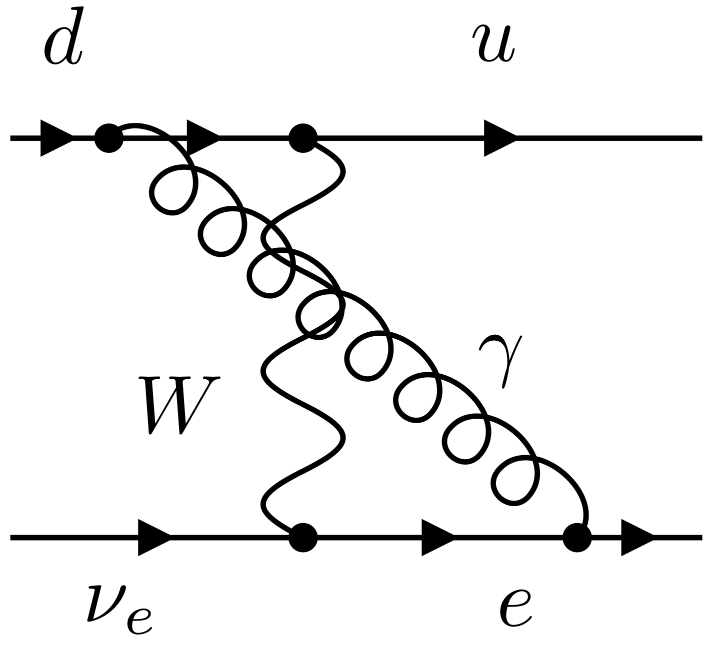
$$\frac{\mathcal{M}_V}{\mathcal{M}_0} = e^2 Q_u Q_d\,(-i)\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\gamma^\alpha \hat{k}\gamma^\mu(1-\gamma_5)\hat{k}\gamma^\beta}{(k^2)^3}\left(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)$$

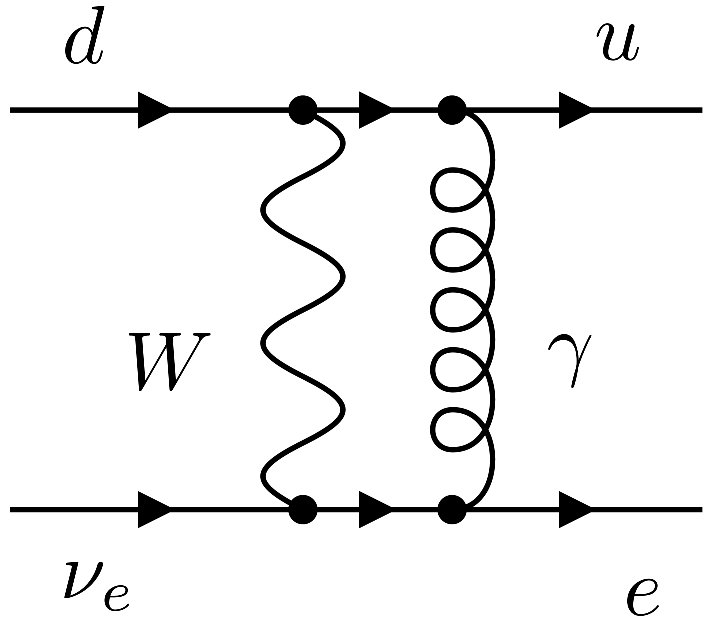
$$\gamma^\alpha \hat{k}\gamma^\mu(1-\gamma_5)\hat{k}\gamma^\beta g_{\alpha\beta} \to \frac{k^2}{4}\gamma^\alpha\gamma^\rho\gamma^\mu(1-\gamma_5)\gamma^\sigma\gamma^\beta\; g_{\rho\sigma}g_{\alpha\beta} = k^2\gamma^\mu(1-\gamma_5)$$

$$\gamma^\alpha \hat{k}\gamma^\mu(1-\gamma_5)\hat{k}\gamma^\beta \frac{k_\alpha k_\beta}{k^2} = k^2\gamma^\mu(1-\gamma_5)$$

$$\mathcal{M}_V \to 0$$

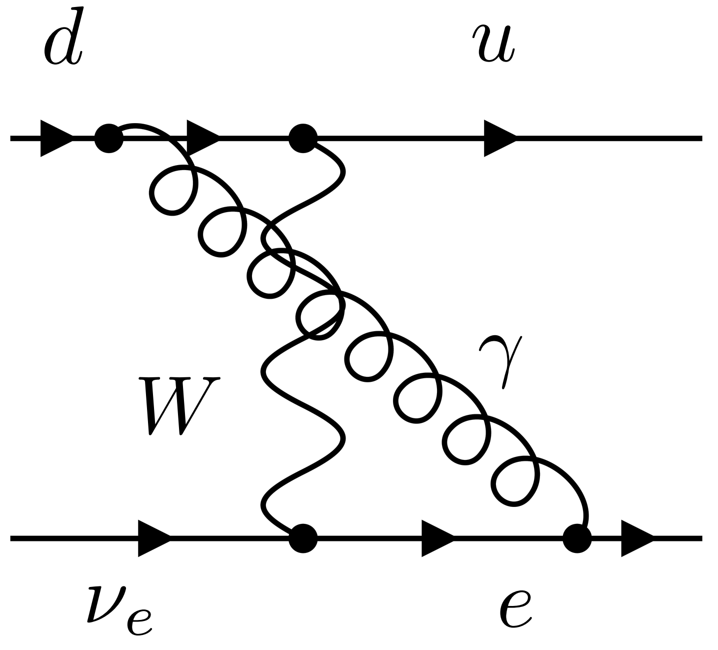






$$\mathcal{L} = S_{EW} \frac{G_F}{\sqrt{2}} J_q^{\alpha}(x) J_{\text{lep},\alpha}(x)$$

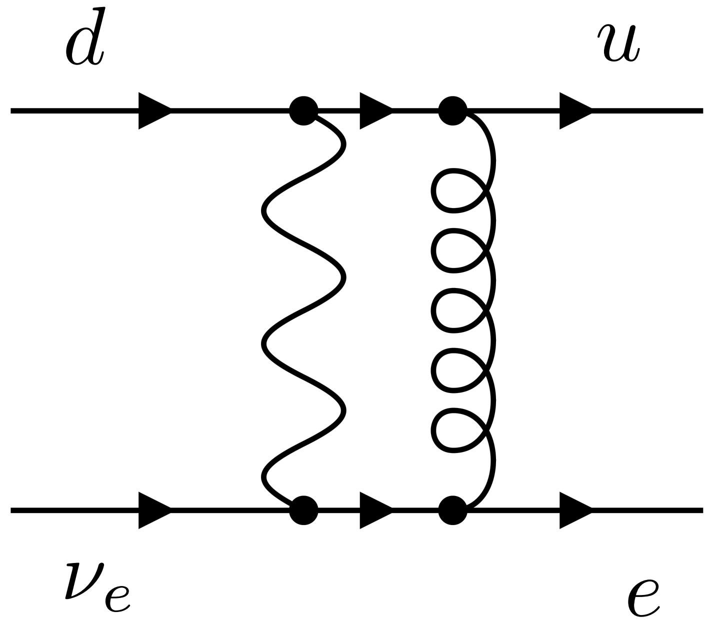
$$S_{EW} = 1 + \frac{\alpha}{\pi} \ln \frac{m_W}{0.3\ \text{MeV}} \approx 1.013$$



$$\frac{\mathcal{M}_B}{\mathcal{M}_0} = e^2 Q_e Q_d\,(-i) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\gamma^\mu(1-\gamma_5)\hat{k}\gamma^\alpha\right] \otimes \left[\gamma^\beta \hat{k}(1-\gamma_5)\right] \left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right) \frac{m_W^2}{(k^2)^3(m_W^2-k^2)}$$

$$\left[\gamma^\mu(1-\gamma_5)\hat{k}\gamma^\alpha\right] \otimes \left[\gamma^\beta \hat{k}\gamma^\mu(1-\gamma_5)\right] g_{\alpha\beta} \to \frac{k^2}{4} \left[\gamma^\mu(1-\gamma_5)\gamma^\rho\gamma^\alpha\right] \otimes \left[\gamma^\beta\gamma^\sigma\gamma^\mu(1-\gamma_5)\right] g_{\alpha\beta}\, g_{\rho\sigma}$$

Using $[\gamma^\mu\gamma^\rho\gamma^\alpha(1-\gamma_5)] \otimes [\gamma_\alpha\gamma_\rho\gamma_\mu(1-\gamma_5)] = 4\,[\gamma^\mu(1-\gamma_5)] \otimes [\gamma_\mu(1-\gamma_5)]$, we find $\mathcal{M}_B = 0$.



$$\frac{\mathcal{M}_B}{\mathcal{M}_0} = -e^2 Q_e Q_u\,(-i) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\gamma^\alpha\hat{k}\gamma^\mu(1-\gamma_5)\right] \otimes \left[\gamma^\beta \hat{k}\gamma^\mu(1-\gamma_5)\right] \left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right) \frac{m_W^2}{(k^2)^3(m_W^2-k^2)}$$

$$\left[\gamma^\alpha\hat{k}\gamma^\mu(1-\gamma_5)\right] \otimes \left[\gamma^\beta \hat{k}\gamma^\mu(1-\gamma_5)\right] g_{\alpha\beta} \to \frac{k^2}{4} \left[\gamma^\alpha\gamma^\sigma\gamma^\mu(1-\gamma_5)\right] \otimes \left[\gamma^\beta\gamma^\sigma\gamma^\mu(1-\gamma_5)\right] g_{\alpha\beta}\, g_{\rho\sigma}$$

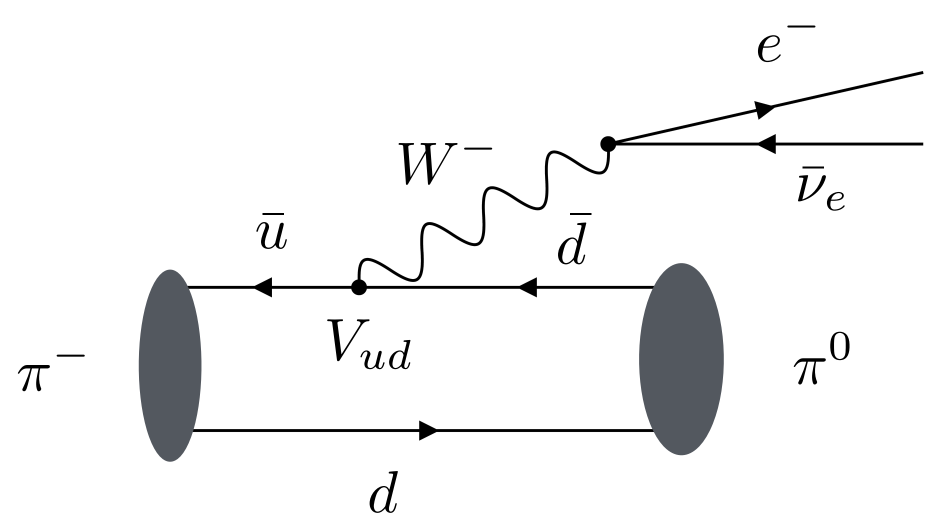
$$[\gamma^\alpha\gamma^\sigma\gamma^\mu(1-\gamma_5)] \otimes [\gamma^\alpha\gamma^\sigma\gamma^\mu(1-\gamma_5)] = 16\,[\gamma^\mu(1-\gamma_5)] \otimes [\gamma^\mu(1-\gamma_5)]$$

$$\frac{\mathcal{M}_B}{\mathcal{M}_0} = -3e^2 Q_e Q_u\ [\gamma^\mu(1-\gamma_5)] \otimes [\gamma^\mu(1-\gamma_5)]\,(-i) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{m_W^2}{(k^2)^2(m_W^2-k^2)}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{m_W^2}{(k^2)^2(m_W^2-k^2)} \approx \frac{i}{8\pi^2} \ln \frac{m_W}{\Lambda_{\text{QCD}}} \qquad \mathcal{M}_B = \mathcal{M}_0\, \frac{\alpha}{\pi} \ln \frac{m_W}{\Lambda_{\text{QCD}}}$$

If we put everything together, including estimates of long-distance QED corrections, we find

$$\pi^- \to \pi^0 + e + \bar{\nu}_e$$



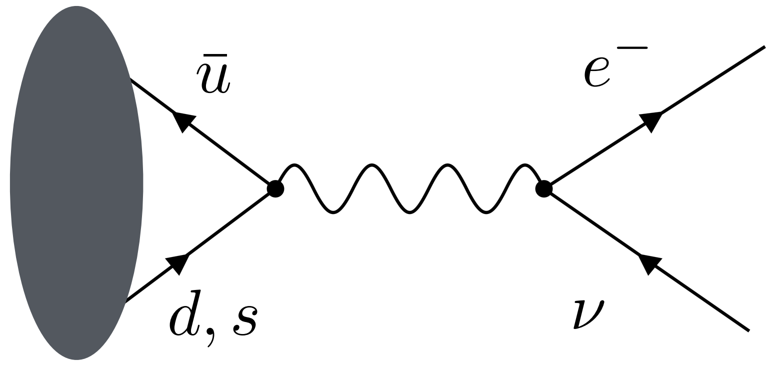
$$\mathcal{L} = \frac{G_F}{\sqrt{2}} J_q^\alpha(x) J_{\mathrm{lep},\alpha}(x) \qquad S_{EW} = 1 + \frac{\alpha}{\pi} \ln \frac{m_W}{0.3\ \mathrm{MeV}} \approx 1.013$$

$$\Gamma = \frac{G_F |V_{ud}|^2 \Delta^5}{30\pi^3} \left(1 - \frac{5m_e^2}{\Delta^2} - \frac{3}{2}\frac{\Delta}{m_\pi}\right) S_{EW}^2 (1 + \delta_{RC})$$

$$\Gamma = \Gamma_{\pi^-,\mathrm{tot}}\ \mathrm{Br} \qquad \Gamma_{\pi^-,\mathrm{tot}} = (2.6033 \pm 0.0005) \times 10^{-8}\ \mathrm{sec}^{-1}, \quad \mathrm{Br} = (1.036 \pm 0.006) \times 10^{-8}.$$

$$|V_{ud}| = 0.9739(29).$$

Getting V_{us} is somewhat easier but we have to rely on lattice computations.



$$\langle 0|\bar{u}\gamma^\alpha\gamma^5 d|\pi^-(p)\rangle = f_\pi p^\mu$$
$$\langle 0|\bar{u}\gamma^\alpha\gamma^5 s|K^-(p)\rangle = f_K p^\mu$$

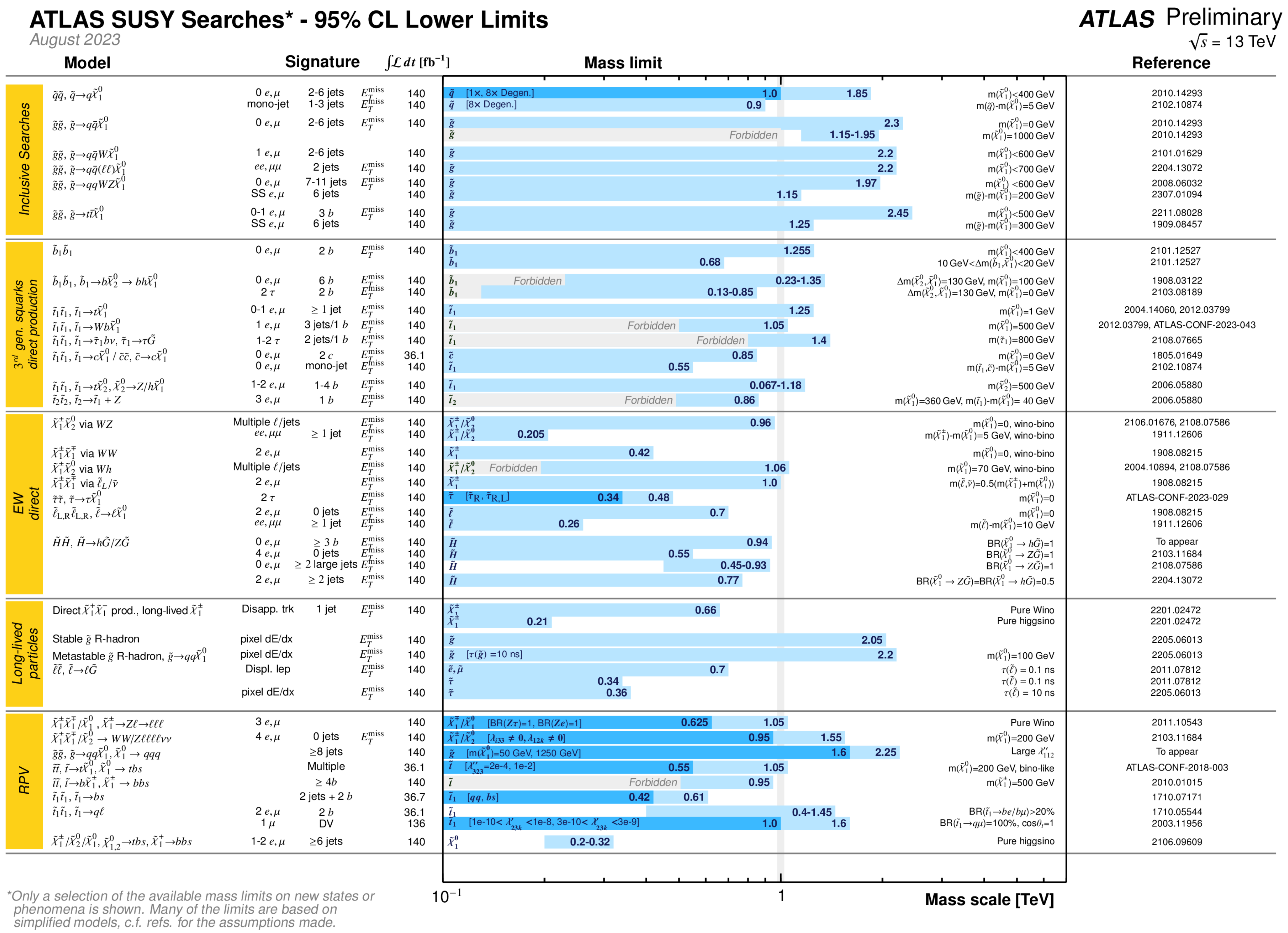
$$\frac{\Gamma(\pi \to \mu\nu)}{\Gamma(K \to \mu\nu)} = \frac{|V_{ud}|^2}{|V_{us}|^2}\frac{f_\pi^2}{f_k^2} F(m_\pi, m_K, m_\mu) \qquad \frac{f_K}{f_\pi} = 1.1932(19)$$

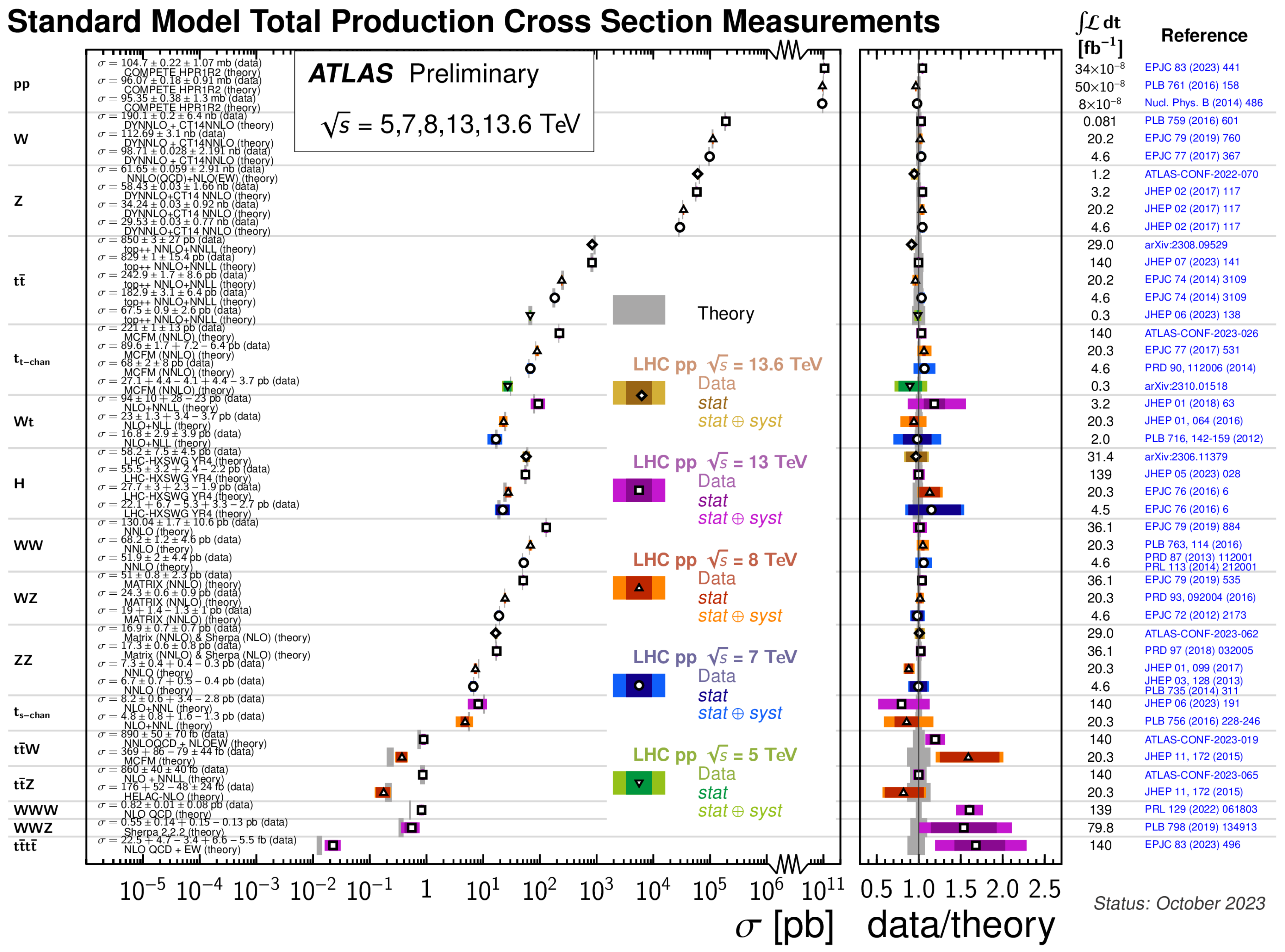
$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23131(45)$$

Finally, we find: $|V_{ud}|^2 + |V_{us}|^2 = 0.9992 \pm 0.006$

Precision physics at the LHC

Physics at the LHC, so far, can be summarized as follows: discovery of the Higgs boson, no new particles or interactions, strong exclusion limits and many measurements of the SM cross sections which by and large show excellent agreement with high-precision theoretical predictions.





In general, the theoretical foundation for describing processes with large momentum transfer in hadron collisions is the collinear factorisation formula. Since non-perturbative corrections are typically small, we need partonic cross sections with high enough precision and parton distribution functions.

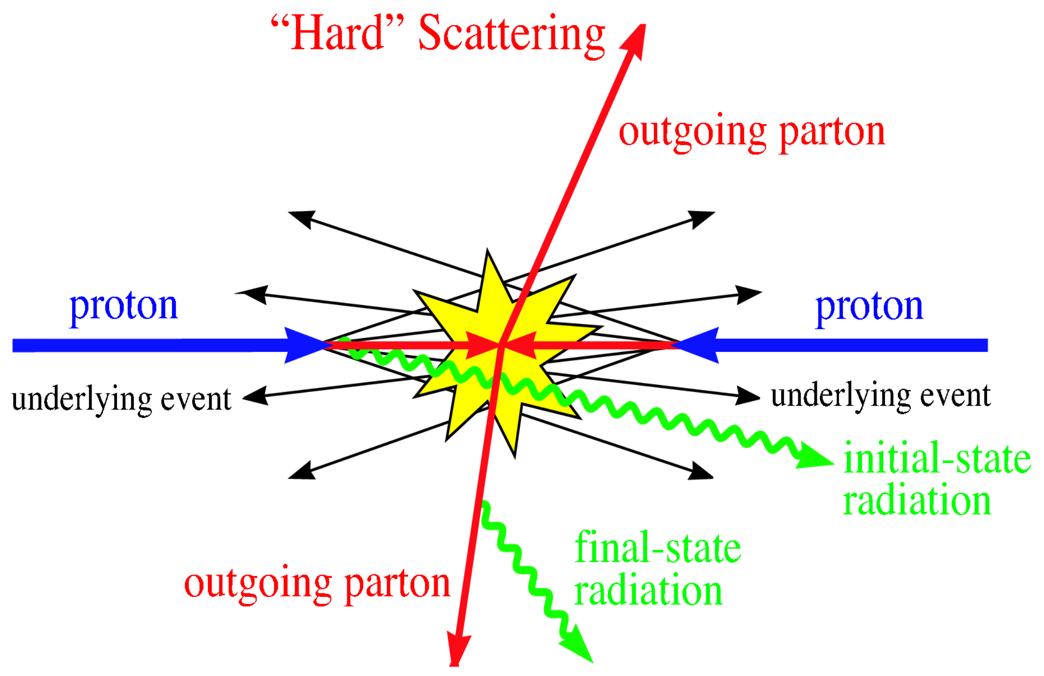
$$d\sigma_{\text{hard}} = \sum_{ij\in\{q,g\}} \int dx_1\, dx_2 f_i(x_1) f_j(x_2)\, d\sigma_{ij}(x_1, x_2), \{p_{\text{fin}}\})\, O_J(\{p_{\text{fin}}\}) \left(1 + \mathcal{O}(\Lambda^n_{\text{QCD}}/Q^n)\right)$$

$$d\sigma_{ij} = d\sigma^{(0)} \left(1 + \frac{\alpha_s}{\pi} c_1 + \left(\frac{\alpha_s}{\pi}\right)^2 c_2 + + \left(\frac{\alpha_s}{\pi}\right)^3 c_3 + \cdots\right)$$

One would expect moderate higher-order effects: $c_1 \sim C_A,\; c_2 \sim C_A^2 \quad \alpha_s/\pi \sim 0.04$.
However, in practice large corrections have been observed for many processes; one reason is that if high powers of the strong coupling constant are involved, the scale dependence of the strong coupling constant plays an important role: if you start with a wrong scale, corrections try to move the result in the right direction and this leads to large radiative corrections.

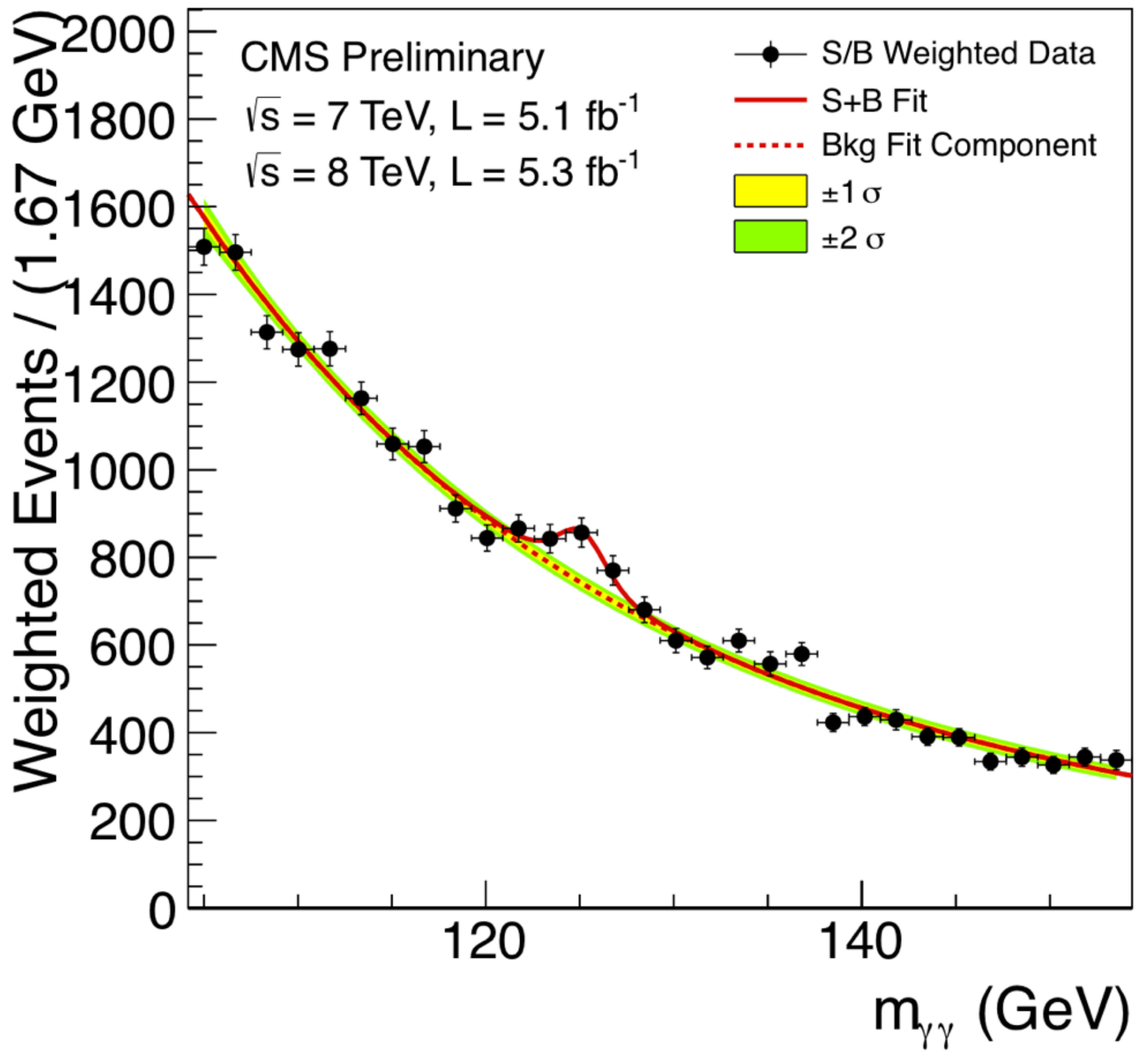
$$\sigma \sim \alpha_s(\mu)^n \to \alpha_s(\mu_1)^n \left(1 - \frac{n\beta_0}{2}\frac{\alpha_s(\mu_1)}{\pi} \ln\frac{\mu}{\mu_1}\right) \qquad \beta_0 = 11/3C_A - 2/3n_f$$

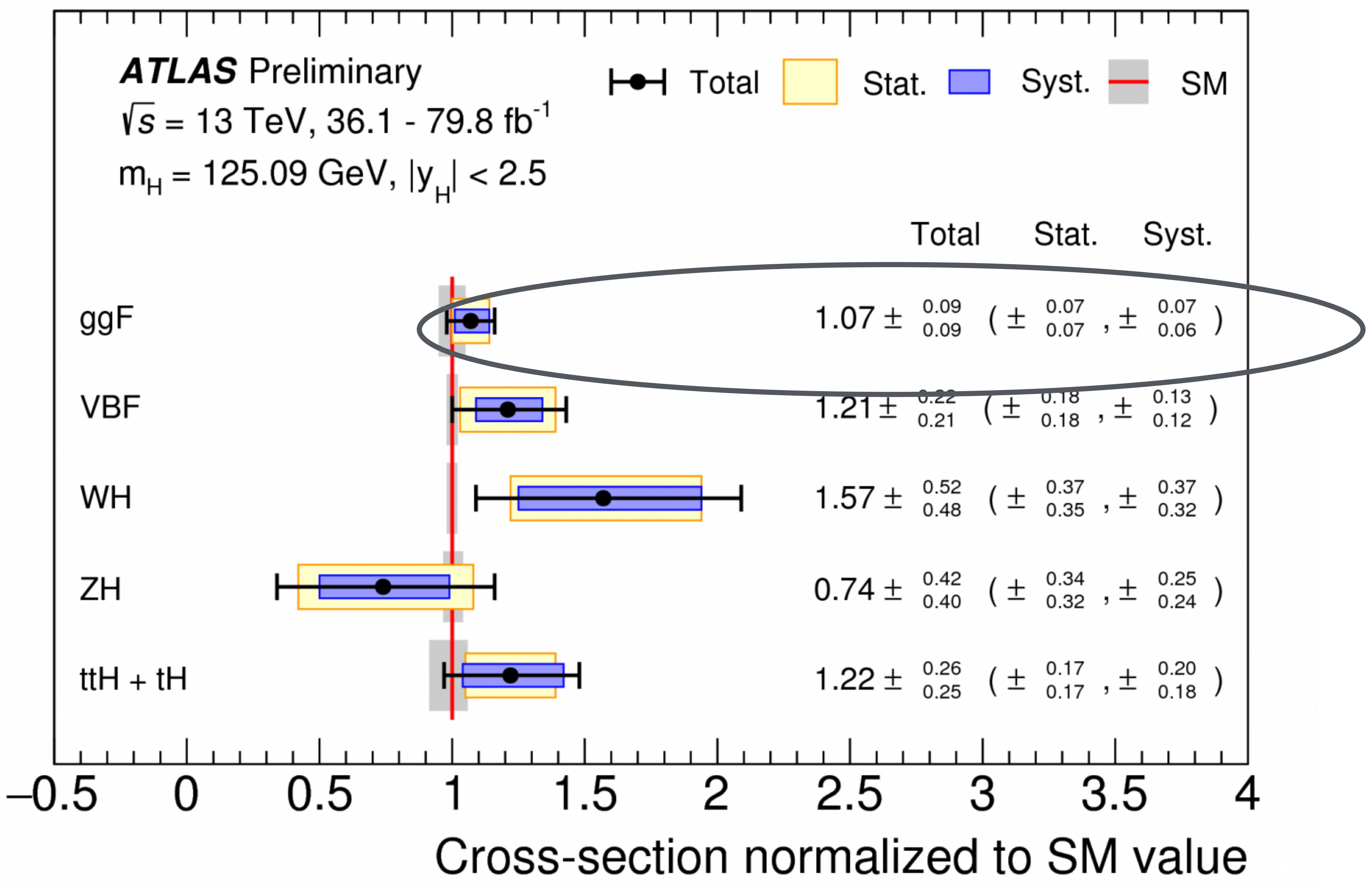
$$\frac{\Delta\sigma_n}{\sigma_n} = n\frac{11}{6}C_A\frac{\alpha_s}{\pi} \approx n\, 2C_A\frac{\alpha_s}{\pi} \approx 0.24\, n$$



Higgs production in gluon fusion

Consider the flagship LHC process — Higgs boson production in the gluon fusion — where the cross section is claimed to be measured to about ten percent precision. This is much worse than a percent precision that we have been entertaining at the electroweak sector.

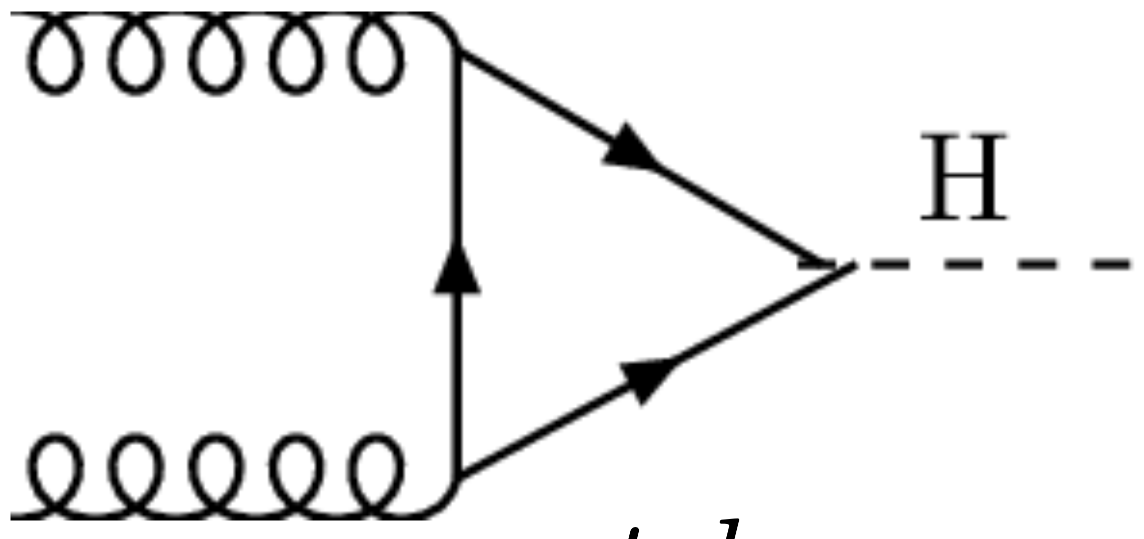




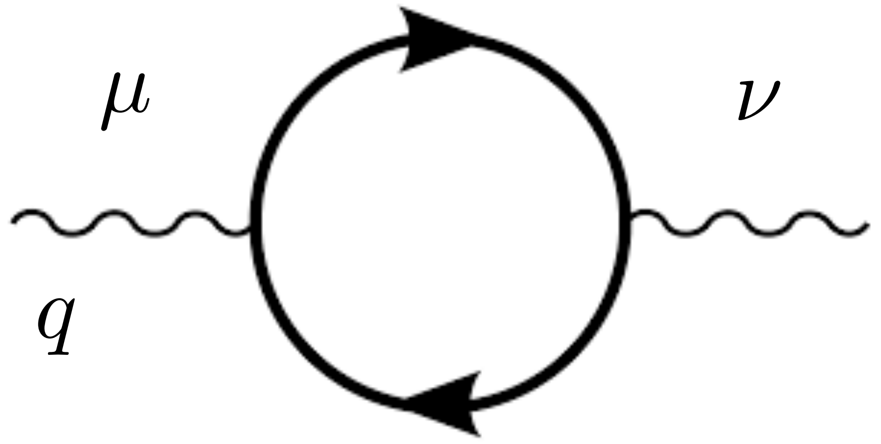
We will discuss how accurately the Higgs boson production cross section in gluon fusion can be predicted theoretically and what it takes to reach the state-of-the-art precision.

At the LHC, Higgs bosons are mainly produced in the gluon fusion. The top quark loop gives the largest contribution

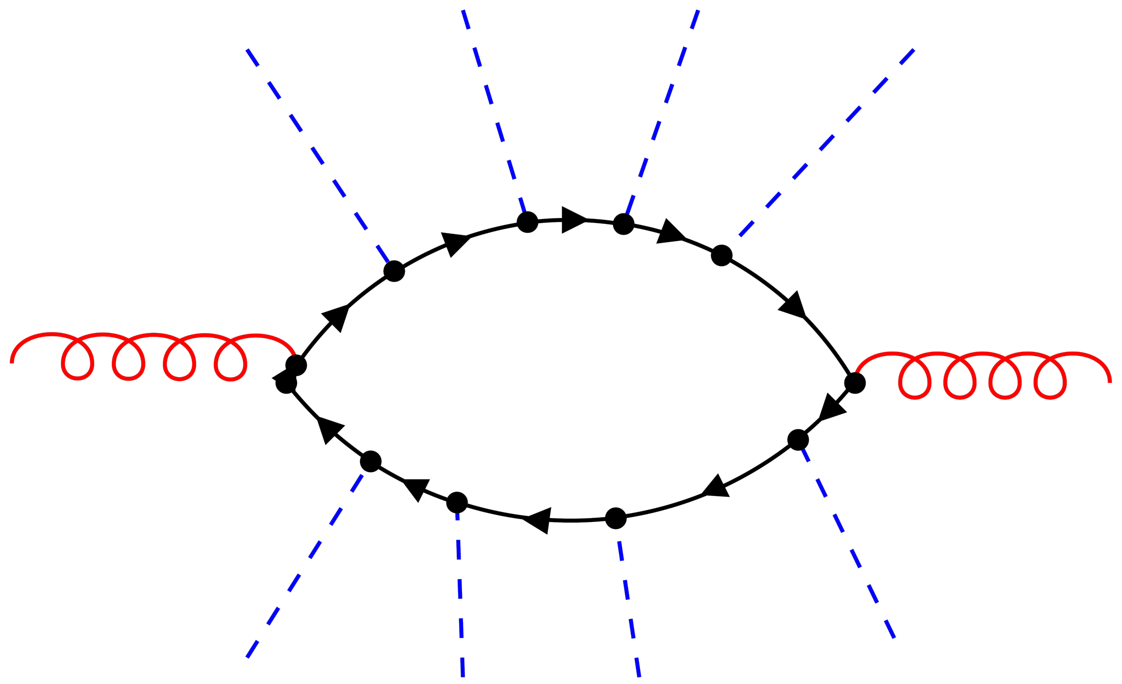
$$\mathcal{L} = m_t \left(1 + \frac{h(x)}{v}\right) \bar{t}t$$



In the limit $m_t \gg m_H$ the Higgs field can be considered as constant x-independent field that shifts the quark mass.



$$\Rightarrow \left[\, m_t \to \tilde{m}_t = m_t \left(1 + \frac{h}{v}\right) \right] \Rightarrow$$

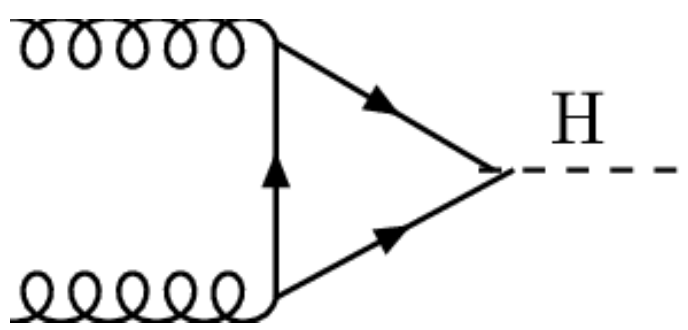


$$\frac{\alpha_s}{4\pi} \delta^{ab} \mathrm{Tr}[T^a T^b](-q^2 g^{\alpha\beta} + q^\alpha q^\beta) \left(\cdots - \frac{8}{3} \ln\left(1 + \frac{h}{v}\right)\right) = \frac{\alpha_s}{3\pi} \delta^{ab} (-q^2 g^{\alpha\beta} + q^\alpha q^\beta) \ln\left(1 + \frac{h}{v}\right)$$

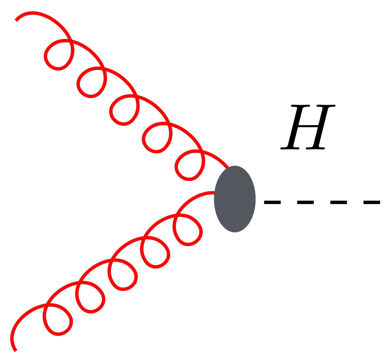
Hence, the effective Lagrangian that describes interaction of gluons with arbitrary number of Higgs bosons reads:

$$\mathcal{L} = \frac{\alpha_s}{12\pi} G^a_{\mu\nu} G^{a,\mu\nu} \ln\left(1 + \frac{h(x)}{v}\right)$$

The limit of the large top quark mass allows us to "remove one loop" and calculate corrections in a theory with a point-like Higgs-gluon-gluon vertex. This gives us O(3%) precision; once this is accomplished, all the "smallish" effects have to be evaluated anew.

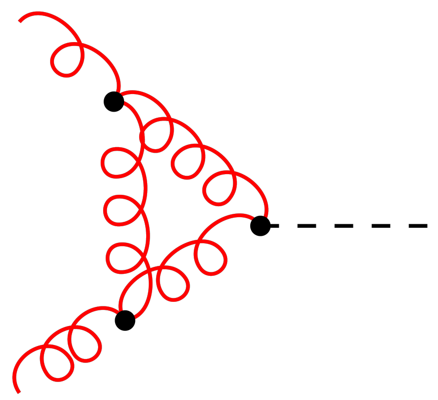


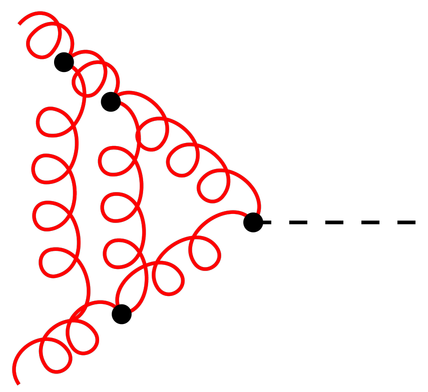
$\Rightarrow$

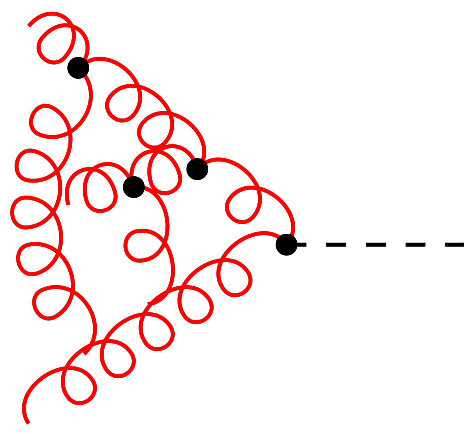


$$\mathcal{L} = \frac{\alpha_s}{12\pi} G^a_{\mu\nu} G^{a,\mu\nu} \ln\left(1 + \frac{h(x)}{v}\right) \to \frac{\alpha_s}{12\pi v} G^a_{\mu\nu} G^{a,\mu\nu} h$$

$$\sigma = 48.58\,\mathrm{pb}^{+2.22\,\mathrm{pb}\,(+4.56\%)}_{-3.27\,\mathrm{pb}\,(-6.72\%)}\ (\mathrm{theory}) \pm 1.56\,\mathrm{pb}\,(3.20\%)\ (\mathrm{PDF}+\alpha_s)\,.$$





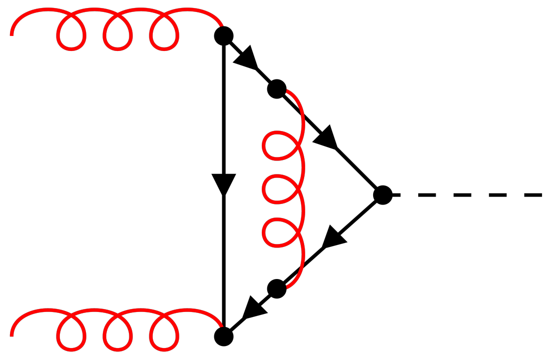


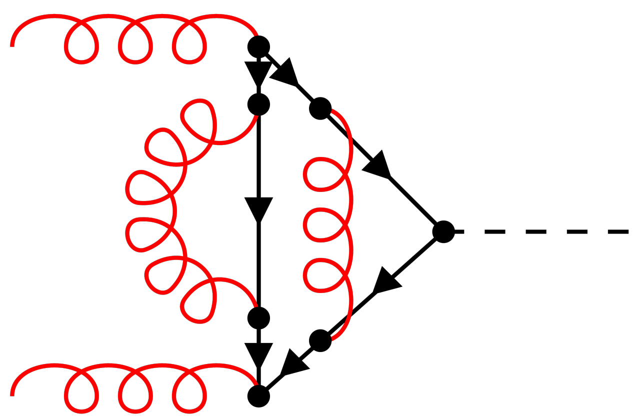
48.58 pb =	16.00 pb	(+32.9%)	(LO, rEFT)
	+ 20.84 pb	(+42.9%)	(NLO, rEFT)
	− 2.05 pb	(−4.2%)	((t, b, c), exact NLO)
	+ 9.56 pb	(+19.7%)	(NNLO, rEFT)
	+ 0.34 pb	(+0.7%)	(NNLO, $1/m_t$)
	+ 2.40 pb	(+4.9%)	(EW, QCD-EW)
	+ 1.49 pb	(+3.1%)	(N^{3}LO, rEFT)

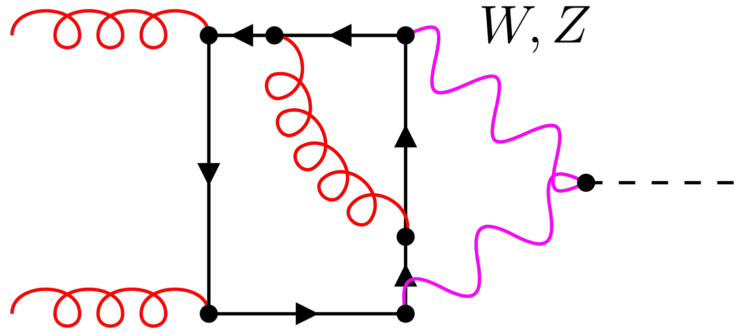
Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger

δ(scale)	δ(trunc)	δ(PDF-TH)	δ(EW)	$\delta(t,b,c)$	$\delta(1/m_t)$
+0.10 pb −1.15 pb	±0.18 pb	±0.56 pb	±0.49 pb	±0.40 pb	±0.49 pb
+0.21% −2.37%	±0.37%	±1.16%	±1%	±0.83%	±1%

Mistlberger, Bonetti, Tancredi, K.M., Becchetti, Bonciani, Del Duca, Hirschi, Moriello, Czakon, Eschment, Schellenberger , Niggetiedt, Poncelet

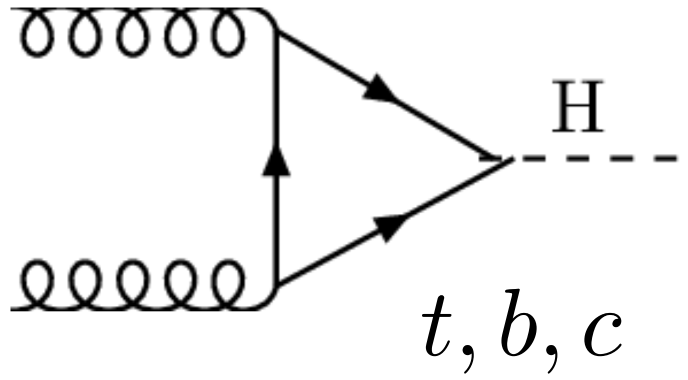






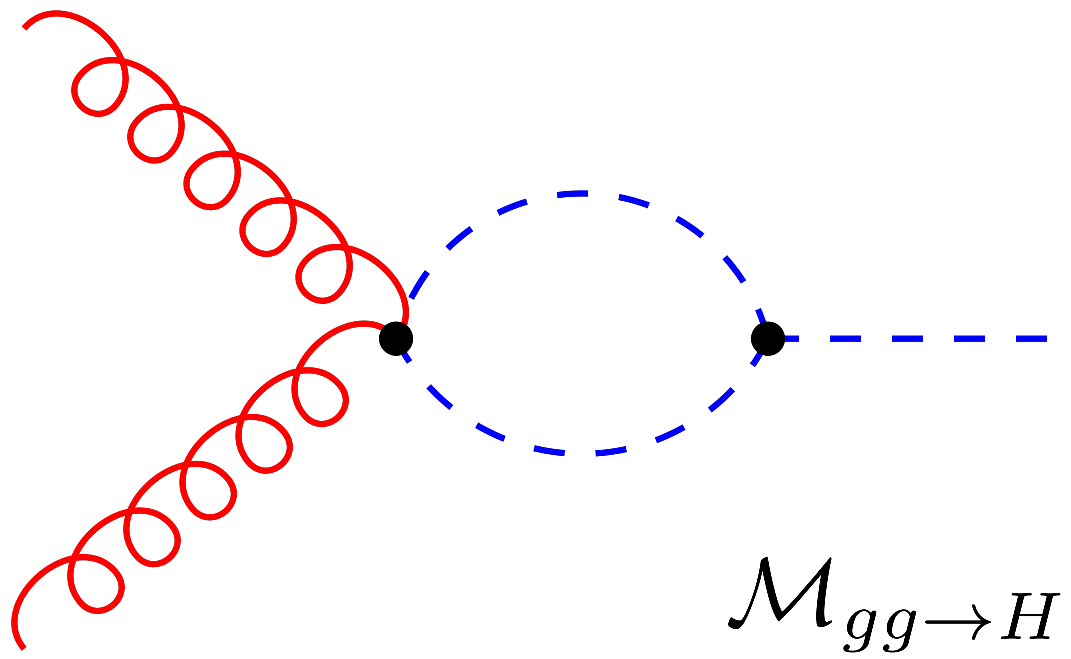
Let us discuss what we can say from the knowledge of the high-precision Higgs production in gluon fusion and assume, for definiteness, that the LHC will, eventually, be able to measure Higgs production in gluon fusion to a 3 percent precision.

The charm Yukawa coupling is poorly known. Its contribution to Higgs production is about two percent. Hence, one can constrain the Yukawa couplings if they are O(2) times larger than the SM.

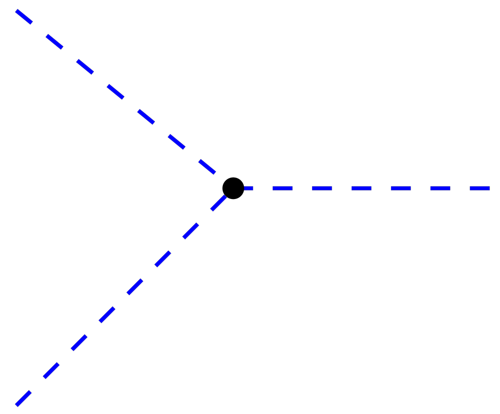


$$\frac{|\mathcal{A}_t + \mathcal{A}_c|^2}{|\mathcal{A}_t|^2} \approx 1 - \frac{3m_c^2}{m_H^2} \ln^2 \frac{m_c^2}{m_H^2} \approx 1 - 2 \times 10^{-2}$$

The Higgs self-coupling changes the Higgs production rate by about a percent. The effect is linear, so a three-percent measurement/theory prediction will constrain h^3 couplings that are O(3) times larger than the SM.



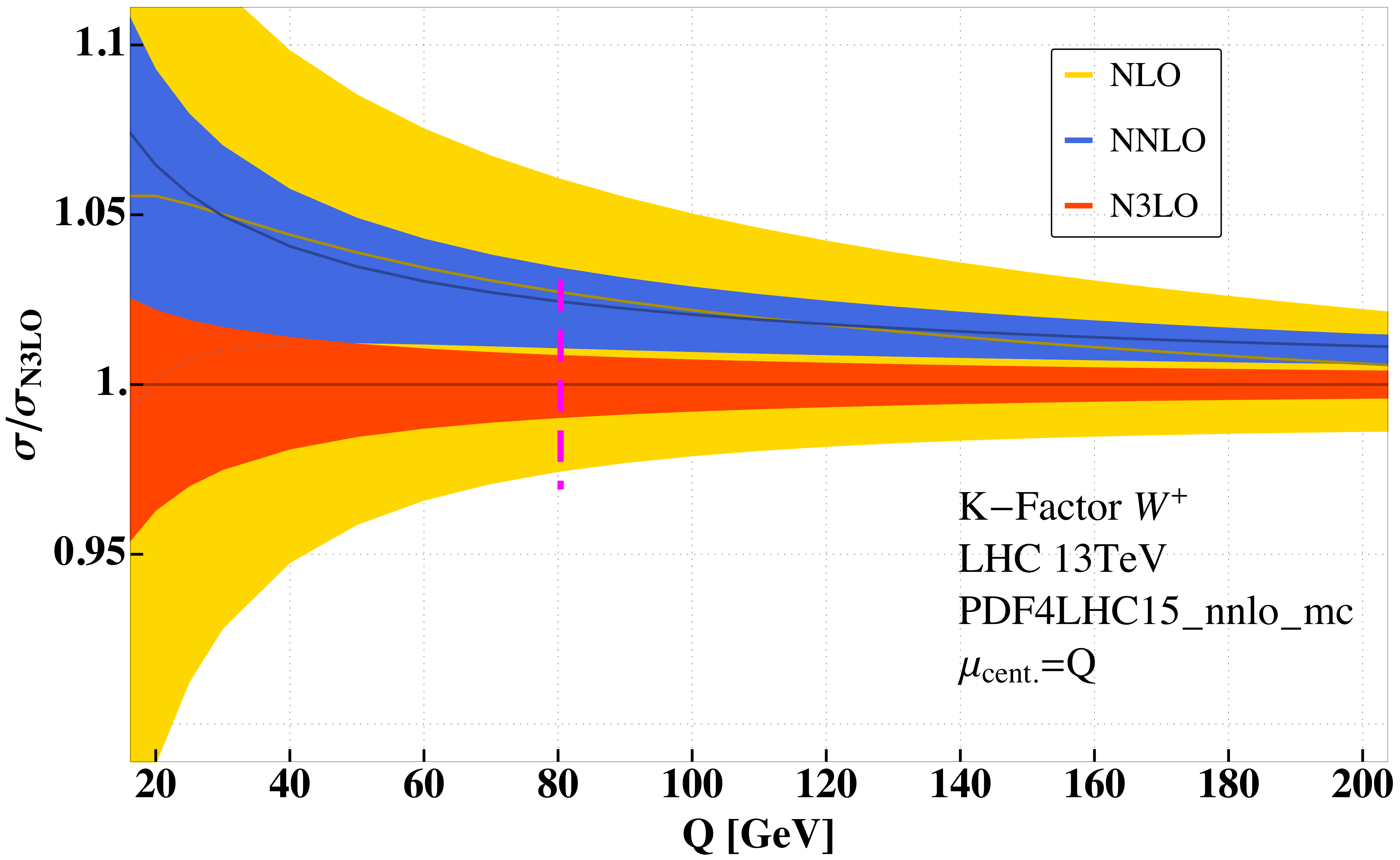
$$\mathcal{L} = \frac{\alpha_s}{12\pi} G^a_{\mu\nu} G^{a,\mu\nu} \ln\left(1 + \frac{h(x)}{v}\right)$$

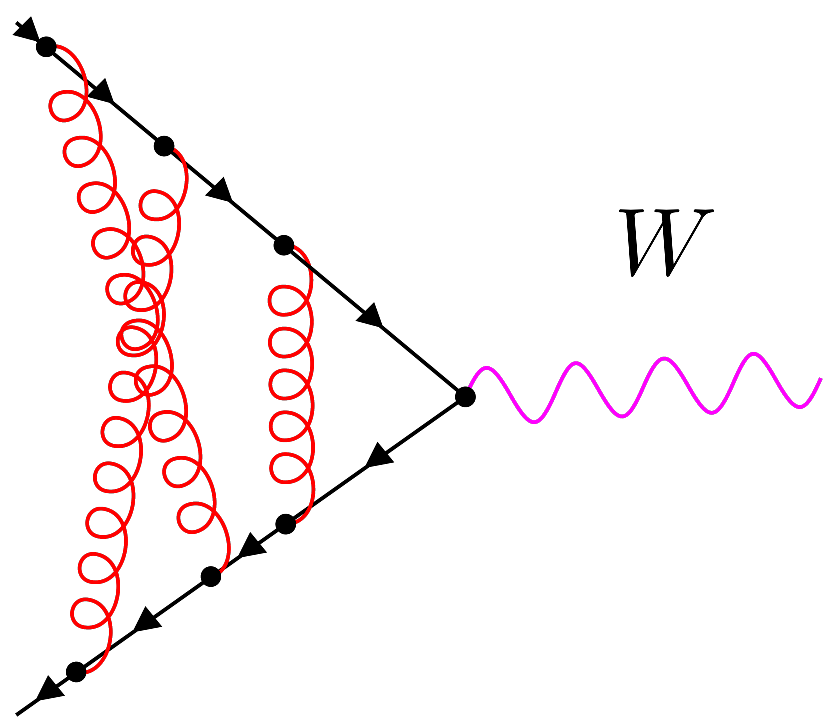


$$-3i\frac{m_H^2}{v}\,\lambda$$

$$\mathcal{M}_{gg\to H} = \mathcal{M}^{(0)}_{gg\to H}\left(1 + c_1 \frac{3m_h^2}{(4\pi)^2 v^2}\right) \approx \mathcal{M}^{(0)}_{gg\to H}\left(1 + 5c_1 \times 10^{-3}\right)$$

The large effects in Higgs boson production in gluon fusion that we discussed are not atypical, although they are somewhat extreme. Furthermore, even in the simplest cases such as the total cross section for single vector boson production at the LHC, the perturbative expansion looks peculiar. At the same time, these results do not include N3LO QCD parton distribution functions, so it is not quite clear if there won't be any changes once higher-order PDFs become available.



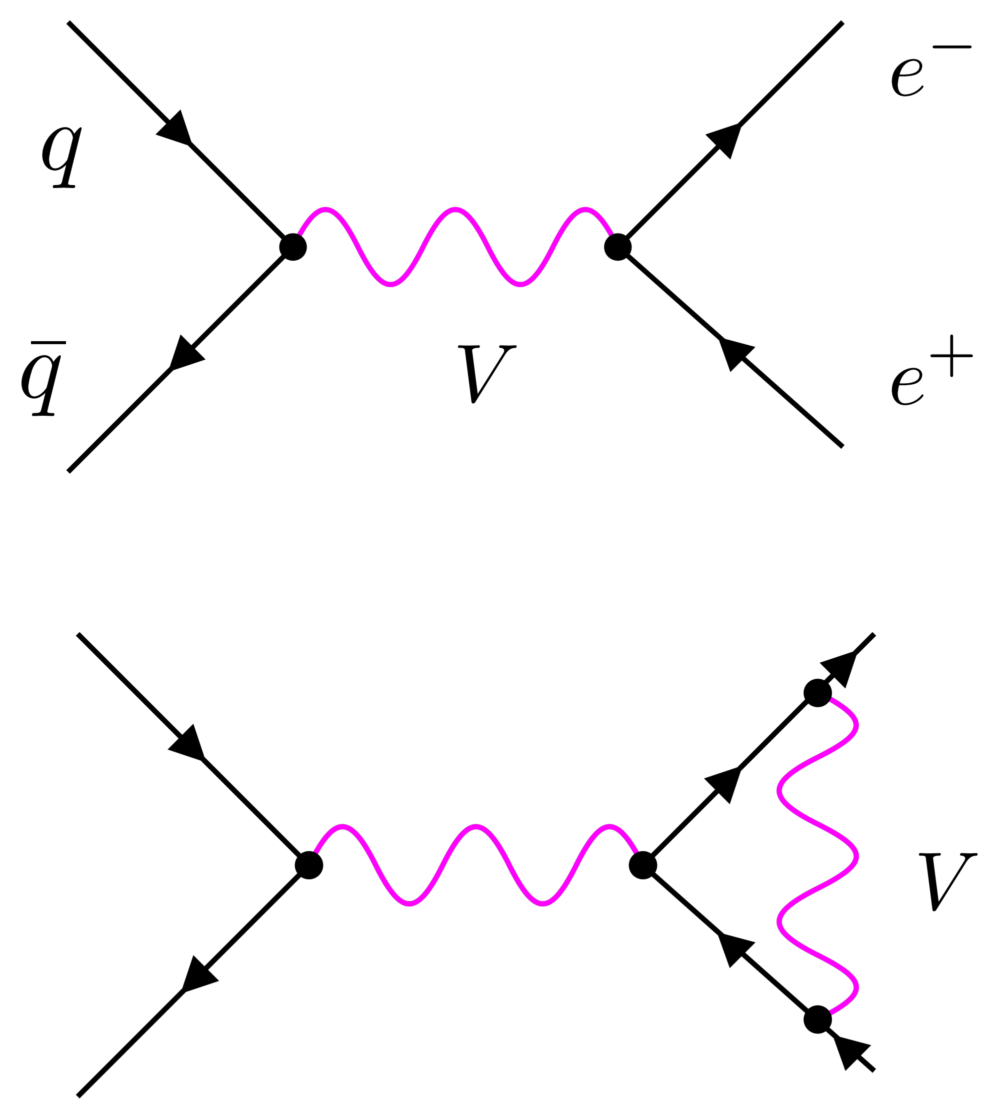


Duhr, Dulat, Mistlberger

Sudakov logarithms

At the LHC, electroweak corrections are usually less important than the QCD ones because of the relation between the couplings $\alpha_s \sim 0.1 \gg \alpha \sim 0.01$. But this is not always the case.

$$\mathcal{M} = H^\rho \frac{-i}{s - m_V^2} L_\rho^{(0)} \qquad L_\rho^{(0)} = i g_{Z,e} \bar{u}_1 \gamma^\rho v_2 \qquad s \gg m_V^2$$



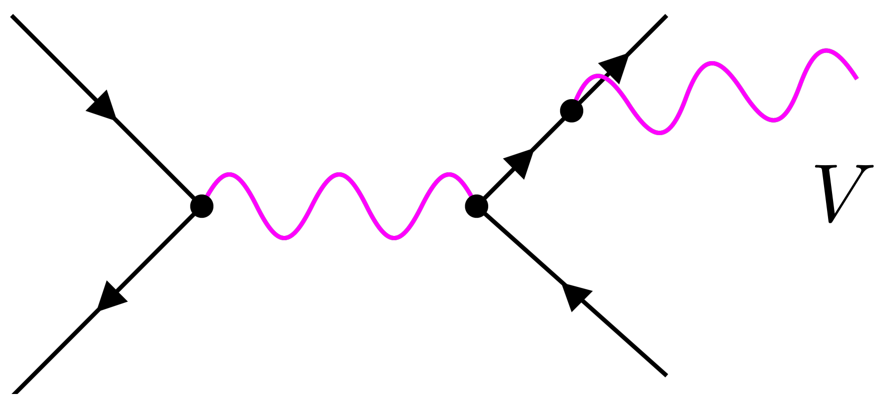
$$L^{(1),\rho} = g_{Z,e} g_{Z,e}^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}_1 \gamma^\mu (\hat{p}_1 + \hat{k}) \gamma^\rho (\hat{k} - \hat{p}_2) \gamma_\mu v(p_2)}{(k+p_1)^2 (k-p_2)^2 (k^2 - m_V^2)}$$

$$L^{(1),\rho} = g_{Z,e} g_{Z,e}^2 \Gamma(3) \int [dx]_3 \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}_1 \gamma^\mu (\hat{p}_1 + \hat{k}) \gamma^\rho (\hat{k} - \hat{p}_2) \gamma_\mu v(p_2)}{((k+P)^2 - \Delta)^3}$$

$$\Delta = m_v^2 x_3 + P^2 - i0 = m_v^2 x_3 - s x_1 x_2 - i0 \qquad L^{(1),\rho} = -L^{(0),\rho} \frac{\alpha_g^2}{2\pi} \frac{1}{2} \ln^2 \frac{s}{m_V^2}$$

$$\mathcal{M} = \mathcal{M}_0 \left(1 - \frac{\alpha_g}{4\pi} \ln^2 \frac{s}{m_V^2}\right) \quad \Rightarrow \quad d\sigma = d\sigma_0 \left(1 - \frac{\alpha_g}{2\pi} \ln^2 \frac{s}{m_V^2}\right)$$

In QED or QCD, enhanced virtual correction gets cancelled with the real emission contributions.



$$\mathcal{M}_R = H^\rho \frac{-i}{s - m_V^2} L_{\rho,R} \qquad L_R^\rho = -i g_B g_{Z,e} \bar{u}_1 \left[\frac{\hat{\epsilon}(\hat{p}_1 + \hat{k})\gamma^\rho}{2p_1 k + m_V^2} + \frac{\gamma^\rho(-\hat{p}_2 + k)}{-2p_2 k + m_V^2} \right] v_2$$

$$k \sim m_V \ll \sqrt{s} \qquad \Longrightarrow \qquad L_R^\rho = -g_{Z,e} L^{\rho,(0)} \left(\frac{p_1 \epsilon_Z}{p_1 k} - \frac{p_2 \epsilon_Z}{p_2 k} \right)$$

$$d\sigma_R = d\sigma_0 \, g_{Z,e}^2 \int [d^3 k] \frac{2 p_1 p_2}{(p_1 k)(p_2 k)} \qquad \Longrightarrow$$

$$d\sigma_R = d\sigma_0 \frac{\alpha_g}{2\pi} \ln^2 \frac{s}{m_V^2}.$$

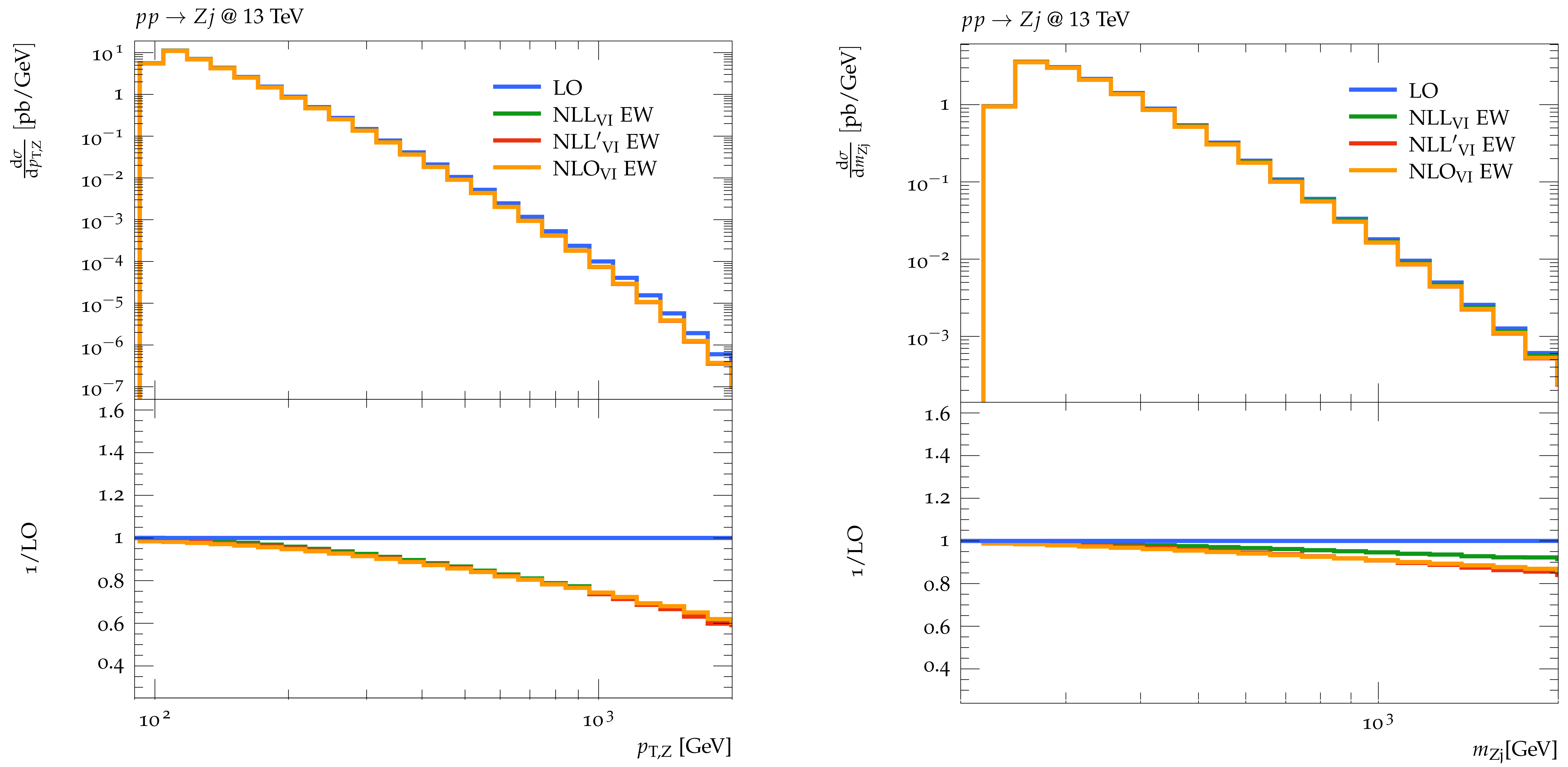
$$d\sigma = d\sigma_0 \left(1 - \frac{\alpha_g}{2\pi} \ln^2 \frac{s}{m_V^2} \right)$$

Comparison with the virtual corrections, makes the cancellation obvious.

However, this cancellation does not quite work for electroweak corrections since process pp->X and pp -> X+ V (V=Z,W) are, typically, treated as different processes in experiment. This implies that there are double logarithmic electroweak corrections that are always negative and they grow with the collision's energy. To estimate the size of these corrections, we write the coupling for the Z-boson and take s = 1 TeV.

$$\alpha_g = \frac{\alpha}{s_W^2 c_W^2} (T_3 - Q \sin^2 \theta_W)^2 \qquad -\frac{\alpha_g}{2\pi} \ln^2 \frac{s}{m_V^2} \approx -0.16 \, (T_3 - Q \sin^2 \theta_W)^2$$

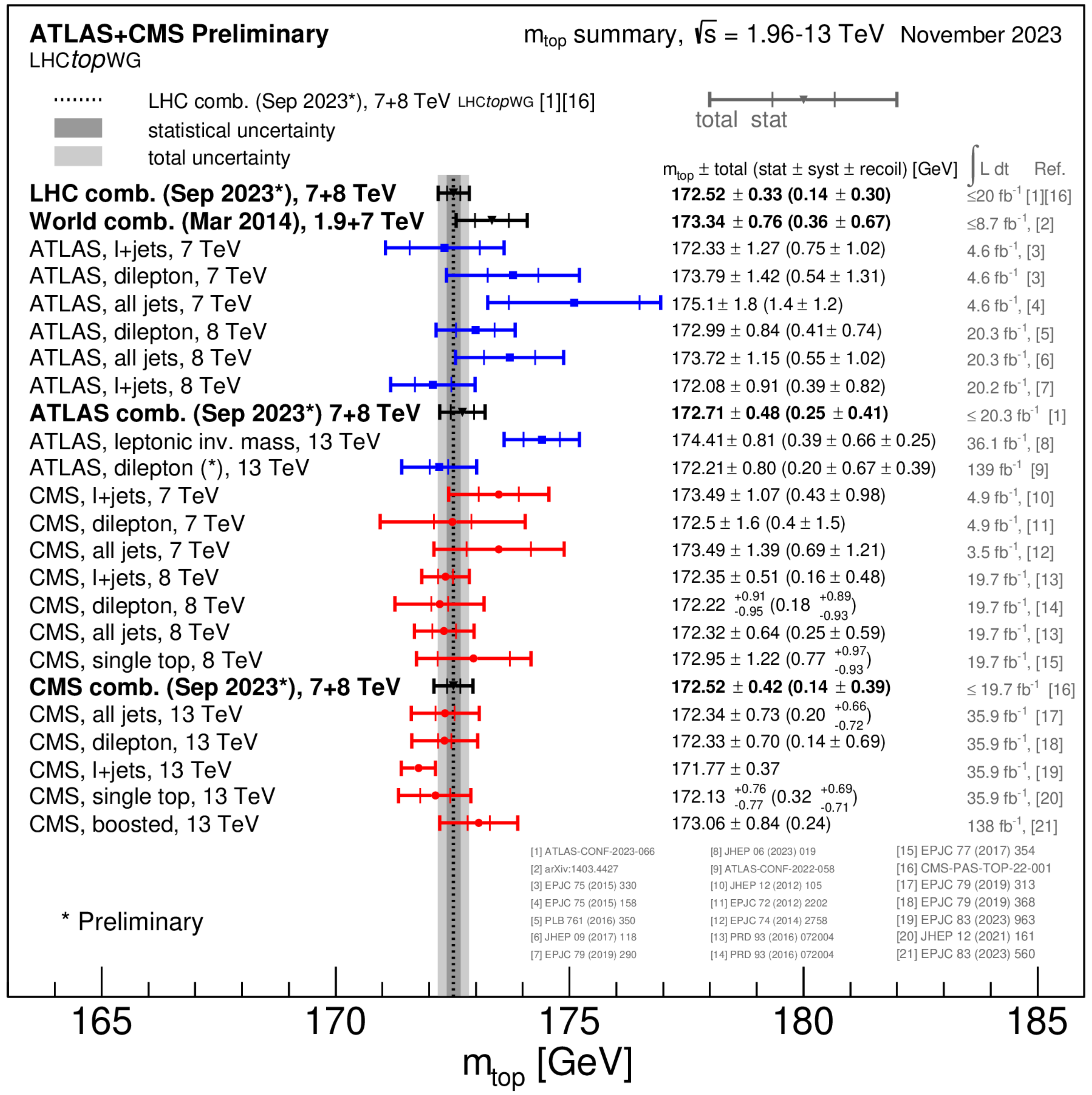
Large effects can indeed be seen in realistic (complete) computations. Below the electroweak corrections to Z+j production at the LHC are shown.



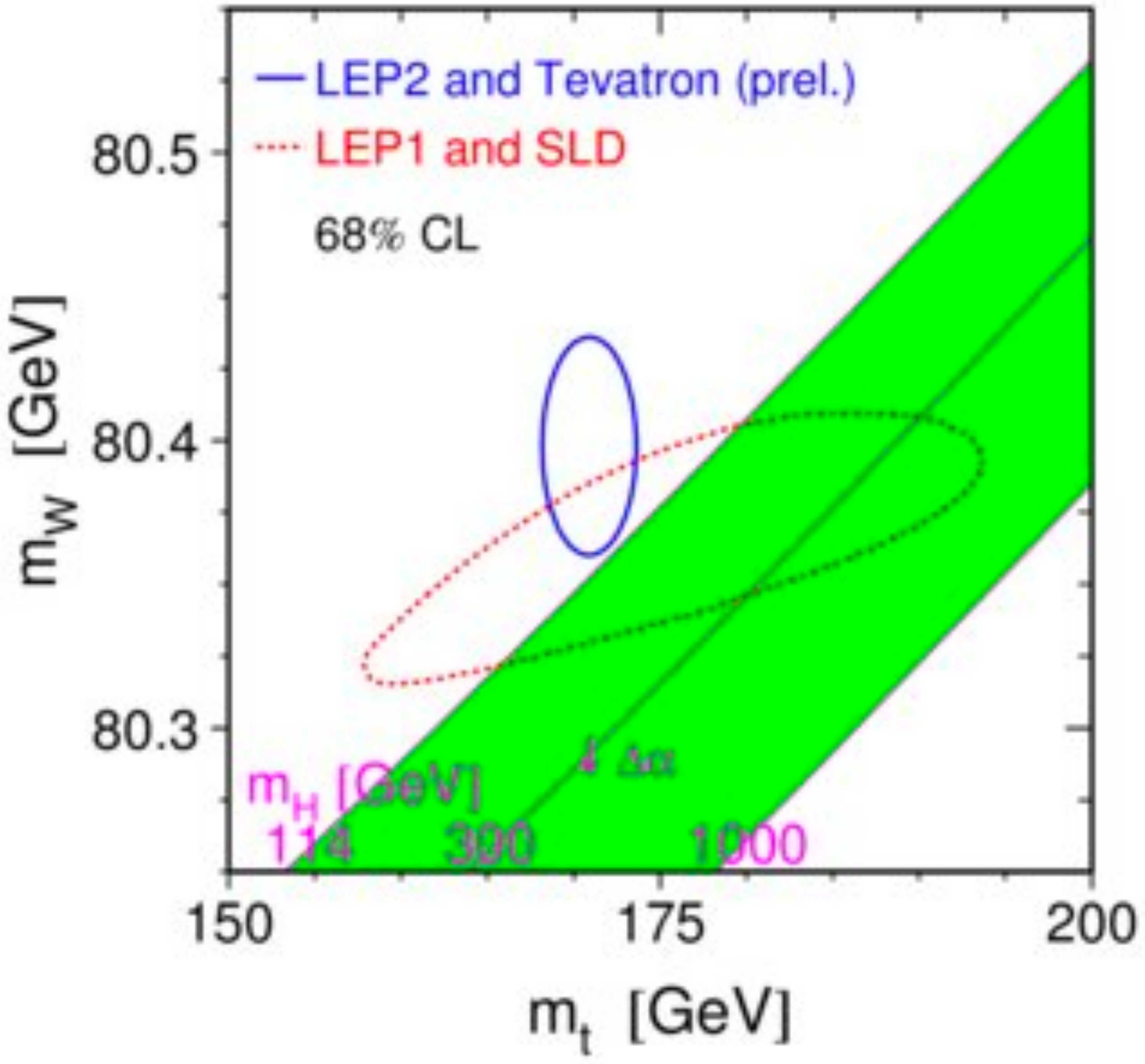
Lindert, Mai

The top quark mass

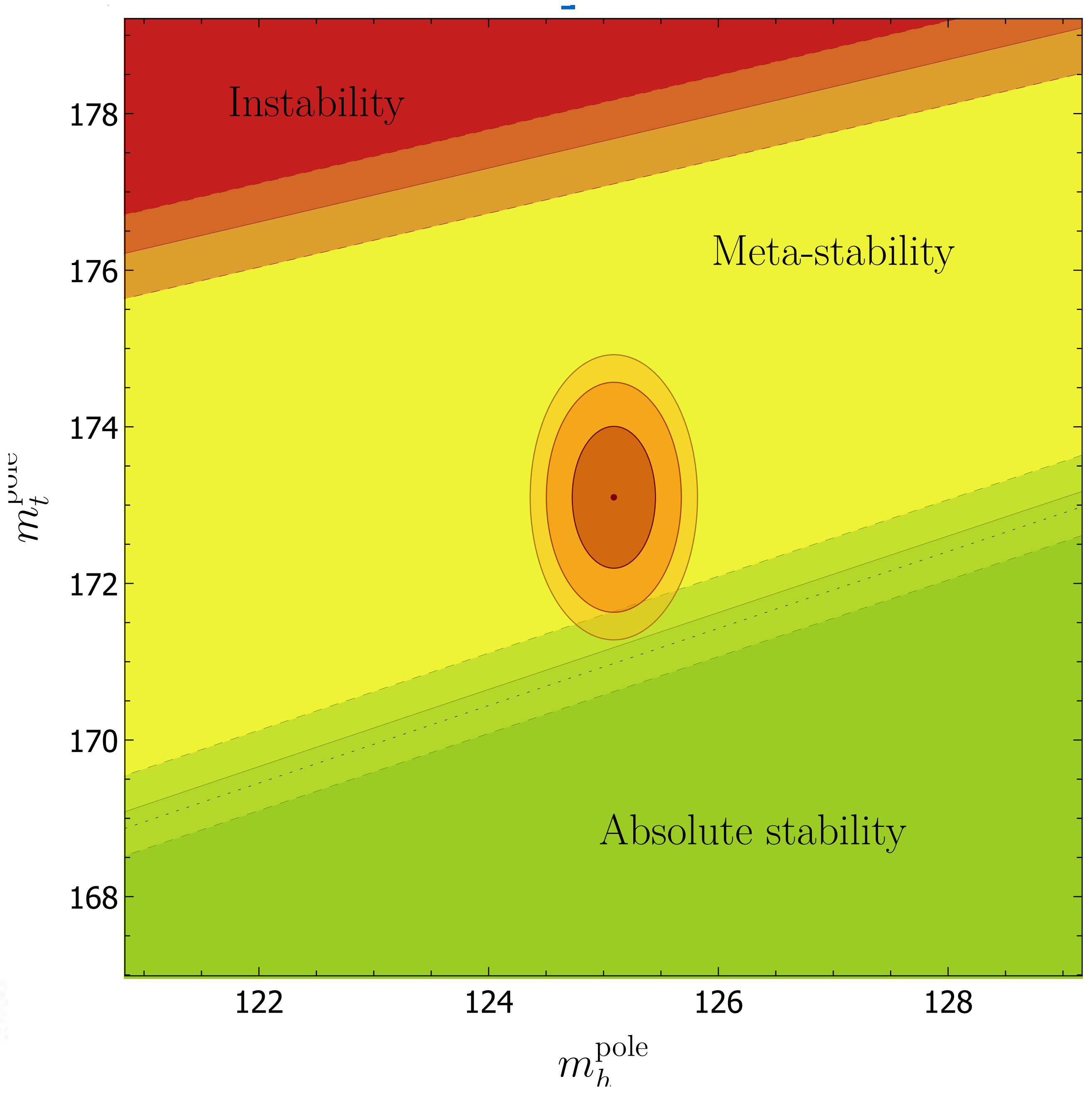
There is another class of observables measured at the LHC that I like to call ultra-high-precision ones since they are measured to a percent or even a permille accuracy. The top quark mass is one of them. An interesting thing is that we start seeing results for the top quark mass with the uncertainty that is comparable to $\Lambda_{\rm QCD}$, the non-perturbative parameter of strong interactions.



$$m_t = 172.52 \pm 0.33 \text{ GeV}$$



$$\tau_{\rm SM} = \left(\frac{\Gamma}{V}\right)^{-1/4} = 10^{139^{+102}_{-51}} \text{ years}$$



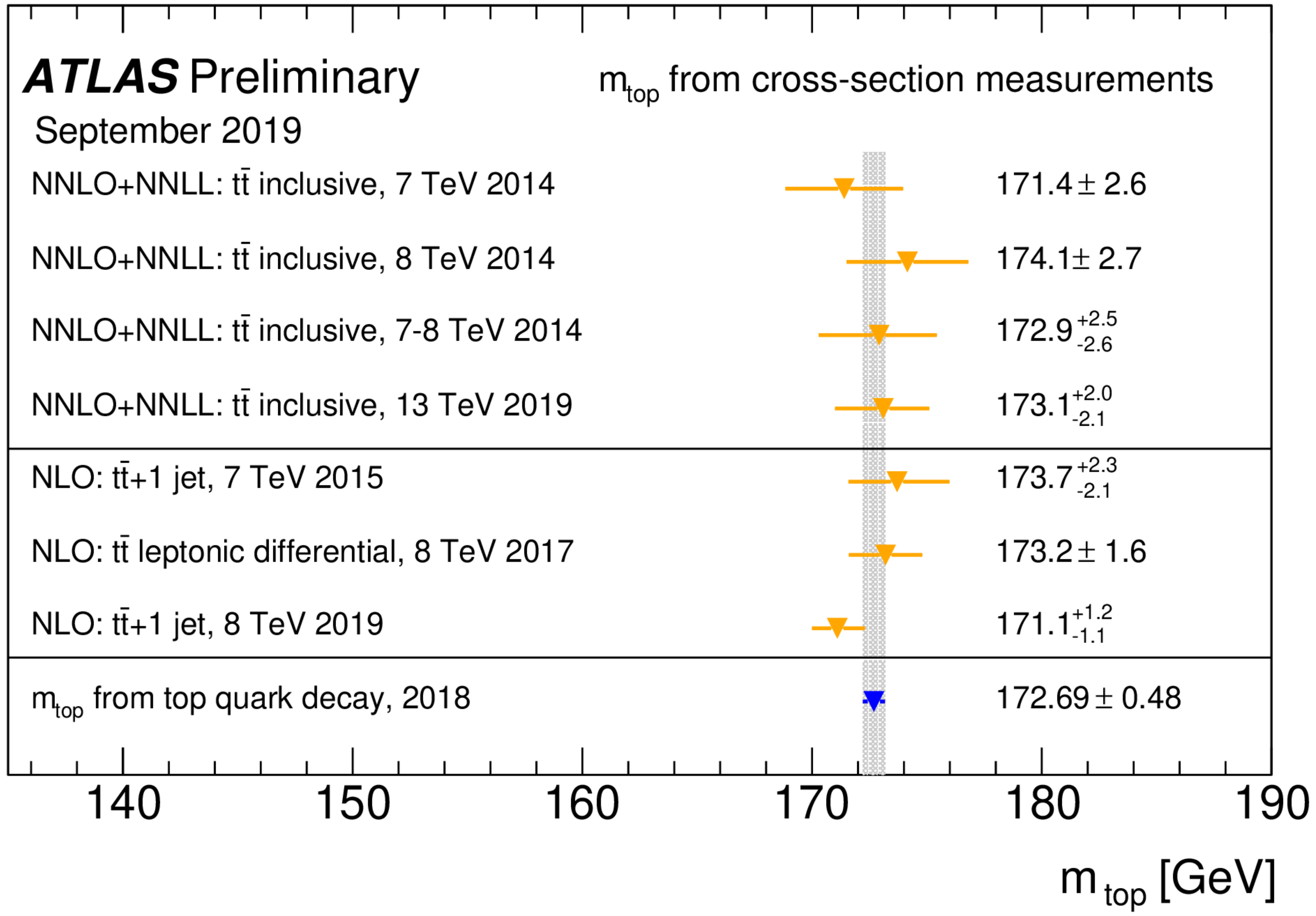
Uncertainty equal parts m_t, α_s, threshold corrections

Stable EW vacuum requires $m_t < 171$ GeV

Top quark mass extractions from cross sections rely on its strong sensitivity to m_t and on higher-order perturbative predictions for top quark pair production cross section. From this observable alone, the top quark mass was measured to about 700 MeV.

To be certain that this works in the right way, we need to know whether there are linear non-perturbative power corrections to the cross section; otherwise they may impact the extracted value of the top quark mass.

$$\mathrm{d}\sigma_{\mathrm{hard}} = \sum_{ij\in\{q,g\}} \int \mathrm{d}x_1\, \mathrm{d}x_2 f_i(x_1) f_j(x_2)\, \mathrm{d}\sigma_{ij}(x_1,x_2),\{p_{\mathrm{fin}}\})\, O_J(\{p_{\mathrm{fin}}\})\left(1+\mathcal{O}(\Lambda_{\mathrm{QCD}}^n/Q^n)\right)$$



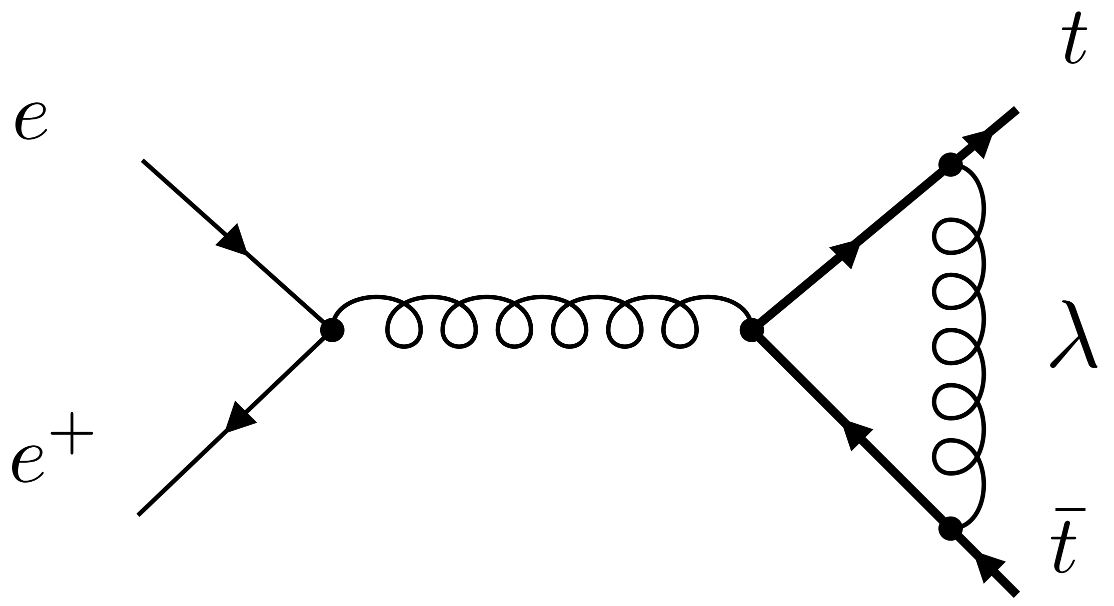
$$\sigma_{t\bar{t}} = \sigma_0 \left(\frac{m_0}{m_t}\right)^5 \left[1 + c_{\mathrm{np}} \frac{\Lambda_{\mathrm{QCD}}}{m_t} + \ldots\right]$$

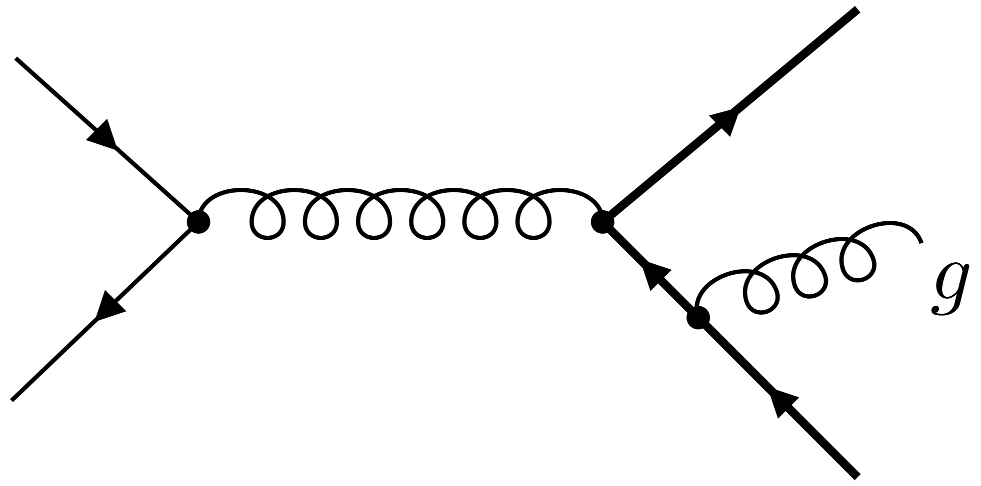
$$m_t \to m_t + \frac{c_{\mathrm{np}}}{5} \Lambda_{\mathrm{QCD}}$$

The existing theory of hard hadron collisions does not allow us to say with confidence whether such corrections exist or not; without this, it is not possible to trust the ultra-precise value of the top quark mass.

One possibility to explore this problem is to connect perturbative and non-perturbative computations, by checking the sensitivity of the former to “non-perturbative” (soft) momenta regions.

It is certainly possible to understand the famous (Kinoshita-Lee-Naunberg) cancellation of soft and collinear singularities in this way; it becomes particularly instructive if the gluon is given a mass.





$$k \sim \lambda \sim \Lambda_{\mathrm{QCD}}$$

$$\mathrm{d}\sigma_{e^+e^-\to t\bar{t}} \sim \mathrm{d}\sigma_0 \left(1 - c_1 \alpha_s \ln \frac{m_t}{\lambda} \cdots \right) \qquad \mathrm{d}\sigma_{e^+e^-\to t\bar{t}+g} \sim \mathrm{d}\sigma_0 \; c_1 \alpha_s \ln \frac{m_t}{\lambda} \cdots$$

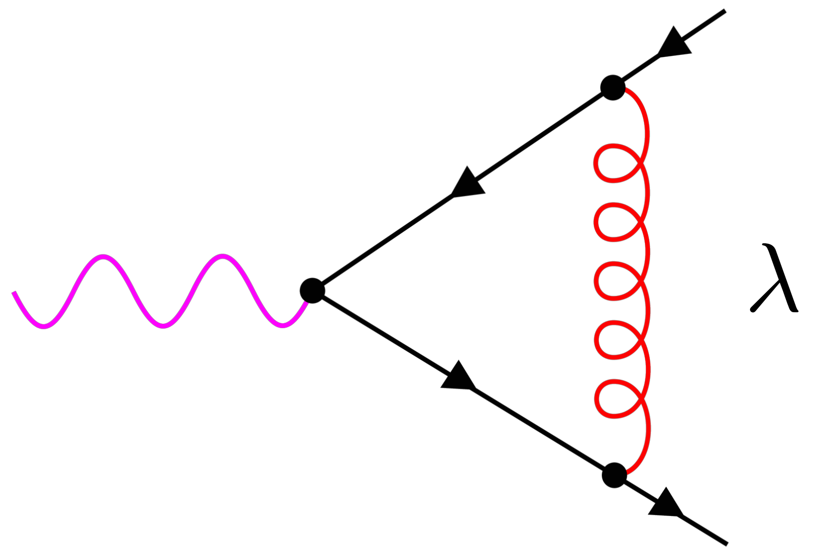
$$\mathrm{d}\sigma_{e^+e^-\to t\bar{t}} + \mathrm{d}\sigma_{e^+e^-\to t\bar{t}+g} \sim \mathrm{d}\sigma_0 \left(1 + \alpha_s \mathcal{O}(\lambda^0)\right)$$

Infra-red safety implies that the sensitivity to long-distance physics, parameterised by the mass of the gluon, is absent, i.e. whatever this mass is, the result is the same, up to power corrections.

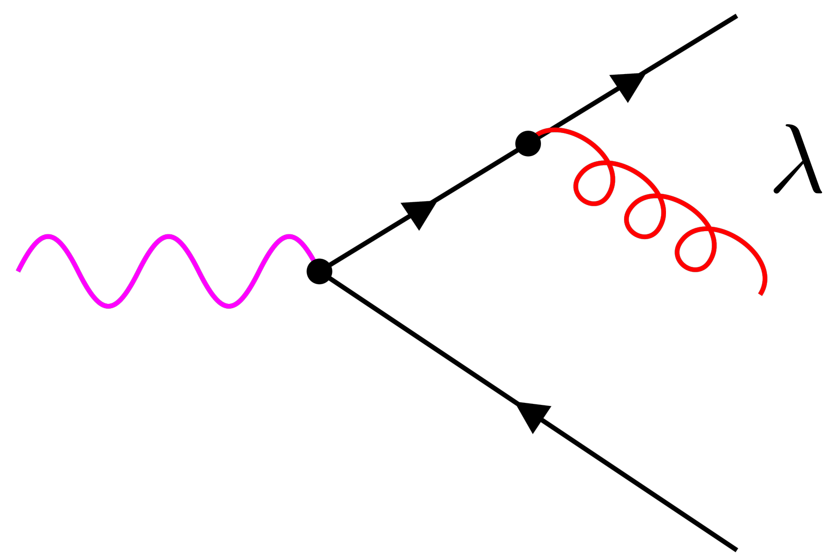
Power corrections $\mathcal{O}(\lambda^n)$ are also interesting if our goal is high-precision predictions; these corrections basically tell us when perturbative predictions alone become insufficient.

We would like to know whether or not there are linear correction to top quark pair production at a hadron collider since, if it is there, it would have important implications for the extraction of the top quark mass with the highest precision.

$$\mathrm{d}\sigma_{t\bar{t}}(E,\lambda) = \mathrm{d}\sigma_{t\bar{t}}^{(0)} + \lambda\, \mathrm{d}\sigma_{t\bar{t}}^{(1)} + \mathcal{O}(\lambda^2)$$



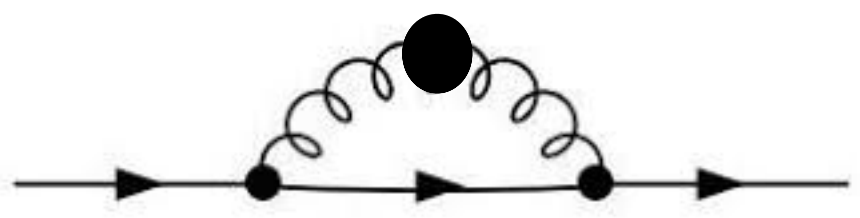
To answer this question, one needs to compute loop corrections and real-emission contributions to top quark pair production cross section with the massive gluon and then expand the result in the small gluon mass up to the linear terms.



In addition, it turns out that for this discussion, it is important to understand what is meant by the "top quark mass" that one tries to measure.

In quantum field theory , particle masses are inferred from poles of propagators. This cannot work for quarks beyond fixed-order perturbation theory (confinement). This issue is quite obscure for the top quark since it is very heavy and very unstable (top is the only quasi-free quark, as we often say).

$$G(p,m) \sim \frac{1}{p^2 - m^2} \quad \overset{\text{pole at}}{\Rightarrow} \quad p^2 = m^2$$



$$m_{\rm pole} = m_{\rm bare} + \frac{4}{3}\int\limits_0^\infty \frac{{\rm d}^3\vec{k}}{4\pi^2}\frac{\alpha_s(|\vec{k}|)}{\vec{k}^2} w(|\vec{k}|, m) \qquad w(|\vec{k}|,m) \approx 1, \quad |\vec{k}| \le m \qquad \alpha_s(|\vec{k}|) \approx \frac{\Lambda_{\rm QCD}^2}{\vec{k}^2 - \Lambda_{\rm QCD}^2}$$

The pole mass of a top quark cannot be determined with the precision better than $\delta m \sim \Lambda_{\rm QCD} \sim 300~{\rm MeV}$.

We must think about the top quark mass as a parameter of the Lagrangian and define it according to a chosen renormalization ``scheme''. Depending on the choice of the scheme and the renormalization scale, we get different mass parameters, that range from the MSbar to the low-scale short-distance masses (kinetic, potential-subtracted, 1S etc.). On the other hand, these short-distance masses can be determined with a much higher precision than the pole mass, at least in theory.

$$m(\mu) = m_{\rm bare} + \frac{4}{3}\int\limits_\mu^\infty \frac{{\rm d}^3\vec{k}}{4\pi^2}\frac{\alpha_s(|\vec{k}|)}{\vec{k}^2} w(|\vec{k}|, m) \qquad \Lambda_{\rm QCD} \ll \mu \qquad m_{\rm pole} = m(\mu) + \frac{4}{3}\int\limits_0^\mu \frac{{\rm d}^3\vec{k}}{4\pi^2}\frac{\alpha_s(|\vec{k}|)}{\vec{k}^2}$$

$$\boxed{m_{\rm pole} = m(\mu) + \frac{4}{3}\alpha_s(\mu)\,\mu}$$

One can show that $\mathcal{O}(\lambda) \sim \mathcal{O}(\Lambda_{\rm QCD})$ power corrections to top quark pair production cross section cancel provided that it is expressed through one of the short distance masses; however, such corrections are present if the cross section is written in terms of the pole mass. Below the way the cancellation works in case of single-top production at the LHC is shown.

$$\sigma = \sigma_{\rm LO}(m_t) + \sigma_R + \sigma_V + \sigma_{\rm ren} = \sigma_{\rm LO}(\tilde{m}_t) + \delta\sigma_{\rm NLO} \qquad \delta\sigma_{\rm NLO} = \sigma_R + \sigma_V + \sigma_{\rm ren} + \delta\sigma_{\rm mass}^{\rm expl} + \delta\sigma_{\rm mass}^{\rm impl}$$

$$\mathcal{T}_\lambda\left[\delta\sigma_{\rm mass}^{\rm expl} + \delta\sigma_{\rm mass}^{\rm impl}\right] = \frac{C_F\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\int {\rm dLips}_{\rm LO}\left[\frac{m_t^2}{p_dp_t}\left[1 + p_d^\mu\left(\frac{\partial}{\partial p_d^\mu} - \frac{\partial}{\partial p_t^\mu}\right)\right]F_{\rm LO} - m_t{\rm Tr}\left[\mathbf{1}\mathbf{N}\not{p}_b\bar{\mathbf{N}}\right] - m_t{\rm Tr}\left[(\not{p}_t + m_t)\left(\frac{\partial\mathbf{N}}{\partial m_t}\not{p}_b\bar{\mathbf{N}} + \mathbf{N}\not{p}_b\frac{\partial\bar{\mathbf{N}}}{\partial m_t}\right)\right]\right],$$

$$\mathcal{T}_\lambda\left[\sigma_{\rm R}\right] = \frac{\alpha_sC_F}{2\pi}\frac{\pi\lambda}{m_t}\int {\rm dLips}(p_u,p_b;p_d,p_t,p_X)\left[\left(\frac{3}{2} - \frac{m_t^2}{p_dp_t} - \frac{m_t^2}{p_tp_b}\right) - \frac{m_t^2}{p_dp_t}p_d^\mu\left(\frac{\partial}{\partial p_d^\mu} - \frac{\partial}{\partial p_t^\mu}\right) - \frac{m_t^2}{p_tp_b}p_b^\mu\left(\frac{\partial}{\partial p_b^\mu} + \frac{\partial}{\partial p_t^\mu}\right)\right]F_{\rm LO},$$

$$\mathcal{T}_\lambda\left[\sigma_V\right] = -\frac{\alpha_sC_F}{2\pi}\frac{\pi\lambda}{m_t}\int {\rm dLips}_{\rm LO}\left[{\rm Tr}\left[\not{p}_t\mathbf{N}\not{p}_b\bar{\mathbf{N}}\right] + \left(\frac{2p_tp_b - m_t^2}{p_tp_b} - \frac{m_t^2}{p_tp_b}p_b^\mu D_{p,\mu}\right)F_{\rm LO}\right],$$

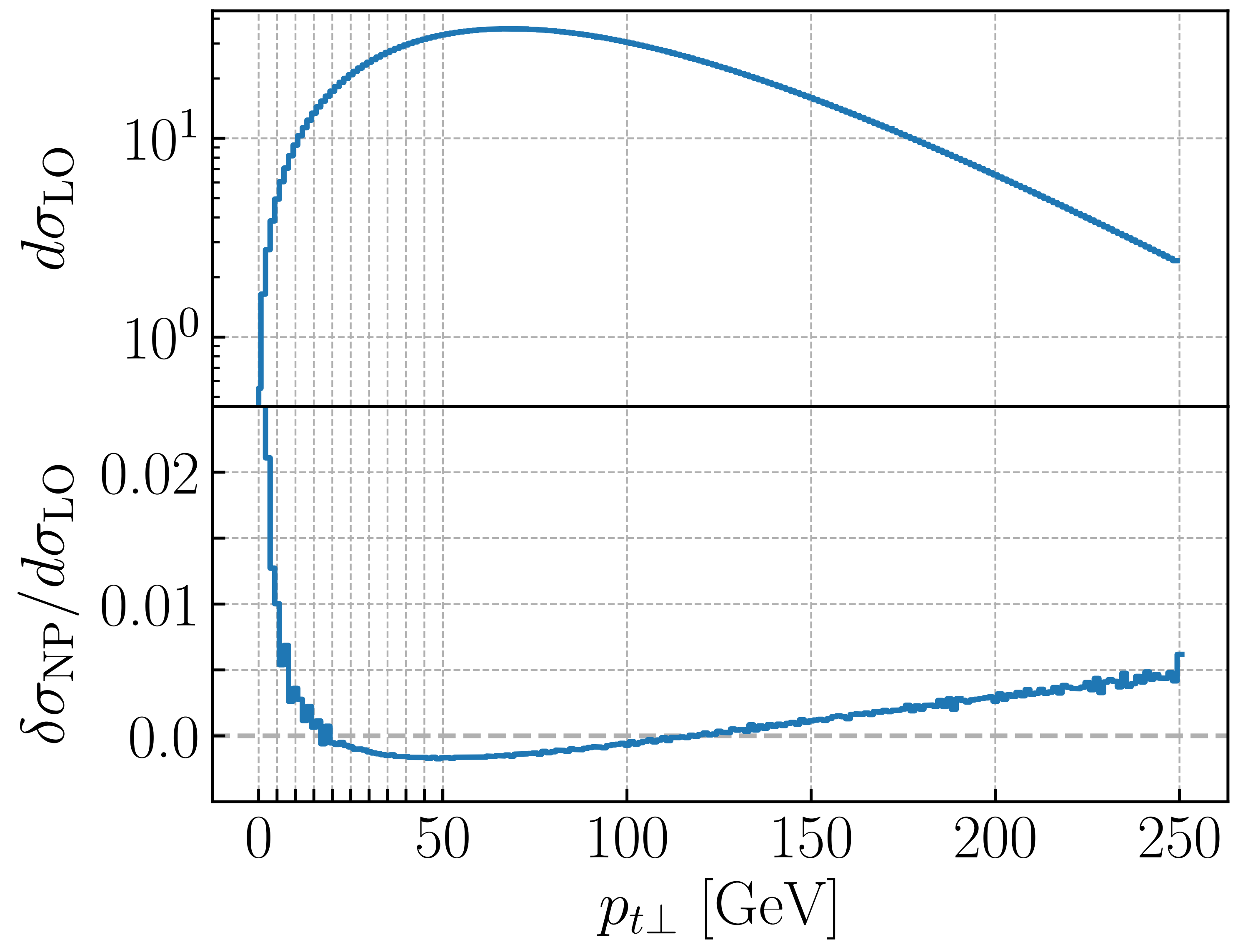
$$\mathcal{T}_\lambda\left[\sigma_{\rm ren}\right] = \frac{\alpha_sC_F}{2\pi}\frac{\pi\lambda}{m_t}\int {\rm dLips}_{\rm LO}\left[\frac{3}{2}F_{\rm LO} + m_t{\rm Tr}\left[(\not{p}_t + m_t)\frac{\partial\mathbf{N}}{\partial m_t}\not{p}_b\bar{\mathbf{N}}\right] + m_t{\rm Tr}\left[(\not{p}_t + m_t)\mathbf{N}\not{p}_b\frac{\partial\bar{\mathbf{N}}}{\partial m_t}\right]\right],$$

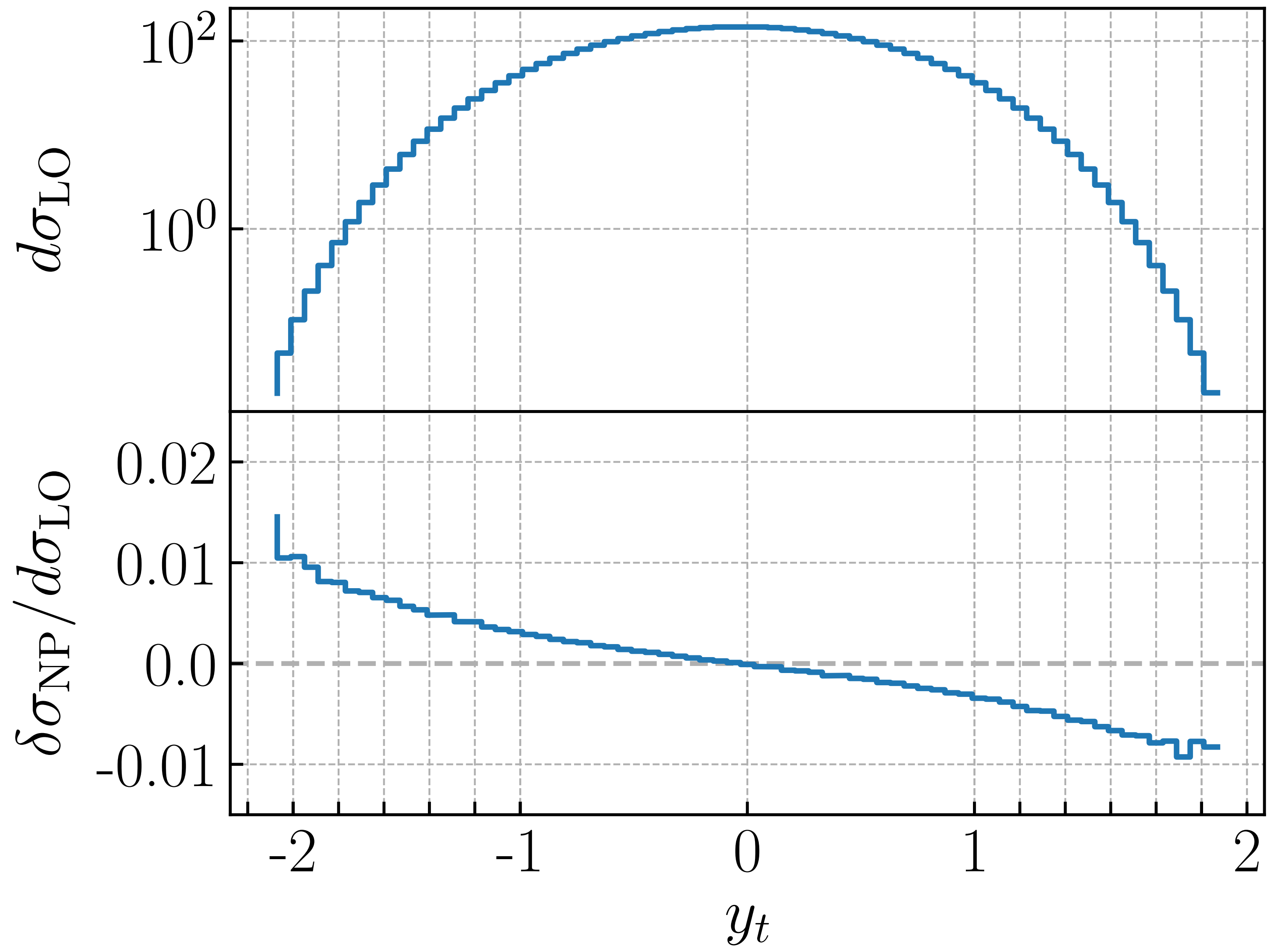
$$\mathcal{T}_\lambda\left[\delta\sigma_{\rm NLO}\right] = \frac{\alpha_sC_F}{2\pi}\frac{\pi\lambda}{m_t}\int {\rm dLips}_{\rm LO}\left(F_{\rm LO} - {\rm Tr}\left[\not{p}_t\mathbf{N}\not{p}_b\bar{\mathbf{N}}\right] - m_t{\rm Tr}\left[\mathbf{1}\mathbf{N}\not{p}_b\bar{\mathbf{N}}\right]\right) = 0.$$

Linear power corrections do exist in kinematic distributions independent of the mass parameter use. In general, shifts are not large but they become enhanced and reach a few percent close to edges of the allowed kinematic regions. Linear power corrections are not universal and exhibit non-trivial dependencies on the kinematic variables.

$$\frac{\delta_{\rm NP}\left[p_{t\perp}\right]}{p_{t\perp}} = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\,\frac{(2C_F - C_A\tau)}{2(1-\tau)}$$

$$\delta_{\rm NP}\left[y_t\right] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\left[(3C_A - 8C_F)\,\tau\cosh^2 y_t - (C_A - 2C_F)\,\frac{\tau(2-\tau)}{4(1-\tau)}\sinh(2y_t)\right]$$





$$\tau = 4m_t^2/s_{t\bar{t}}$$

$$\alpha_s\lambda = \frac{0.4\ \text{GeV}}{C_F} = 0.3\ \text{GeV}$$

Results for the Tevatron where quark annihilation channel dominates.

Makarov, K.M. , Nason,Oczelik

Conclusions

The Standard "Model" of particle physics, supplemented with general theory of relativity, is the current version of the "ultimate theory of everything". The SM is not a model where you have enough knobs to turn to achieve an agreement with the observation.

Treating it as such, we would like to find how far we should push it before it starts breaking, ideally in the controllable environment (colliders, low-energy experiments etc.).

We do this in the most natural way — by comparing the best measurements of various quantities and elementary particle's reactions with the best theoretical predictions that we have for them. This is called precision SM physics.

In principle, since the SM is a renormalizable theory, we just need a few parameters to describe (or fail to describe) every single measurement that it out there.

In practice, we start seeing limits of what we can do with perturbative physics and that non-perturbative physics starts playing more and more important role in the deliberations about the validity of the SM.