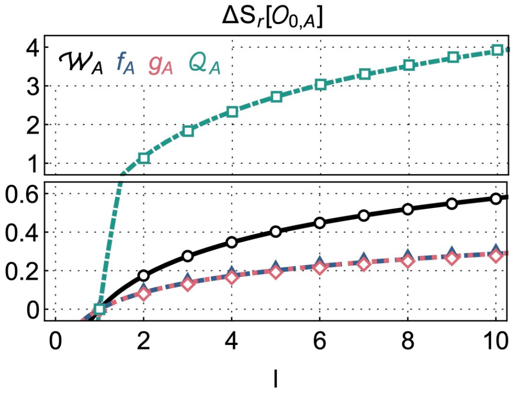
Quantum features from classical entropies



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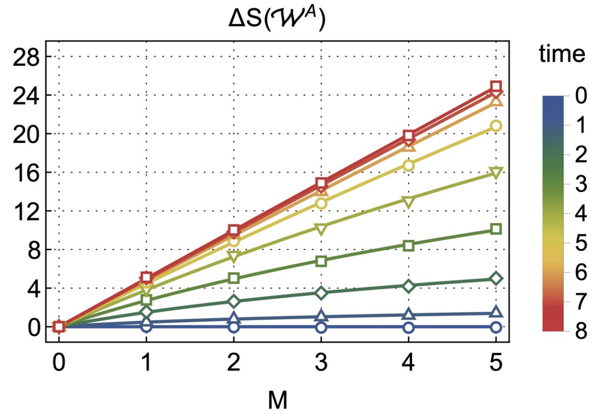

arXiv:2403.12320, arXiv:2404.12321, arXiv:2404.12323

QuIC

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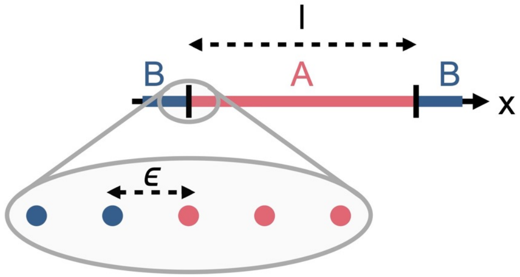
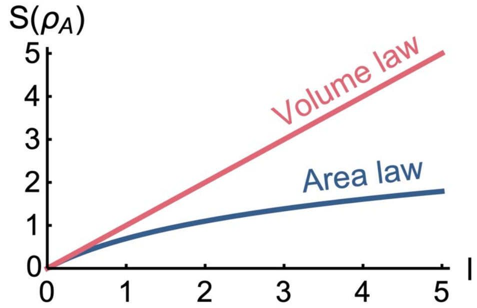
Big picture

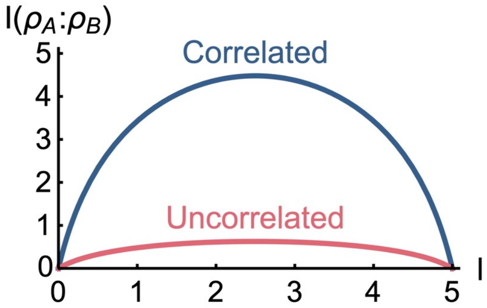
- Entanglement of a subregion





thermal

ground,
quenched
states, …

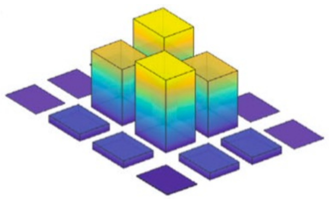


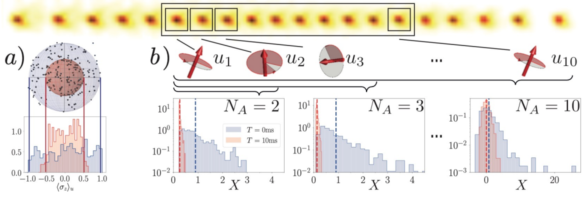
local state $\boldsymbol{\rho}_A = \mathbf{Tr}_B\{\boldsymbol{\rho}\}$

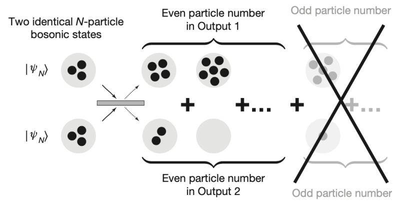
$S(\boldsymbol{\rho}_A) = -\mathbf{Tr}_A\{\boldsymbol{\rho}_A \ln \boldsymbol{\rho}_A\}$

$I(\boldsymbol{\rho}_A : \boldsymbol{\rho}_B) = S(\boldsymbol{\rho}_A) + S(\boldsymbol{\rho}_B) - S(\boldsymbol{\rho})$

- Quantum state tomography *or* clever readout







→ Quantum entropies $S(\boldsymbol{\rho}_A)$ and local state $\boldsymbol{\rho}_A$ needed?

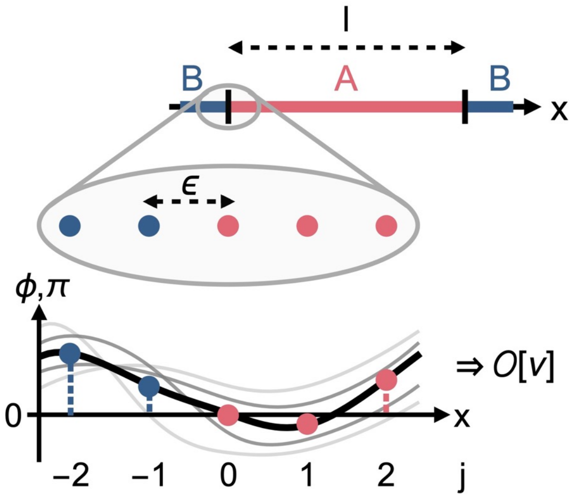
Outline

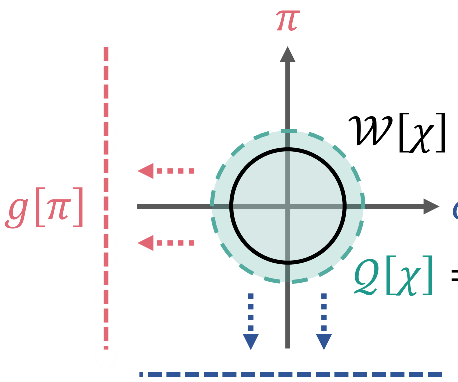
Theory	Experimental perspective
• Subtracted classical entropies	• Bose-Einstein condensate
• Scalar field	• Area law → Volume law

Measurement distributions

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

- Fundamental fields $\chi = (\phi, \pi)$; $\boldsymbol{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$; $[\boldsymbol{\phi}(x), \boldsymbol{\pi}(x')] = i\delta(x - x')$





$$\mathcal{W}[\chi] = \int \frac{\mathcal{D}\phi'}{\pi} \langle \phi - \phi' | \boldsymbol{\rho} | \phi + \phi' \rangle e^{2i \int \mathrm{d}x \, \pi(x) \phi'(x)}$$

$$\mathcal{Q}[\chi] = \int \mathcal{D}\chi' \, \mathcal{W}[\chi'] \, \overline{\mathcal{W}}(\chi - \chi')$$

$$f[\phi] = \langle \phi | \boldsymbol{\rho} | \phi \rangle = \int \mathcal{D}\pi \, \mathcal{W}[\chi]$$

- Measurement distributions $\mathcal{O}[v]$

  → Local distributions $\mathcal{O}_A[v_A] = \int \mathcal{D}v_B \, \mathcal{O}[v]$

Classical entropy & uncertainty

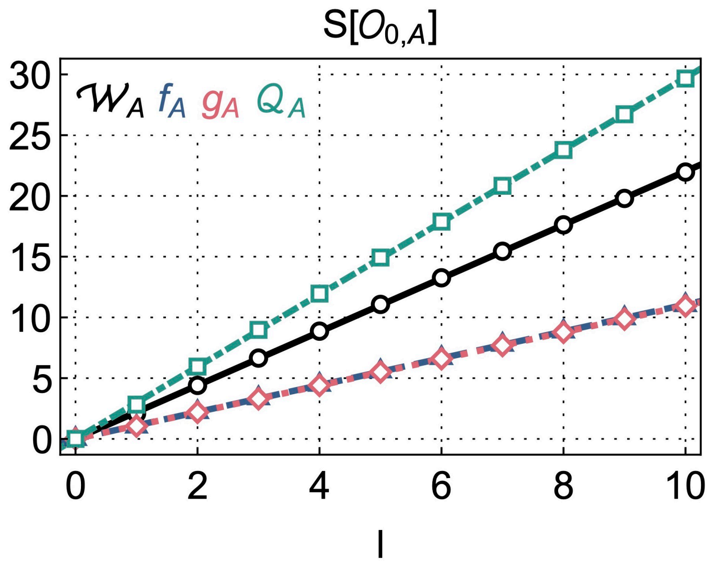
Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

- Local classical Rényi entropy

$$S_r[\mathcal{O}_A] = \frac{1}{1-r}\ln\left[\int \mathcal{D}v_A\, \mathcal{O}_A^r\right] \geq S_r[\bar{\mathcal{O}}_A] \sim \frac{l}{\epsilon}$$

Volume law

Entropic uncertainty



- Subtract extensive contribution

$$\Delta S_r[\mathcal{O}_A] = S_r[\mathcal{O}_A] - S_r[\bar{\mathcal{O}}_A] \sim \text{local uncertainty}$$

- Classical Rényi mutual information

$$I_r[\mathcal{O}_A : \mathcal{O}_B] = S_r[\mathcal{O}_A] + S_r[\mathcal{O}_B] - S_r[\mathcal{O}] \sim \text{correlations } A \leftrightarrow B$$

→ Let's check these out!

Scalar quantum field

- Hamiltonian

$$\boldsymbol{H} = \int \mathrm{d}x\, [\boldsymbol{\pi}^2 + (\partial_x \boldsymbol{\phi})^2 + m^2 \boldsymbol{\phi}^2]$$

- Typical local distribution

$$\mathcal{O}_A[\nu_A] = \underbrace{\frac{1}{Z_A^{\mathcal{O}}}\, e^{-\frac{1}{2}\int \mathrm{d}x\mathrm{d}x' \nu_A^T(x) \left(\gamma_A^{\mathcal{O}}\right)^{-1}(x,x') \nu_A(x')}}_{\text{Gaussian}} \times \underbrace{\kappa_A^{\mathcal{O}}[\nu_A]}_{\text{polynomial}}$$

$$\gamma_A^{\mathcal{O}} = \begin{pmatrix} \langle \boldsymbol{\phi}_A \boldsymbol{\phi}_A \rangle_{\mathcal{O}} & \langle \boldsymbol{\phi}_A \boldsymbol{\pi}_A + \boldsymbol{\pi}_A \boldsymbol{\phi}_A \rangle_{\mathcal{O}} \\ \langle \boldsymbol{\phi}_A \boldsymbol{\pi}_A + \boldsymbol{\pi}_A \boldsymbol{\phi}_A \rangle_{\mathcal{O}} & \langle \boldsymbol{\pi}_A \boldsymbol{\pi}_A \rangle_{\mathcal{O}} \end{pmatrix}$$

General formulae

- Classical Rényi entropy

Volume law

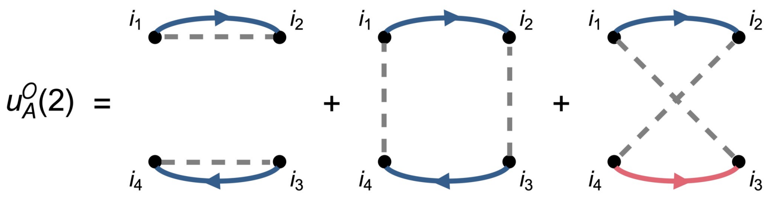
$$S_r[\mathcal{O}_A] = \underbrace{\frac{1}{2}\ln\det\left(2\pi\gamma_A^{\mathcal{O}}\right) + \boxed{\frac{1(2)}{2}\frac{\ln r}{r-1}\frac{l}{\epsilon}}}_{\text{Gaussian}} + \underbrace{\frac{\ln U_A^{\mathcal{O}}}{1-r}}_{\text{non-Gaussian}}$$

Gaussian non-Gaussian

$$U_A^{\mathcal{O}}(r) = \left(\kappa_A^{\mathcal{O}}[\partial_\zeta]\right)^r e^{\frac{1}{2r}\int \mathrm{d}x\mathrm{d}x'\zeta^T(x)\gamma_A^{\mathcal{O}}(x,x')\zeta(x')}\Big|_{\zeta=0}$$

- Subtracted classical Rényi entropy

$$\Delta S_r[\mathcal{O}_A] = \frac{1}{2}\ln\det\left[\gamma_A^{\mathcal{O}}\left(\bar{\gamma}_A^{\mathcal{O}}\right)^{-1}\right] + \frac{\ln U_A^{\mathcal{O}}}{1-r}$$



- Classical Rényi mutual information

$$I_r[\mathcal{O}_A:\mathcal{O}_B] = \frac{1}{2}\ln\frac{\det\left(\gamma_A^{\mathcal{O}}\right)\det\left(\gamma_B^{\mathcal{O}}\right)}{\det\left(\gamma^{\mathcal{O}}\right)} + \frac{1}{1-r}\ln\frac{U_A^{\mathcal{O}}U_B^{\mathcal{O}}}{U^{\mathcal{O}}}$$

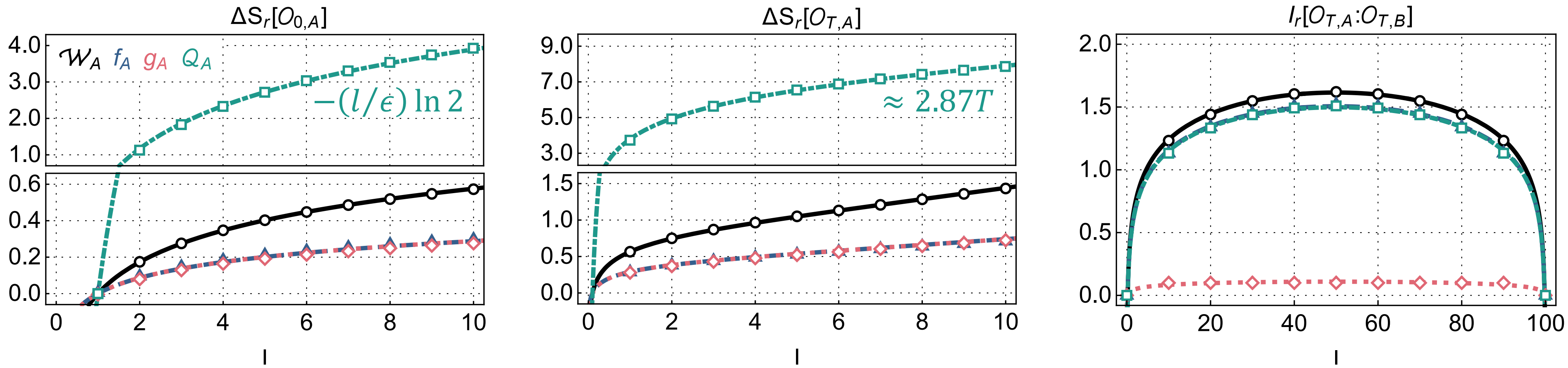
Gaussian states

- Classical and quantum entropies are related

$$S_2(\boldsymbol{\rho}_A) = \Delta S_r[\mathcal{W}_A] = \Delta S_r[f_A] + \Delta S_r[g_A]$$

$$I_2[\boldsymbol{\rho}_A : \boldsymbol{\rho}_B] = I_r[\mathcal{W}_A : \mathcal{W}_B] = I_r[f_A : f_B] + I_r[f_A : g_B]$$

if $\mathcal{W} = f \times g$



$$\Delta S_r[\mathcal{W}_A] = \frac{c}{4} \ln \frac{l}{\epsilon}$$

$$\Delta S_r[\mathcal{W}_A] = \frac{c}{4} \ln \frac{\sinh(\pi T l)}{\pi \epsilon T}$$

$$I_r[\mathcal{O}_{T,A} : \mathcal{O}_{T,B}] \leq a\, |\partial A|$$

Particles

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

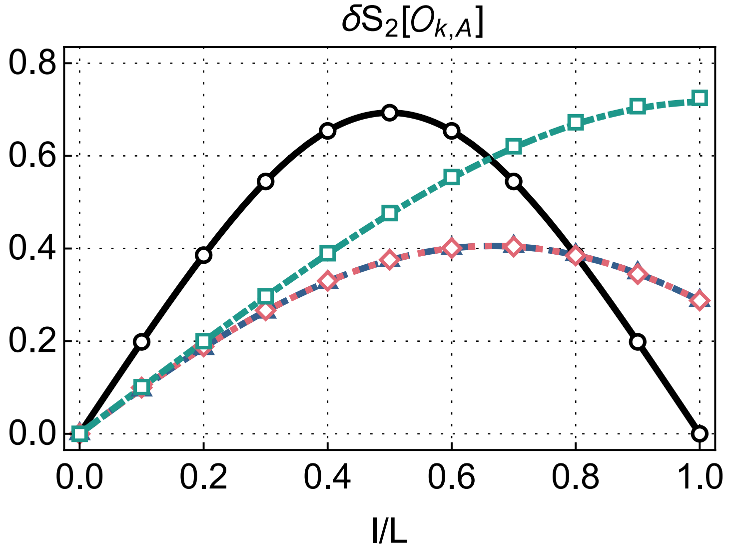
$$\delta S_r[\mathcal{O}_{k,A}] = \frac{1}{1-r} \ln\left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i\right]$$

→ Area laws also for non-Gaussian states

- Entanglement surplus of a particle

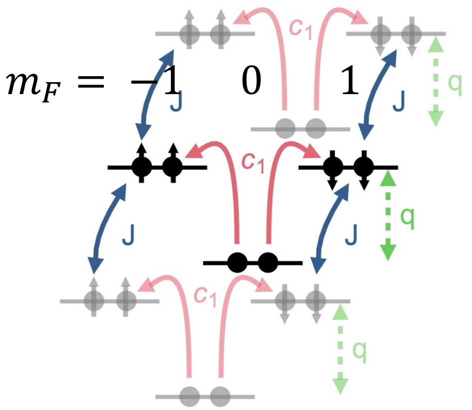
$$\delta S_2[\mathcal{W}_{k,A}] = -\ln\left[\left(\frac{l}{L}\right)^2 + \left(1 - \frac{l}{L}\right)^2\right]$$

→ Like a Qbit with $\mathrm{p} = \frac{l}{L}$



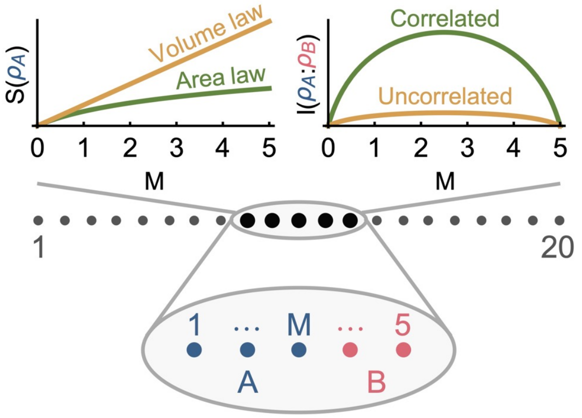
Spin-1 BEC

- Dynamics: Quench



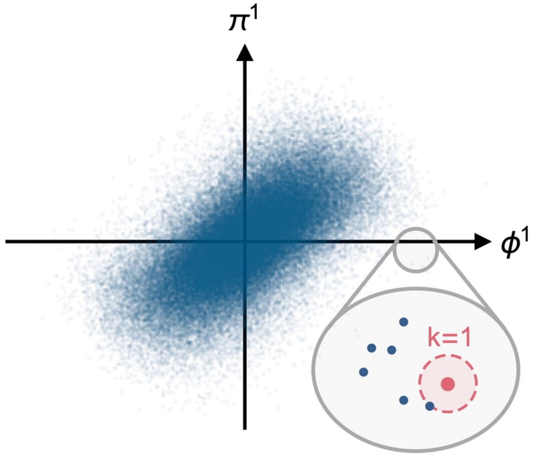
$$\boldsymbol{H} = \sum_{j=1}^{20} q\left(\boldsymbol{N}_1^j + \boldsymbol{N}_{-1}^j\right) + c_0 \boldsymbol{N}^j\left(\boldsymbol{N}^j - \mathbb{I}\right)$$
$$+c_1\left[\left(\boldsymbol{N}_0^j - \tfrac{1}{2}\mathbb{I}\right)\left(\boldsymbol{N}_1^j + \boldsymbol{N}_{-1}^j\right)\right.$$
$$\left.+\boldsymbol{a}_0^{j\dagger}\boldsymbol{a}_0^{j\dagger}\boldsymbol{a}_1^j\boldsymbol{a}_{-1}^j + h.c.\right]$$
$$-J\sum_{j=1}^{19}\sum_{m_F=\pm 1}\left(\boldsymbol{a}_{m_F}^{j\dagger}\boldsymbol{a}_{m_F}^{j+1} + h.c.\right)$$

- Readout: Spins



$$\boldsymbol{\phi}^j \equiv \frac{1}{2\sqrt{n}}\left[\boldsymbol{a}_0^{j\dagger}\left(\boldsymbol{a}_1^j + \boldsymbol{a}_{-1}^j\right) + h.c.\right]$$
$$\boldsymbol{\pi}^j \equiv \frac{-i}{2\sqrt{n}}\left[\boldsymbol{a}_0^{j\dagger}\left(\boldsymbol{a}_1^j + \boldsymbol{a}_{-1}^j\right) - h.c.\right]$$

- Entropy estimation: kNN



$$\hat{S}(k, N_s) = g(k, N_s, d) + \frac{d}{N_s}\sum_{i=1}^{N_s} \ln \epsilon^i(k)$$

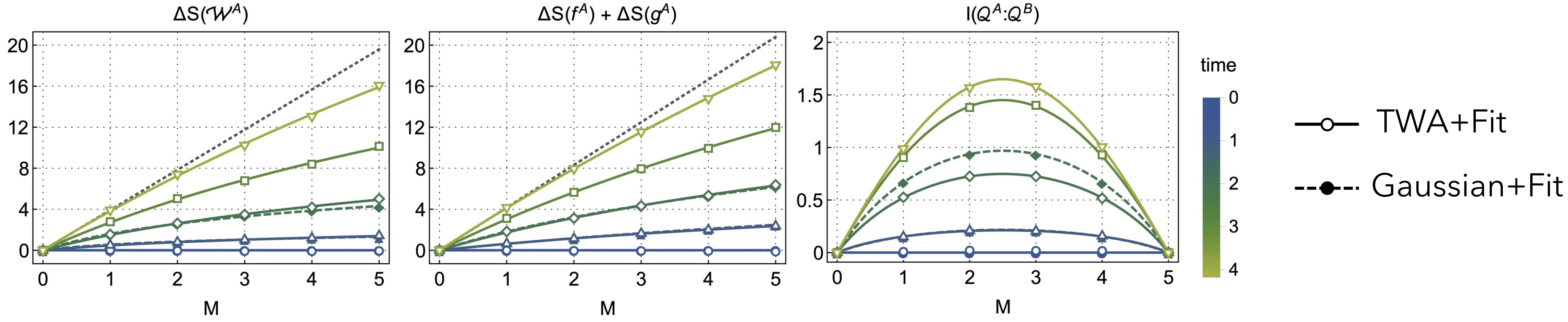
→ Asymptotically unbiased

→ Entropy estimated *directly* from data

→ *No* assumptions on $\boldsymbol{\rho}_A$ or $\mathcal{O}_A$

Area law

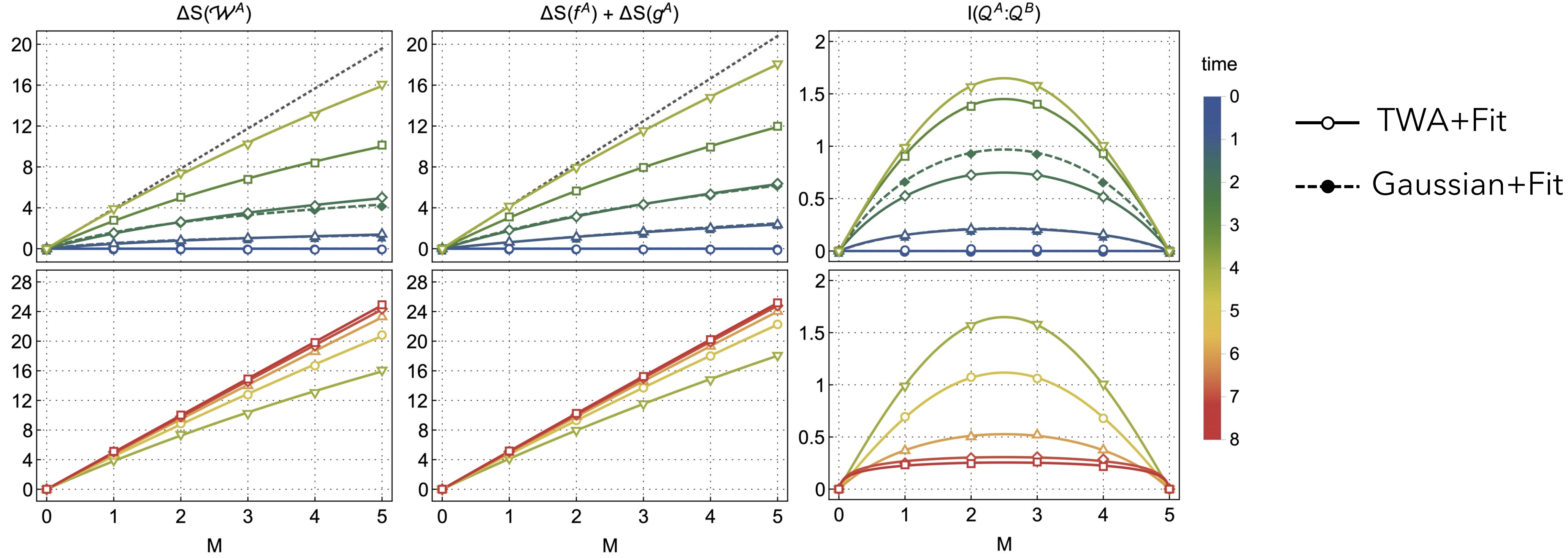
- Early times



→ Area laws signal build-up of quantum correlations

Area law → Volume law

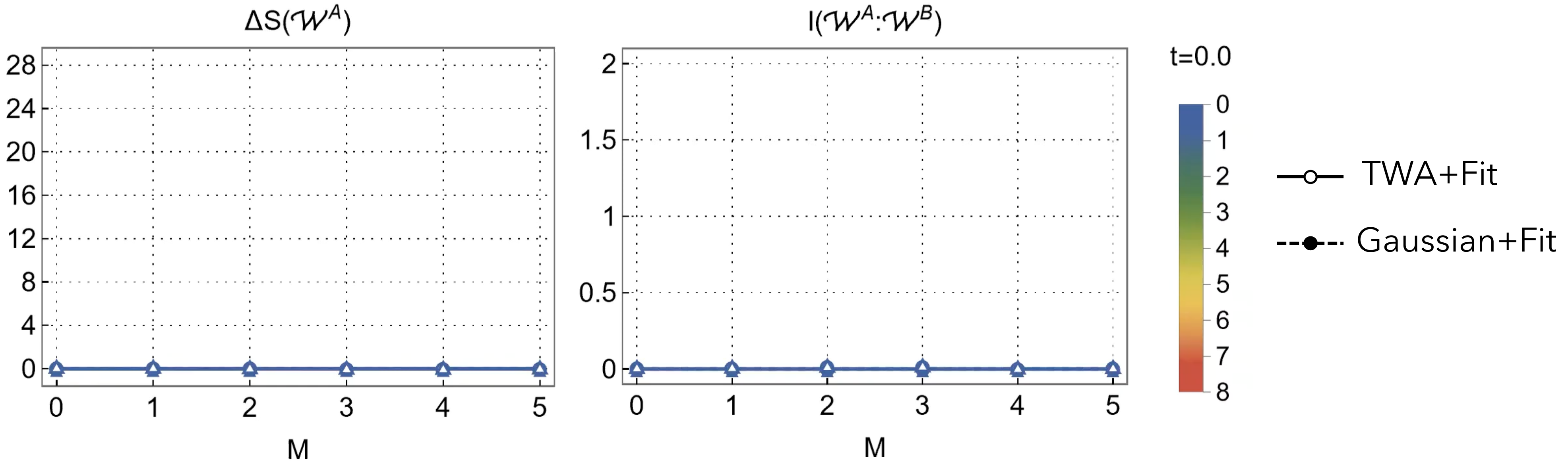
- Early times + late times



→ Volume laws reveal local thermalization, incline = $1/T$

Area law → Volume law

- Full time evolution of $\mathcal{W}$-quantities



Summary

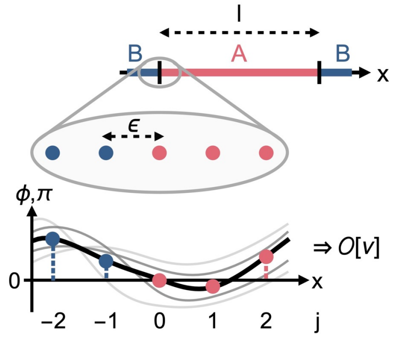
Theory

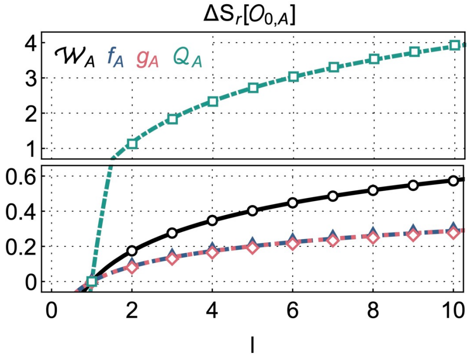
- Subtracted classical entropies

$$\Delta S_r[\mathcal{O}_A] = S_r[\mathcal{O}_A] - S_r[\bar{\mathcal{O}}_A]$$

$$I_r[\mathcal{O}_A : \mathcal{O}_B] = S_r[\mathcal{O}_A] + S_r[\mathcal{O}_B] - S_r[\mathcal{O}]$$

- Scalar field



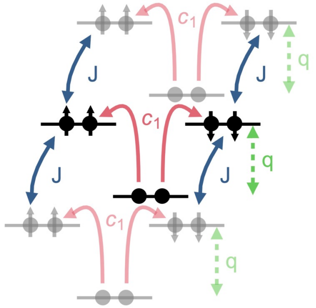


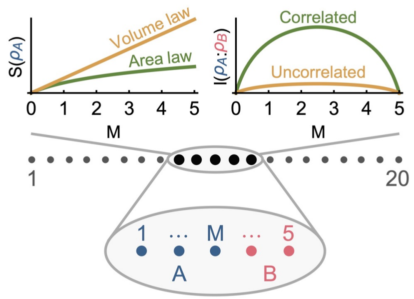
Distributions $\mathcal{O}[v]$

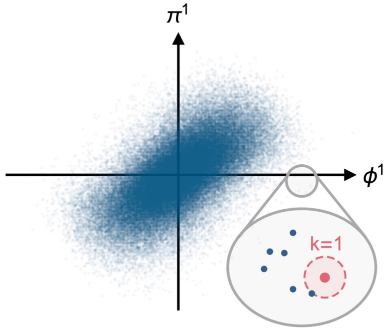
$\Delta S_r[\mathcal{O}_A] \propto c \ln \frac{l}{\epsilon}$

Experimental perspective

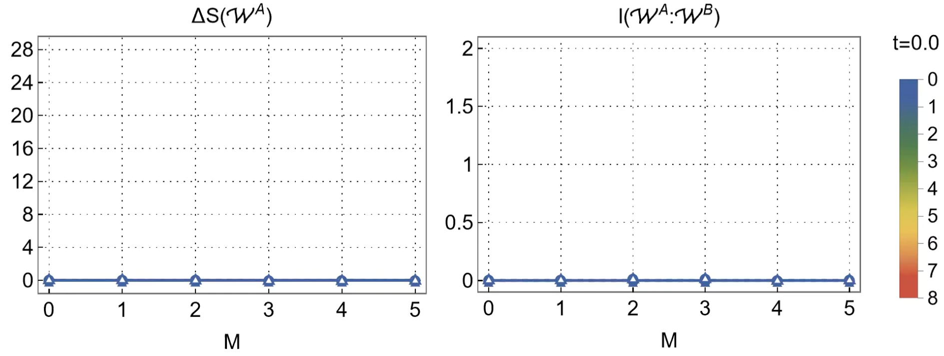
- Spin-1 BEC





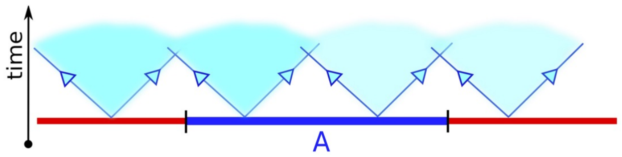


- Area law → Volume law

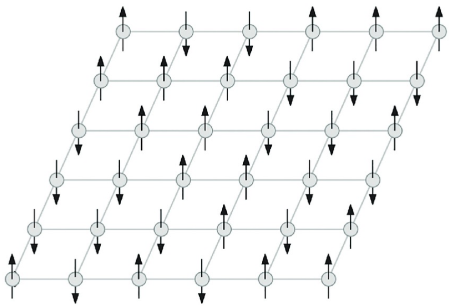


Outlook

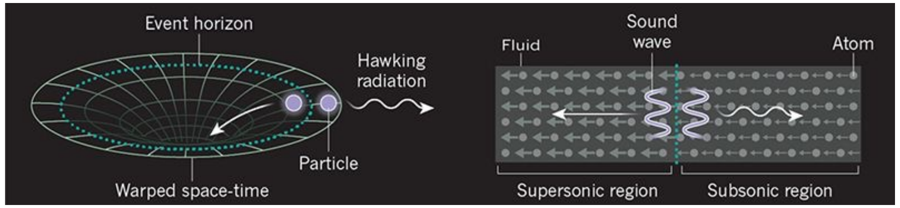
- Classical entropies *should* solve *most* of their quantum analog's problems



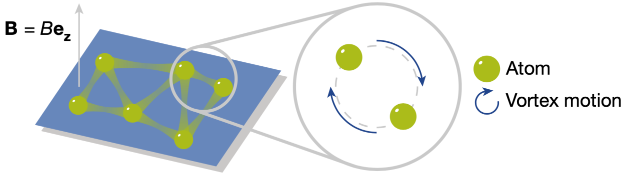
Jena Information spreading/scrambling,
Lieb-Robinson bounds for MIs



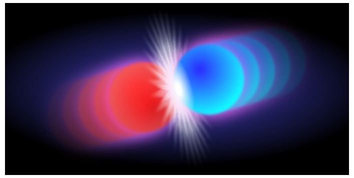
Higher-order correlations,
QPTs, MBL, Disorder,
Symmetry-resolved entanglement



Entropy of (sonic) black holes



Topological EE,
Central charges, Brussels
Chern numbers,
Bulk ↔ Edges



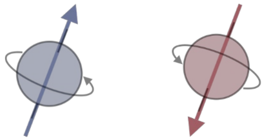
Non-Gaussian (interacting) theories
Nottingham & Wien

Thanks
for your
attention :)

Backup

Quantifying quantum information

- A single QBit



Pure states $|1\rangle$ $|0\rangle$ $\in \mathcal{H}$ Mixed states $\boldsymbol{\rho} = p|\mathbf{1}\rangle\langle\mathbf{1}| + (1-p)|\mathbf{0}\rangle\langle\mathbf{0}| \in B(\mathcal{H})$

- Quantum entropy = Missing information $S(\boldsymbol{\rho}) = -\mathbf{Tr}\{\boldsymbol{\rho} \ln \boldsymbol{\rho}\}$
  - Increases with states' mixedness

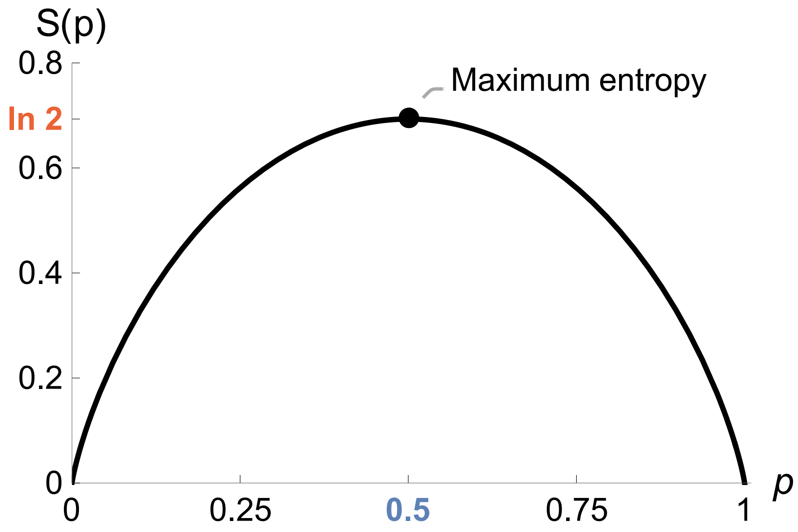
    $S(\boldsymbol{\rho}) \sim \boldsymbol{\rho}^{-1}$

  - Information of independent systems adds up

    $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B \to S(\boldsymbol{\rho}_A) + S(\boldsymbol{\rho}_B)$

  - Non-negative (zero iff state is pure)

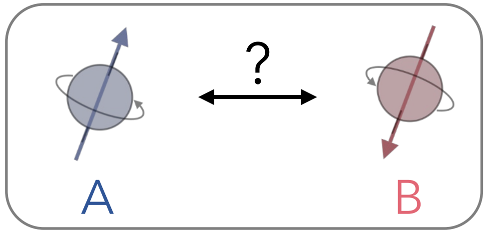
    $S(\boldsymbol{\rho}) \geq 0, \;\; S(\boldsymbol{\rho}) = 0 \Leftrightarrow \boldsymbol{\rho} = |\psi\rangle\langle\psi|$



Entanglement and quantum entropy

- Two QBits

Global $\boldsymbol{\rho}$



Local $\boldsymbol{\rho}_A = \mathbf{Tr}_B\{\boldsymbol{\rho}\}$ $\boldsymbol{\rho}_B = \mathbf{Tr}_A\{\boldsymbol{\rho}\}$

- Classical theories: classical correlations (separable states, e.g. $\boldsymbol{\rho} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$)

$$S(\boldsymbol{\rho}) \geq S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) \quad \rightarrow \quad S(\boldsymbol{\rho}) = 0 \Rightarrow S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) = 0$$

$\rightarrow$ *Know the system and thus know **all** about its parts*

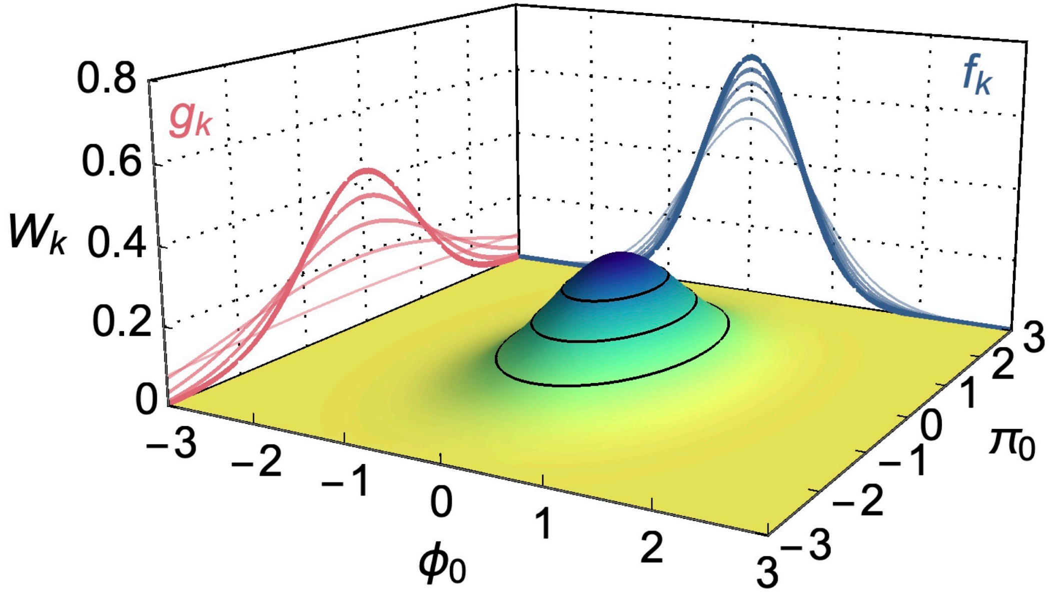
- Quantum theories: entanglement

$$S(\boldsymbol{\rho}) < S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) \quad \rightarrow \quad S(\boldsymbol{\rho}) = 0 \quad \wedge \quad S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) > 0$$

$\rightarrow$ *Know the system and may know **nothing** about its parts*

Particles in phase space

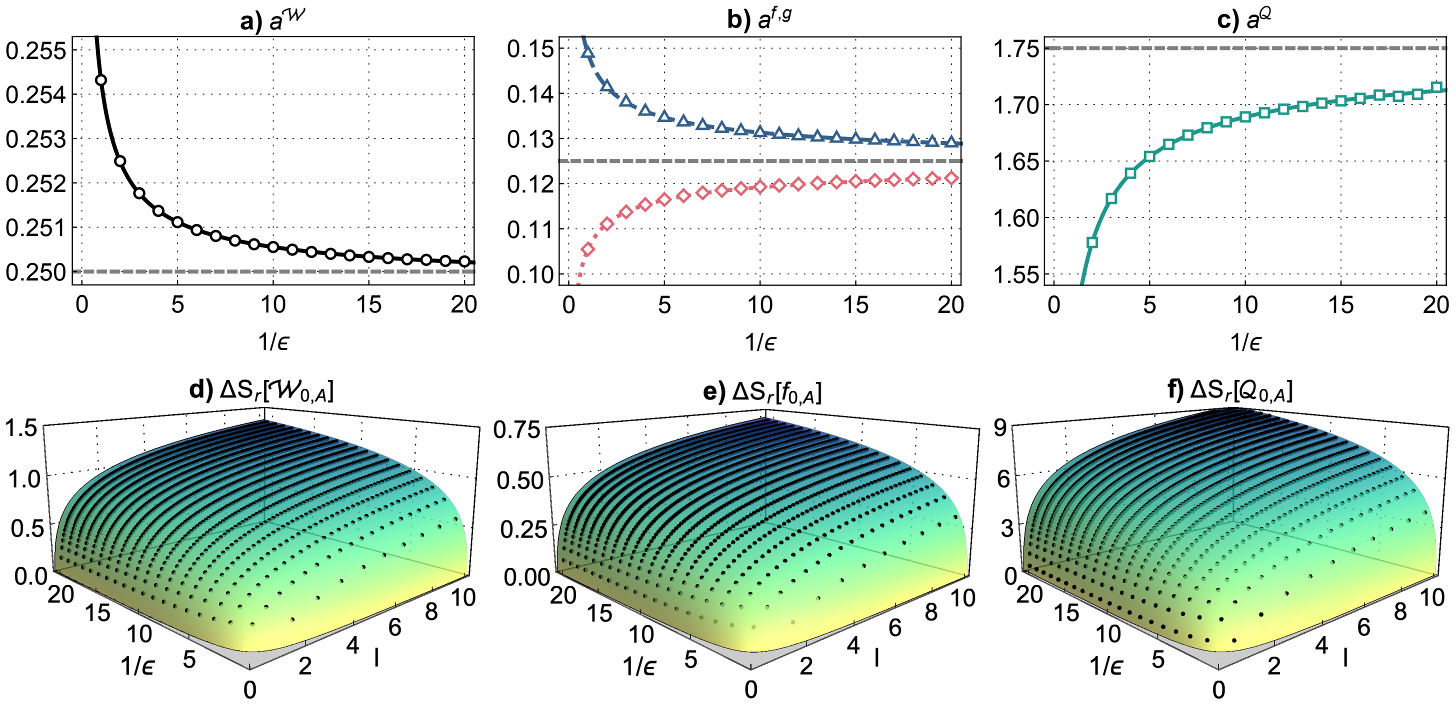
- Wigner $\mathcal{W}$-distribution for free particle



Central charge

- Ground state entropies

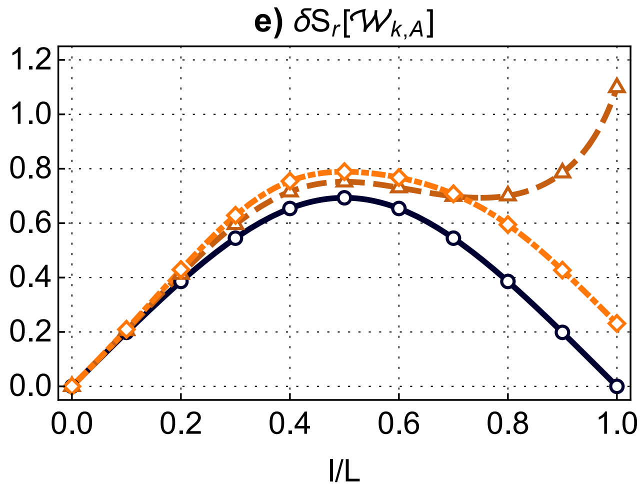
$$\Delta S_r[\mathcal{W}_{0,A}] = \Delta S_r[f_{0,A}] + \Delta S_r[g_{0,A}] = \frac{c}{4}\ln\frac{l}{\epsilon}$$



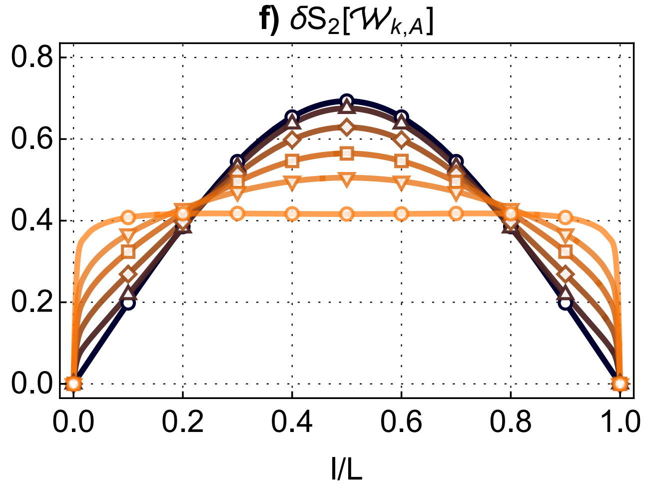
Particles – Other quantities I

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

$$\delta S_r[\mathcal{O}_{k,A}] = \frac{1}{1-r} \ln \left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i\right]$$



$$\delta S_r[\mathcal{O}_{k,A}] \sim (2)\frac{l}{L}$$
$$\sim \delta S_2[\boldsymbol{\rho}_{k,A}]$$



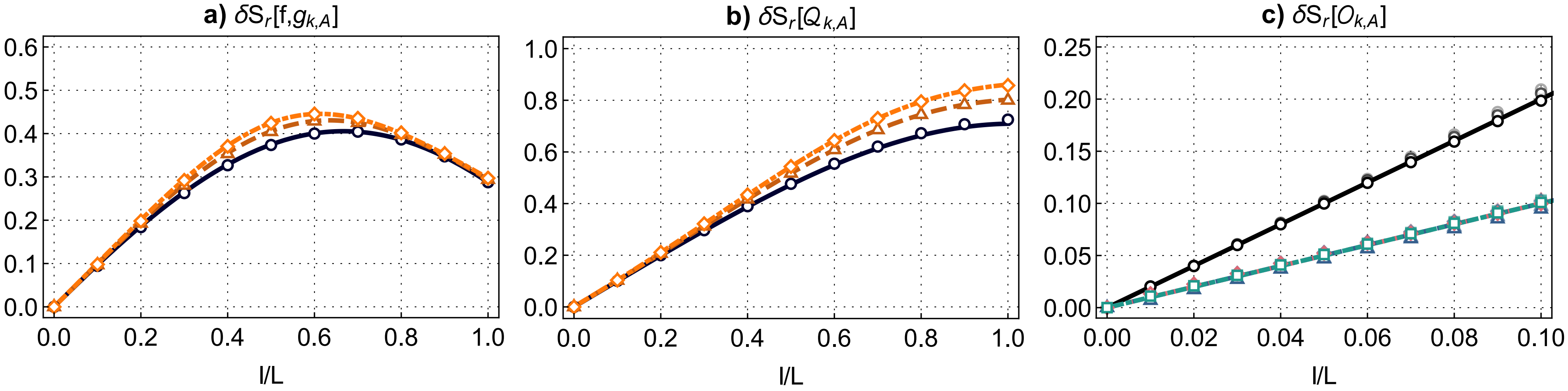
small energies

$$\delta S_2[\mathcal{W}_{k,A}] \to const.$$

Particles – Other quantities II

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

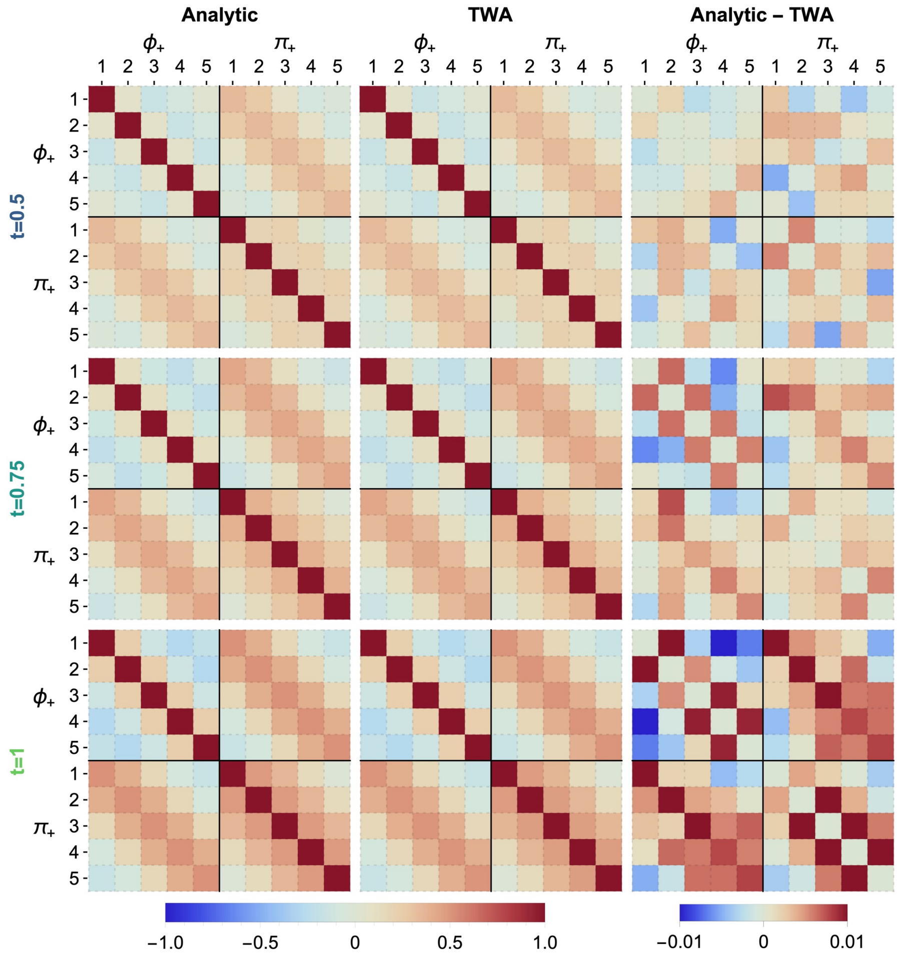
$$\delta S_r[\mathcal{O}_{k,A}] = \frac{1}{1-r} \ln\left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i\right]$$



Parameters

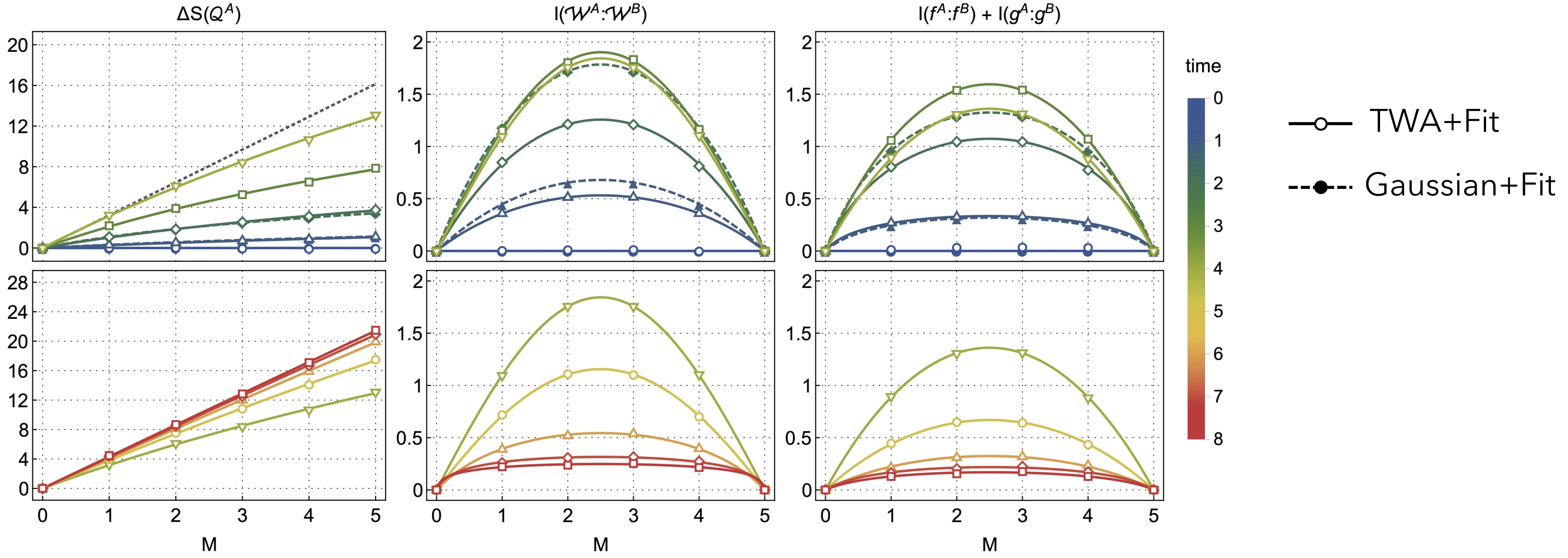
- Experimental values
  - Total system size $N = 20$
  - Subsystem size $M = 5$
  - Energy scale $c_1 = -1/n$
  - Number of atoms $n = 10^3$ per well
  - Spin coupling $c_0 = -2c_1$ (^{7}Li)
  - 2nd order Zeemann shift $q = 2J$
  - Coupling between wells $J = 2$
  - Samples $N_s = 10^4$

Correlation matrices: Gaussian vs. TWA



Time evolution - Other quantities

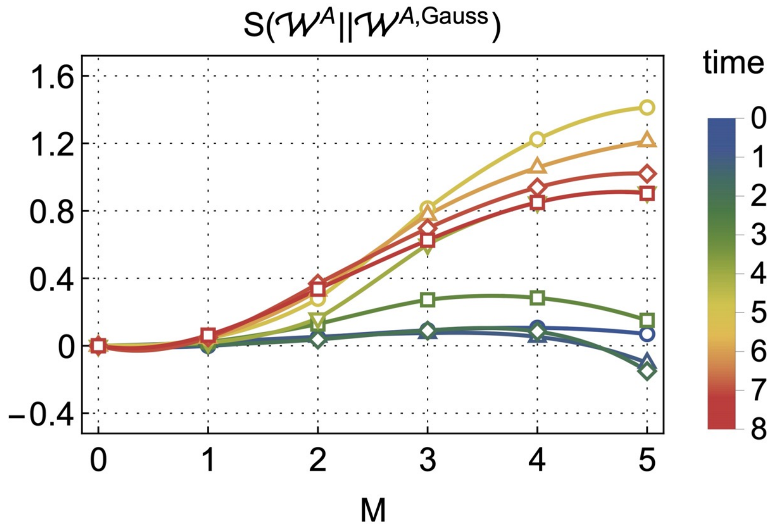
- Early times + late times



Non-Gaussianity

- Relative Wigner entropy

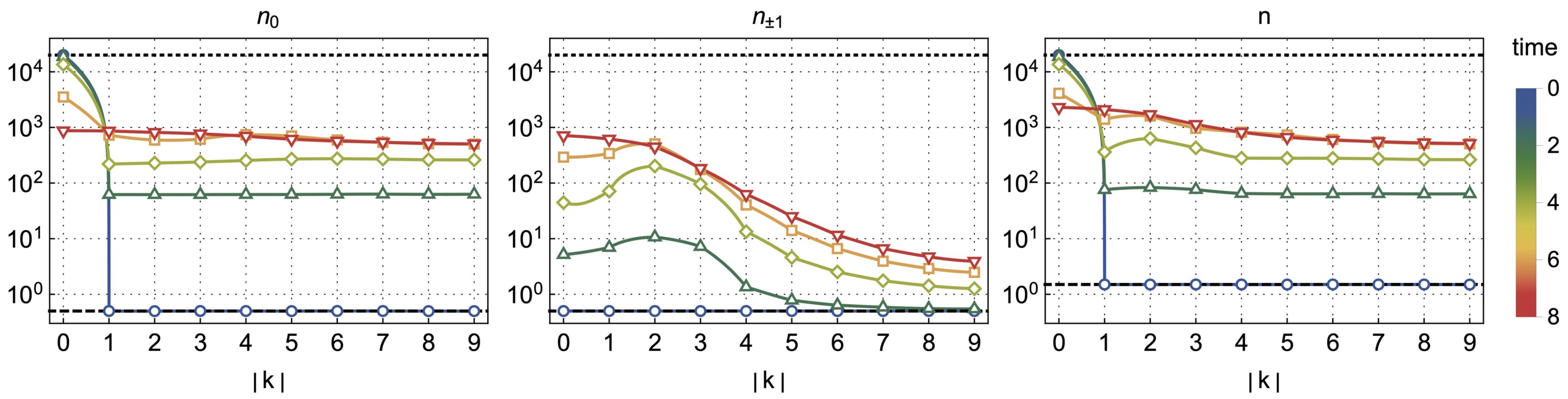
$$S[\mathcal{W}^A \| \mathcal{W}^{A,\text{Gauss}}] = \int \mathcal{D}\nu_A \, \mathcal{W}^A \left(\ln \mathcal{W}^A - \ln \mathcal{W}^{A,\text{Gauss}} \right)$$



→ Non-Gaussian features in higher dimensions

Mode occpuations

- Populations of momentum modes



→ Mesoscopic occupations justify TWA for late times