# Metrological usefulness of entanglement and nonlinear Hamiltonians

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# Motivation

- A central task is to prove *quantum advantages*.
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**Our goal**

1. Characterize useful entanglement via **nonlinear** Hamiltonians.
2. Compare linear and nonlinear **useful** Hamiltonians.

# Multipartite entanglement

An $N$-partite state is *fully separable* if

$$\varrho_{\mathrm{sep}} = \sum_k p_k \, |\phi_k^{(1)}\rangle\langle\phi_k^{(1)}| \otimes \cdots \otimes |\phi_k^{(N)}\rangle\langle\phi_k^{(N)}| \, .$$

Otherwise, it is *multipartite entangled*.

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## Examples of four qubits

GHZ state: $|\mathrm{GHZ}\rangle = |0000\rangle + |1111\rangle$

W state: $|\mathrm{W}\rangle = |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle$

Cluster state: $|\mathrm{CL}\rangle = |0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle$

Singlet state: $|S_4\rangle = |0011\rangle + |1100\rangle - \frac{1}{2}(|10\rangle + |10\rangle) \otimes (|10\rangle + |10\rangle)$

Biseparable state: $|\psi^+\rangle \otimes |\psi^+\rangle$ with $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

# Quantum Fisher information

## The situation

An input state $\varrho$ is transformed by the unitary dynamics

$$\varrho \rightarrow \varrho_\theta = e^{-i\theta H} \varrho e^{+i\theta H}.$$

How fast does $\varrho$ evolve under the unitary generated by a given $H$?

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## Quantum Fisher information

$$F_Q(\varrho, H) = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|H|l\rangle|^2, \quad \varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

This is related to **metrological sensitivity** via Cramér-Rao bound.

# Metrological usefulness

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$$s(H) \equiv \frac{\max_{\varrho} F_Q(\varrho, H)}{\max_{\varrho_{\rm sep}} F_Q(\varrho_{\rm sep}, H)} > 1.$$

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- If $s(H_1) > s(H_2)$, then $H_1$ is **more useful** than $H_2$.

# Several facts

$$s(H) = \frac{\max_{\varrho}\ F_Q(\varrho, H)}{\max_{\varrho_{\rm sep}}\ F_Q(\varrho_{\rm sep}, H)}$$

- **Linear (Local)** Hamiltonians $H_L = \sum_{i=1}^{N} H_i \otimes I_{\bar{i}}$ obey

$$s(H_L) = N.$$

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- **No exact computation** of $s(H_{NL})$, so far.

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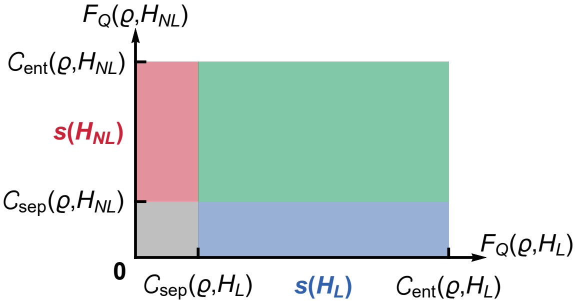
**Q2.** How does Q1 relate to entanglement classification?

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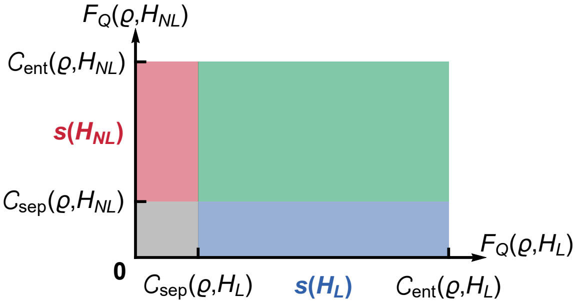


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**Focus on:** $H_L = J_\alpha = \frac{1}{2}\sum_{i=1}^{N} \sigma_\alpha^{(i)}$ and $H_{NL} = J_\alpha^k$.

e.g., $N = 3,\ J_z = \frac{1}{2}(\sigma_z \otimes I \otimes I + I \otimes \sigma_z \otimes I + I \otimes I \otimes \sigma_z)$

# Our result 1: Separability bounds

- Consider an $N$-qubit Hamiltonian $J_\alpha = \frac{1}{2}\sum_{i=1}^{N} \sigma_\alpha^{(i)}$.
- Separability bound: $\mathcal{C}_{\rm sep}(J_\alpha) = N$ Pezzé and Smerzi, PRL 2009

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- Our separability bounds:

$$\mathcal{C}_{\rm sep}(J_\alpha^2) = \frac{(N-1)^3 N}{2(2N-3)},$$

$$\mathcal{C}_{\rm sep}(J_\alpha^3) = \frac{9N^5 - 18N^4 - 120N^3 - 180N^2 - 1020N + c_1 + c_2}{216},$$

where

$$c_1 = \frac{380(164-71N)}{3(N-5)N+20} + \frac{12800(N-1)}{[3(N-5)N+20]^2} - 3084,$$

$$c_2 = 3\sqrt{\frac{N^2[N(N(3N(N(3(N-9)N+128)-360)+1720))-1440)+480]^3}{(N-2)(N-1)[3(N-5)N+20]^4}}.$$

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## Remark

- **Known scaling:** $\mathcal{C}_{\rm sep}(J_\alpha^k) = \mathcal{O}(N^{2k-1})$ ✓

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**Remark**

- **Known scaling:** $\mathcal{C}_{\text{sep}}(J_\alpha^k) = \mathcal{O}(N^{2k-1})$ ✓
- Proof's idea: Lagrange multipliers for symmetric polynomials
- Technical difficulty: **Lack of QFI's additivity**

# Our result 2: Most useful entanglement

- We show that for any direction $\alpha$, the bound

$$\mathcal{C}_{\text{ent}}(J_\alpha^k) = \begin{cases} \frac{N^{2k}}{4^{k-1}}, & \textbf{odd } k, \\ \frac{N^{2k}}{4^k}, & \textbf{even } k, \end{cases}$$

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$$|\Phi\rangle = \sqrt{\lambda_1}\,|\alpha_+\rangle^{\otimes N} + \sqrt{\lambda_2}\,|\alpha_-\rangle^{\otimes N} + \sqrt{1-\lambda_1-\lambda_2}\,|S_N\rangle\,,$$

when $\lambda_1 = \lambda_2 = 1/2$ for **odd** $k$, or $\lambda_1 + \lambda_2 = 1/2$ for **even** $k$. Here $\sigma_\alpha\,|\alpha_\pm\rangle = \pm\,|\alpha_\pm\rangle$, singlet state $U^{\otimes N}\,|S_N\rangle = e^{i\varphi}\,|S_N\rangle$, $\forall U$.

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- **Particularly:** our state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|\mathrm{GHZ}\rangle + |S_N\rangle)$ has

$$F_Q(|\Phi\rangle) = \left(\frac{1}{2}\delta_{k,\textbf{odd}} + \delta_{k,\textbf{even}}\right)\mathcal{C}_{\rm ent}(J_\alpha^k).$$

# Our result 3: Useful Hamiltonians

For the quantity $s(H) = \frac{\max_\varrho F_Q(\varrho, H)}{\max_{\varrho_{\rm sep}} F_Q(\varrho_{\rm sep}, H)}$, we find

$$s(J_\alpha) > s(J_\alpha^3) > s(J_\alpha^2), \quad \forall N \geq 7,$$
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## Interpretations

- $J_\alpha$ is **more metrologically useful** than $J_\alpha^2$ and $J_\alpha^3$, for large $N$.
- The hierarchy $s(J_\alpha^k) > s(J_\alpha^{k+1})$ does not exist.

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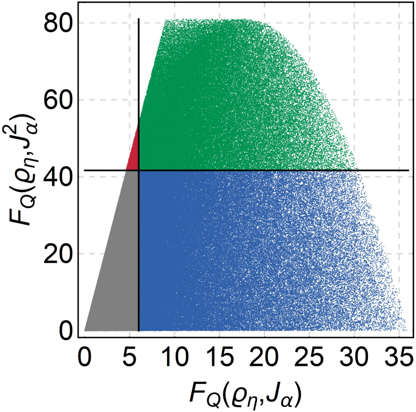
- The hierarchy $s(J_\alpha^k) > s(J_\alpha^{k+1})$ does not exist.
- **Conjecture:** $s(J_\alpha^k) > s(J_\alpha^{k+2})$ exists, for some $k$ and large $N$,

$$s(J_\alpha) > s(J_\alpha^3) > \cdots > s(J_\alpha^{2k+1}) > s(J_\alpha^2) > s(J_\alpha^4) > \cdots > s(J_\alpha^{2k}) \ ?$$

# Our result **4**: Entanglement classification

Consider: $\varrho_\eta = \eta\,|\Phi\rangle\langle\Phi| + \frac{1-\eta}{2^N} I$, for $N = 6$,

where $|\Phi\rangle = \sqrt{\lambda_1}\,|0\rangle^{\otimes N} + \sqrt{\lambda_2}\,|1\rangle^{\otimes N} + \sqrt{1-\lambda_1-\lambda_2}\,|S_N\rangle$.




**Larger $N$, smaller red area!**

# Conclusion

**Summary**

- We proved: $s(J_\alpha) > s(J_\alpha^3) > s(J_\alpha^2)$.
- We showed: $|\Phi\rangle = \frac{1}{\sqrt{2}}(|\mathrm{GHZ}\rangle + |S_N\rangle)$ is very useful.

**Open questions**

- Other computations of $\max_{\varrho_{\rm sep}} F_Q(\varrho_{\rm sep}, H_{NL})$?
- More linear $H$, more useful? Is a positive $H$ not very useful?

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**Open questions**

- Other computations of $\max_{\varrho_{\rm sep}} F_Q(\varrho_{\rm sep}, H_{NL})$?
- More linear $H$, more useful? Is a positive $H$ not very useful?
- **Weak** conjecture: $s(J_\alpha^k) > s(J_\alpha^{k+2})$, for large $N$?
- **Strong** conjecture: $s(H_L^k) > s(H_L^{k+2})$, $\forall H_L$ and large $N$?

# Technical details about separability bounds

$$\mathcal{C}_{\rm sep}(J_\alpha^2) = \max_{\varrho_{\rm sep}} \; F_Q(\varrho_{\rm sep}, J_\alpha^2) = \cdots = \max_{\vec{\alpha}} P(\vec{\alpha}),$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)^T$ with $\alpha_i \in [-1, 1]$ and

$$P(\vec{\alpha}) = \frac{1}{2}\Big[N(N-1) + 2(N-2)\sum_{i\neq j}\alpha_i\alpha_j - \sum_{i\neq j}\alpha_i^2\alpha_j^2 - 2\sum_{i\neq j\neq k}\alpha_i^2\alpha_j\alpha_k\Big].$$

Using the Lagrangian multiplier

$$\mathcal{L}(\vec{\alpha}, \kappa_1, \kappa_2, \kappa_3) = P(\vec{\alpha}) + \sum_{m=1,2,3}\kappa_m\left(\sum_{i=1}^{N}\alpha_i^m - p_m\right),$$

we showed that the maximal $P(\vec{\alpha})$ can be attained **only** by the symmetric case $\vec{\alpha}_* = (\alpha_*, \ldots, \alpha_*)$ with $\alpha_* = \pm\sqrt{\frac{(N-2)}{(2N-3)}}$.