Estimation of entanglement monotones in permutationally invariant spin systems

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Outline

- Entanglement detection and quantification
  - Entanglement witnesses
  - Entanglement monotones → bounds
- Permutationally invariant spin systems
- Methods and results
  - Case study: XXX model
  - Other models and generalizations
- Outlook

Problem statement

Given a (generally non-pure) **thermal state** of N spin-$\frac{1}{2}$ particles

$$\varrho = \frac{e^{-H/T}}{Z}$$

Is this state entangled, i.e. it cannot be written as

$$\varrho_{\mathrm{sep}} = \sum_k p_k \left(|\uparrow\uparrow\uparrow\uparrow \cdots\rangle\langle\uparrow\uparrow\uparrow\uparrow \cdots| \otimes \cdots \otimes |\downarrow\downarrow\uparrow\uparrow \cdots\rangle\langle\downarrow\downarrow\uparrow\uparrow \cdots| \right)$$

with $\sum_k p_k = 1$, and, if so, **how much entangled** is it?

Motivation

(Multipartite) entanglement is interesting:

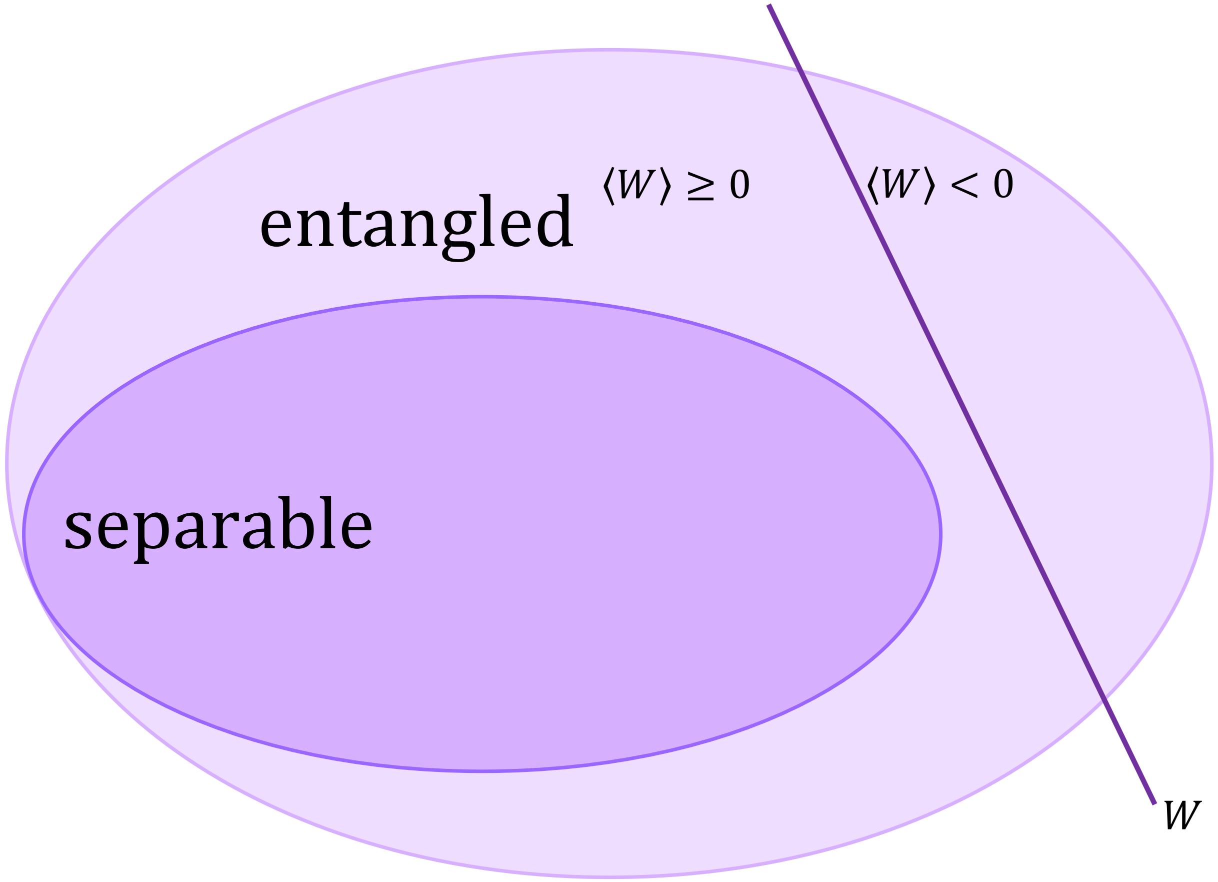
- Study of quantum correlations and complex phases
- Implementations in quantum simulation, sensing, communication

→Need for entanglement detection and quantification

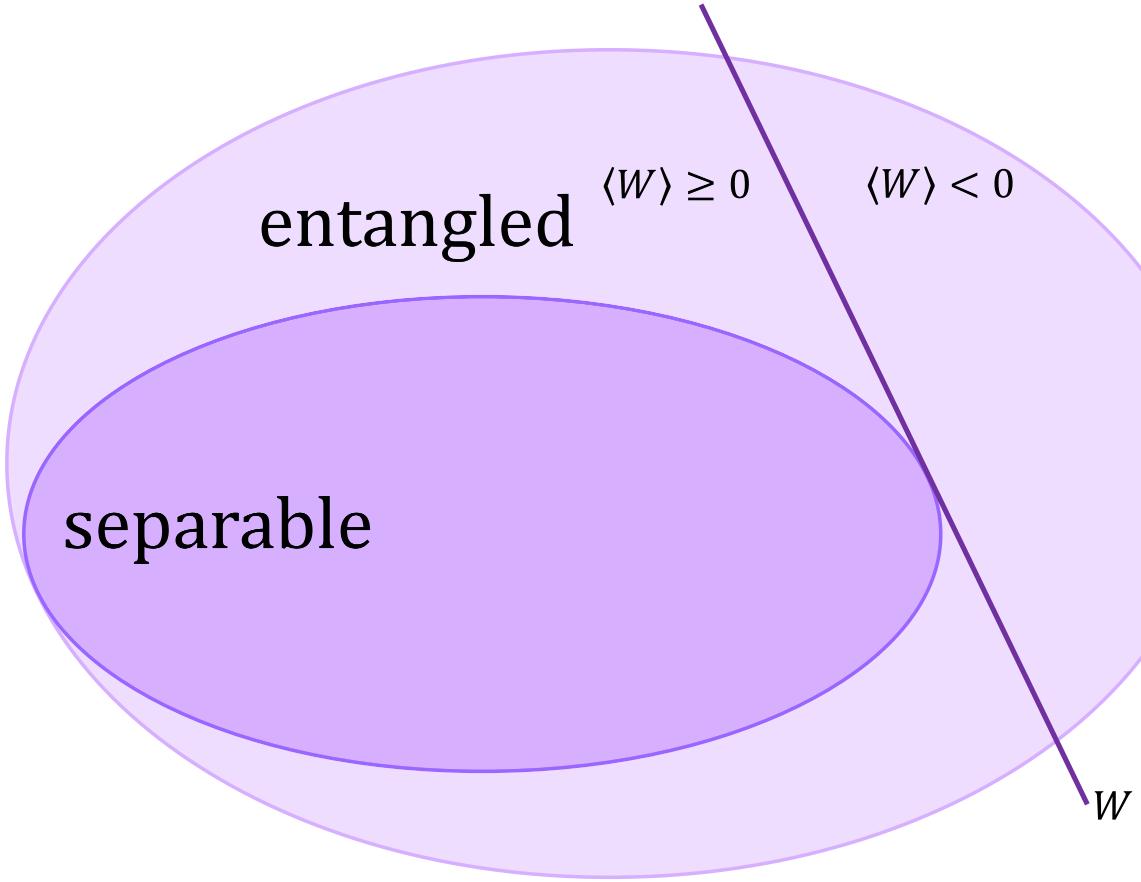
In most systems full-state tomography is not possible

→ Methods based on *entanglement witnesses*

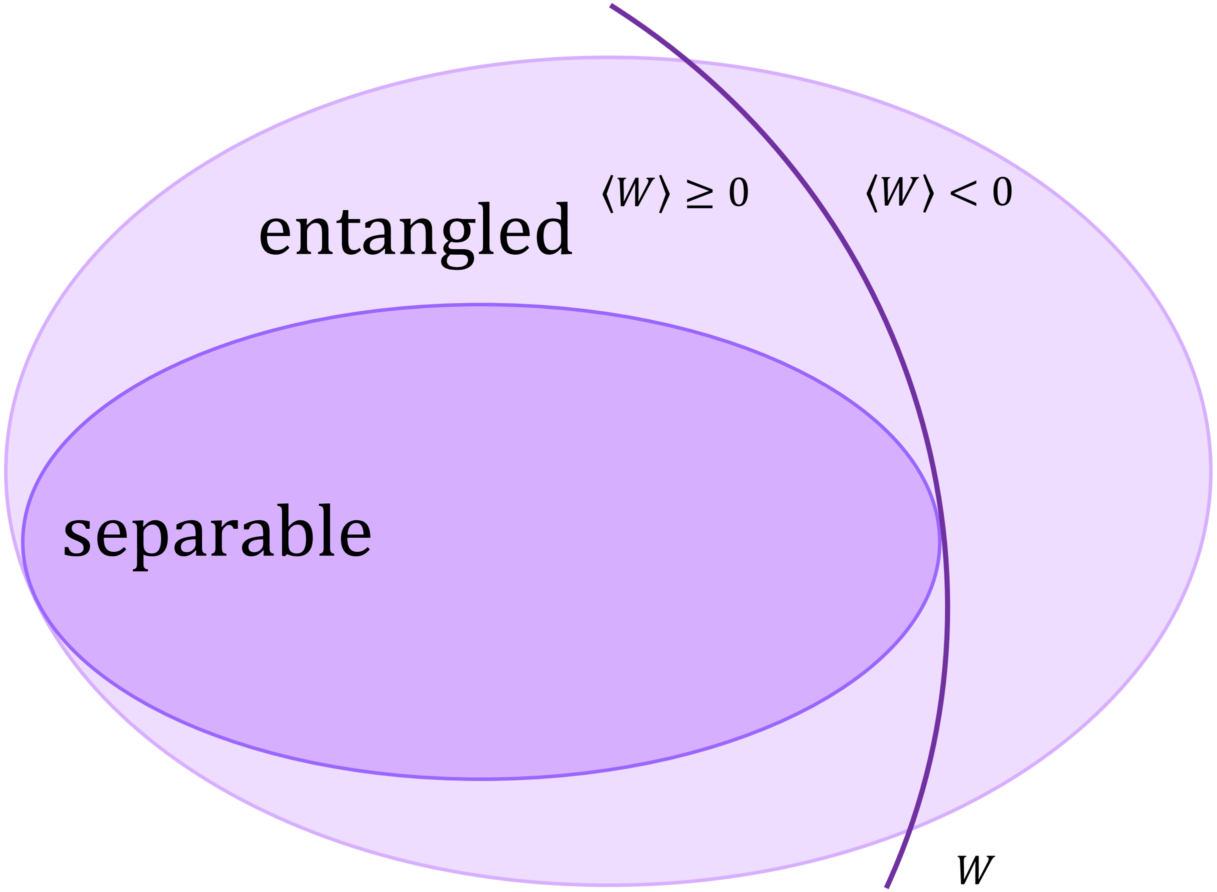
Entanglement witnesses



Optimal entanglement witnesses



Nonlinear optimal entanglement witnesses



Variance-based optimal entanglement witnesses

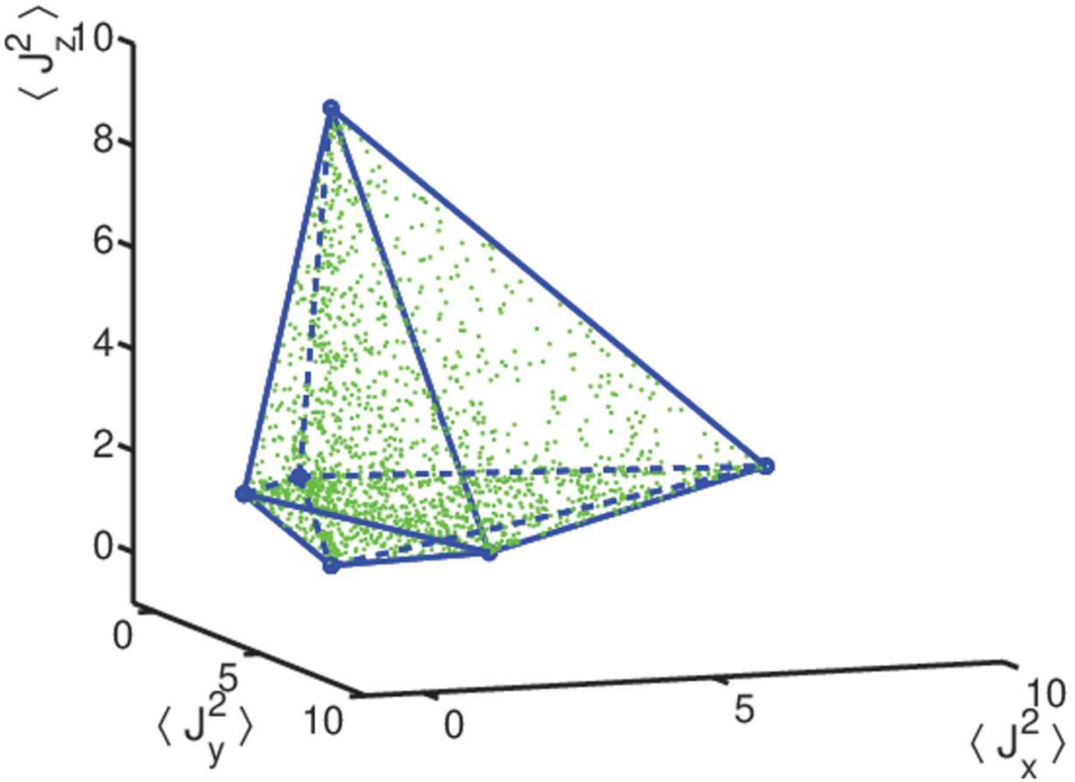
→Spin-squeezing inequalities (SSIs) [1]:

$$\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle \leq N(N+2)/4$$

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq N/2$$

$$(N-1)(\Delta S_k)^2 - \langle S_l^2\rangle - \langle S_m^2\rangle \geq -N/2$$

$$(N-1)[(\Delta S_k)^2 + (\Delta S_l)^2] - \langle S_m^2\rangle \geq N(N-2)/4$$



[1] G. Tóth et al., Phys. Rev. A 79, 042334 (2009)

Witness-based entanglement monotones

Non-negative real functions of density matrices that are zero for all separable states σ and non-increasing under LOCC:

$$\varepsilon_M = \max\{0, -\min_{W \in M} \mathrm{tr}\,(W\varrho)\} \quad [2]$$

Example: Best Separable Approximation (BSA)

$$BSA(\varrho) = \min(t) : \varrho = (1-t)\sigma + t\,\nu, \quad t \in [0,1]$$

target state (ϱ) — separable (σ) — entangled (ν)

→Special cases:

$$t = 0 \rightarrow \varrho = \sigma \quad \text{and} \quad t = 1 \rightarrow \varrho = \nu$$

[2] F. G. S. L. Brandao, Phys. Rev. A 72, 022310 (2005)

Bounds on the BSA

$$BSA(\varrho) = \min(t) : \varrho = (1-t)\sigma + t\,\nu, \quad t \in [0,1]$$

For a thermal state $\varrho = \frac{e^{-H/T}}{Z}$ we can find

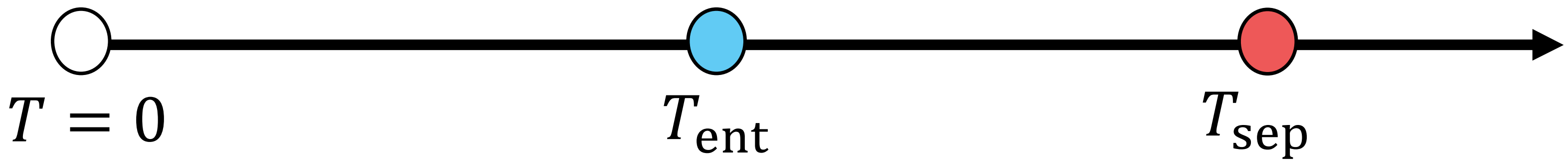
- A lower bound based on an entanglement witness

$$BSA(\varrho) \geq \max\{0, -\langle W \rangle\}$$

- An upper bound based on a separable ansatz state

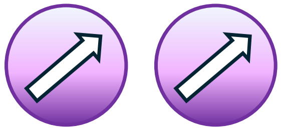
$$BSA(\varrho) \leq BSA(\sigma)$$

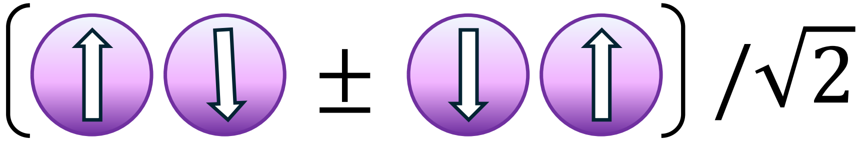
$$\sigma = \sum_k p_k \left(|\uparrow\uparrow\uparrow\uparrow \cdots\rangle\langle\uparrow\uparrow\uparrow\uparrow \cdots| \otimes \cdots \otimes |\downarrow\downarrow\uparrow\uparrow \cdots\rangle\langle\downarrow\downarrow\uparrow\uparrow \cdots| \right)$$



Permutationally invariant spin states

Example:

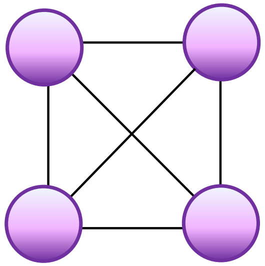




Permutational invariance (PI): $U_\pi \varrho U_\pi^\dagger = \varrho$ with $\pi \in \mathfrak{S}_N$

Relevance: Thermal states of fully connected spin systems

Case study: Fully connected XXX model



$$H_{\mathrm{XXX}} = g\left(S_x^2 + S_y^2 + S_z^2\right)$$

with $S_k = \sum_{n=1}^{N} \frac{1}{2} \sigma_k^{(n)}$ and $g = 1$ in the upcoming calculations

Interlude: Schur-Weyl duality

For $A^{\otimes N}$ acting on $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ we can factorize the Hilbert space

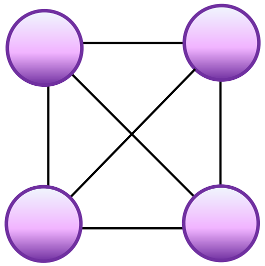
$$\mathcal{H}_N = \bigoplus_{S=0}^{N/2} \mathcal{H}_S \otimes \mathbb{C}^{\mu_S}$$

where $\mathcal{H}_S = \mathbb{C}^{2S+1}$ is the Hilbert space of a single spin- S particle and μ_S is the spin multiplicity.

With this, our density matrices also factorize

$$\varrho = \bigoplus_{S=0}^{N/2} p_S \, \varrho_S \otimes \frac{1}{\mu_S} \mathbb{1}_{\mu_S}$$

Case study: Fully connected XXX model



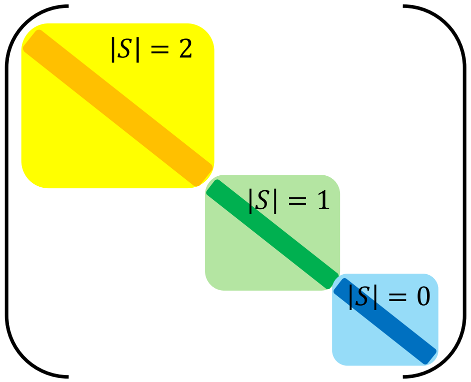
$$H_{\mathrm{XXX}} = g\left(S_x^2 + S_y^2 + S_z^2\right) = \bigoplus_{S=0}^{N/2} S(S+1)\,\mathbb{1}_S \otimes \mathbb{1}_{\mu_S}$$

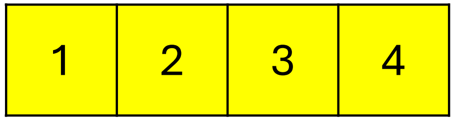
with $S_k = \sum_{n=1}^{N} \frac{1}{2} \sigma_k^{(n)}$ and $g = 1$ in the upcoming calculations

The $S_k = \sum_{n=1}^{N} \frac{1}{2} \sigma_k^{(n)}$ split into the sectors $\mathcal{H}_S$ of the associated irrep of $SU(2)$ that contains PI states belonging to the representation of the symmetric group S_N.

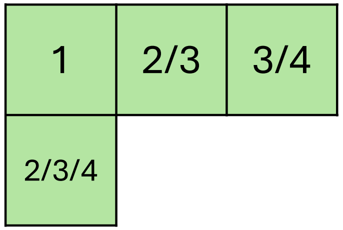
Example: $N = 4$

Note:
$\mathcal{H}_\lambda \leftrightarrow \mathcal{H}_S$

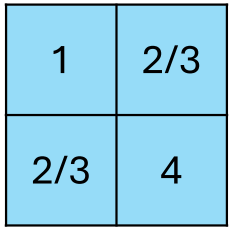




$|S| = 2$
$\lambda = \{4\}$

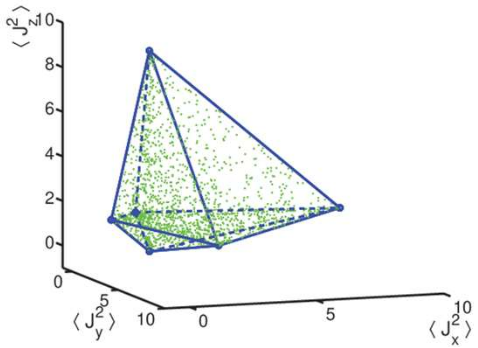


$|S| = 1$
$\lambda = \{3,1\}$



$|S| = 0$
$\lambda = \{2,2\}$

Lower bound: BSA for H_{XXX}



With $W = \langle H \rangle - N/2$:

normalized witness

$$BSA(\varrho) \geq \max\{0, -2W/N\}$$

Interlude: calculations for large N

$$Z(T) = \sum_{S=0}^{N/2} \mu_S (2S+1) e^{-gS(S+1)/(TN)} \qquad \langle H \rangle = \sum_{S=0}^{N/2} \mu_S (2S+1) S(S+1) e^{-gS(S+1)/(TN)}$$

$$= -\frac{N}{g} \partial_\beta \log\big(Z(\beta)\big), \beta = 1/T$$

Multiplicity:

$$\mu_S = \binom{N}{N/2-S} - \binom{N}{N/2-S-1}$$

$$\mu_x \approx (1+2x)\mathrm{e}^{-2x(x+1)/N} \text{ [3]}$$

$$\rightarrow Z(\mathrm{T}) = \int_0^\infty dx\,(2x+1)\mu_x e^{-gx(x+1)/(TN)}$$

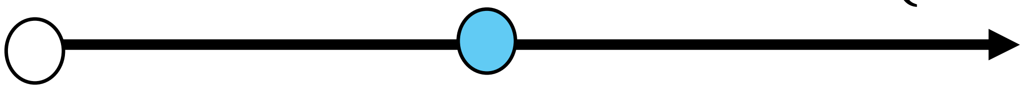
[3] T. Curtright et al. , Phys. Lett. A 381 5, 0375-9601 (2017)

Lower bound: BSA for H_{XXX}

With $W = \langle H \rangle - N/2$:

$$BSA(\varrho) \geq \max\{0, -2W/N\}$$

$$\rightarrow \; BSA(T) \geq \max\left\{0, 1 - \frac{3}{g/T + 2}\right\}$$



$T = 0$:
$BSA(0) = 1$

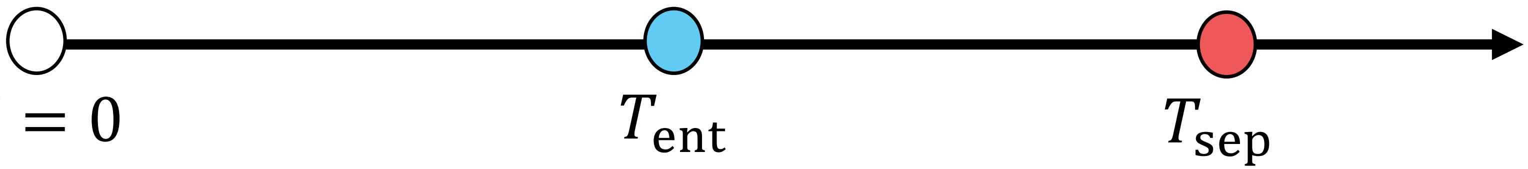
$T_{\mathrm{ent}}/N = g$:
$BSA(T_{\mathrm{ent}}) = 0$

Analogously for $H_{\mathrm{XX}} = g\left(S_x^2 + S_y^2\right)$: $T_{\mathrm{ent}}/N = g/2$

Check: [1] G. Tóth et al., Phys. Rev. A 79, 042334 (2009)

Upper bound: separable state ansatz

Idea:

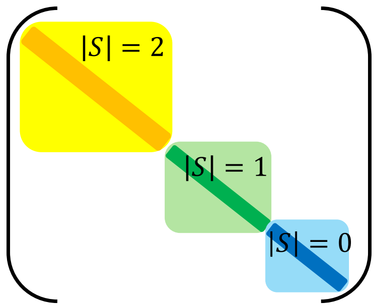


Ansatz based on states that are both PI and separable:

$$\sigma = \sum_k p_k \left(|\uparrow\uparrow\uparrow\uparrow \cdots\rangle\langle\uparrow\uparrow\uparrow\uparrow \cdots| \otimes \cdots \otimes |\downarrow\downarrow\uparrow\uparrow \cdots\rangle\langle\downarrow\downarrow\uparrow\uparrow \cdots| \right)$$

Target state: $\varrho = \frac{e^{-H/T}}{Z}$ diagonal in basis of global spin $|S\ S_z\ \mu_S\rangle$

→ Ansatz states should also be diagonal



How to obtain separable PI ansatz states I

Separable PI states can be written as

$$\varrho_{\mathrm{PI}}^{\mathrm{sep}} = \frac{1}{N!} \sum_{\pi \in \mathfrak{S}_N} U_\pi \left(\sum_k p_k \left(|\uparrow\uparrow \cdots\rangle\langle\uparrow\uparrow \cdots| \otimes \cdots \otimes |\downarrow\uparrow \cdots\rangle\langle\downarrow\uparrow \cdots| \right)_k \right) {U_\pi}^\dagger$$

Most general case: map the boundary of the set of separable states (spanned by pure states) into PI states:

$$\varrho_{\mathrm{PI}}^{\mathrm{boundary}} = \frac{1}{N!} \sum_{\pi \in \mathfrak{S}_N} U_\pi \left(|\uparrow\uparrow \cdots\rangle\langle\uparrow\uparrow \cdots| \otimes \cdots \otimes |\downarrow\uparrow \cdots\rangle\langle\downarrow\uparrow \cdots| \right) {U_\pi}^\dagger$$

How to obtain separable PI ansatz states II

→ For our PI model, we only need to find the coefficients corresponding to operators in each $\mathcal{H}_S$.

For an ansatz of product states of N single particle s_z states and a corrsponding S_z they are

$$\alpha_{S,S_z}^{m_1,\ldots,m_N} = \sum_{i_S} |\langle\, m_1, \ldots, m_N | S, S_z, i_S \rangle|^2$$

which can be obtained from combinatorics and d-dimensional SU(2) generators (or, equivalently, the Schur matrix).

Note: Equivalent method

Method: Write ansatz states as *Werner states* [4]:

coordinates of product states in spin subspaces

$$\alpha_\lambda = \frac{1}{\dim(\mathcal{H}_\lambda)} \frac{d(\lambda)}{N!} \mathrm{Imm}_\lambda G(\psi_1, \dots, \psi_N)$$

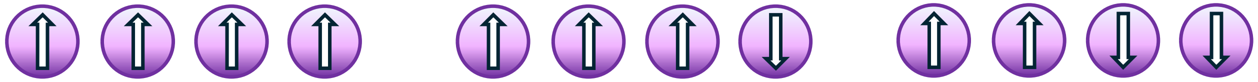
$$\varrho_W = U^{\otimes N} \varrho_W \left(U^{\otimes N}\right)^\dagger = \sum_{\pi \in \mathfrak{S}_\mathfrak{N}} \alpha_\pi U_\pi = = \sum_\lambda \alpha_\lambda P_\lambda$$

projectors into spin subspaces

central young projectors

We can characterize product states patterns using *Gram matrices*.

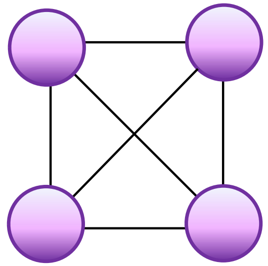
Example: $H_{XXX}, N = 4$:



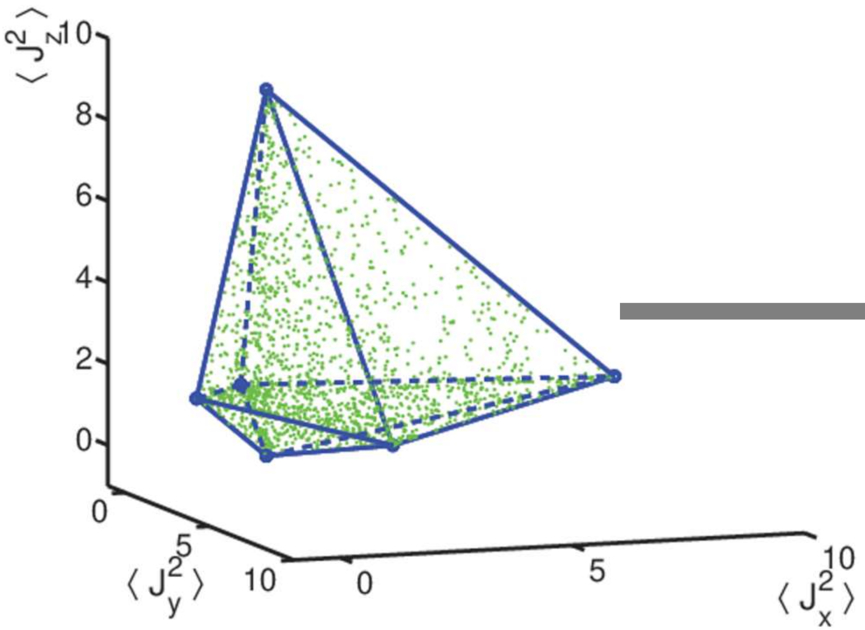
$$\{G_1, G_2, G_3\} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

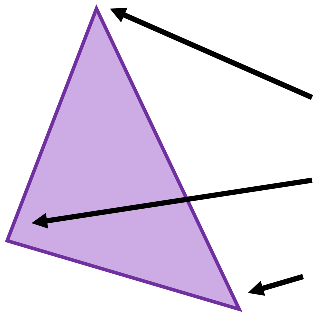
[4] R. F. Werner, Phys. Rev. A 40, 4277 (1989)

Case study: XXX model



$$\sigma^i = \frac{1}{2^N}\left(1 - \sum_{i=1}^{N/2+1} p_i\right) + \sum_{\lambda=1}^{N/2+1} p_i\, \alpha_\lambda^i$$





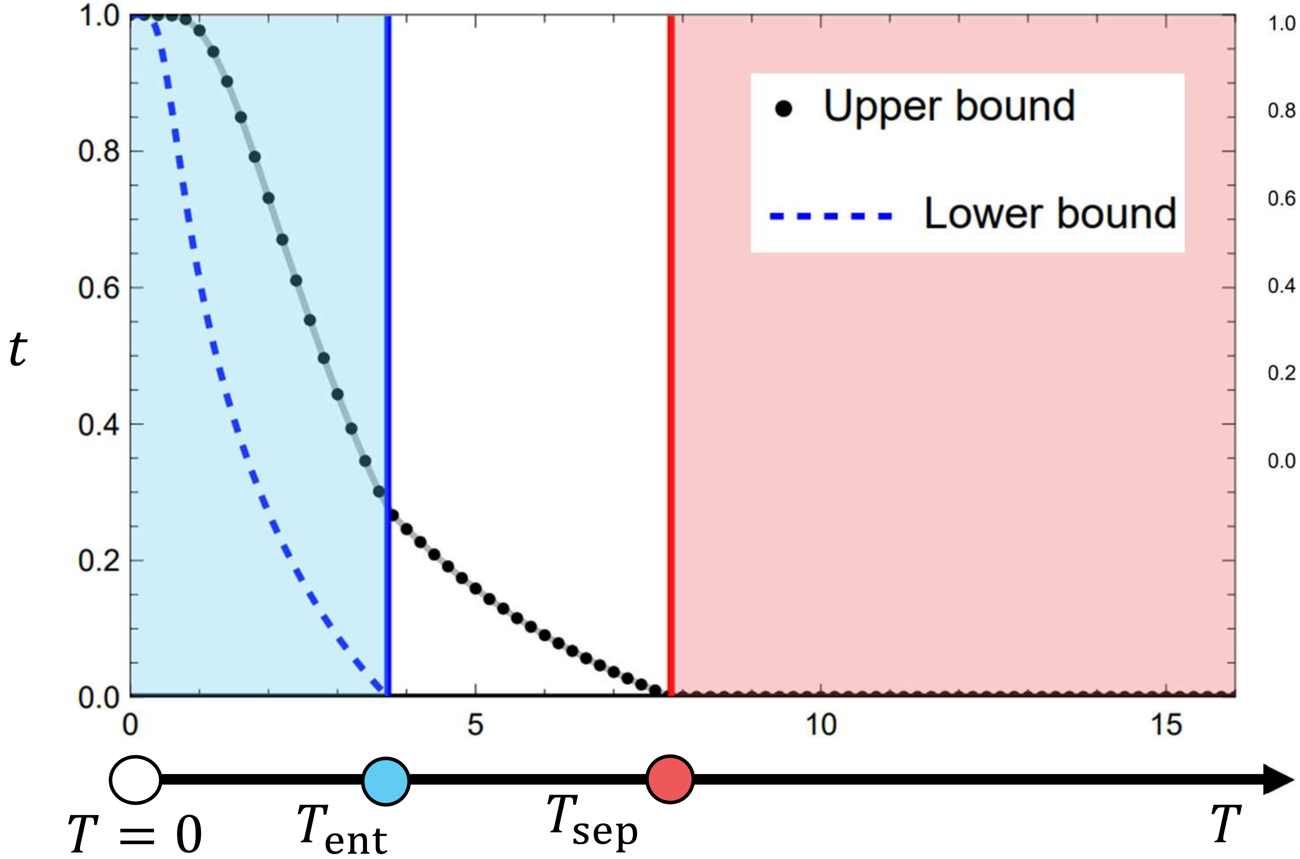
$(N = 4)$:

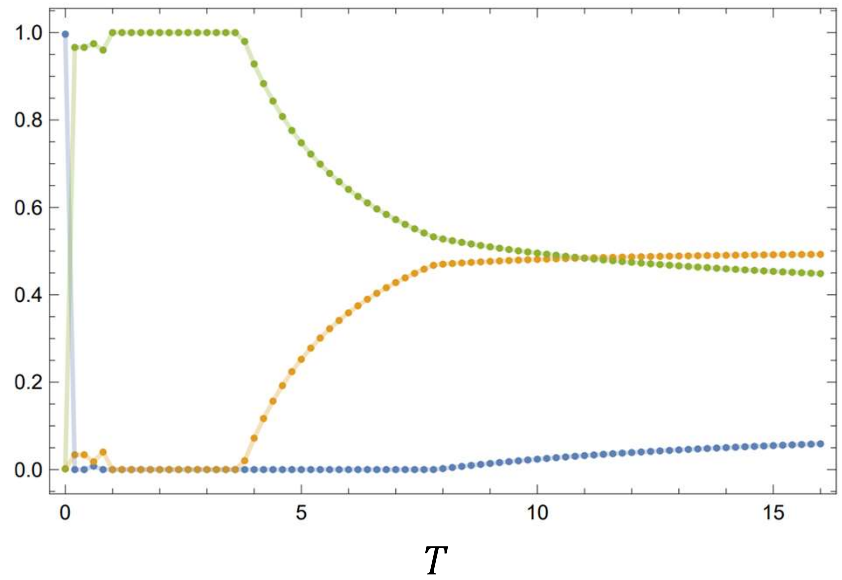
$\alpha_{\lambda=\{4\}} = \{1/5, 1/20, 1/30\}$

$\alpha_{\lambda=\{3,1\}} = \{0, 1/18, 1/12\}$

$\alpha_{\lambda=\{2,2\}} = \{0, 0, 1/6\}$

BSA for the XXX model ($N = 4$)



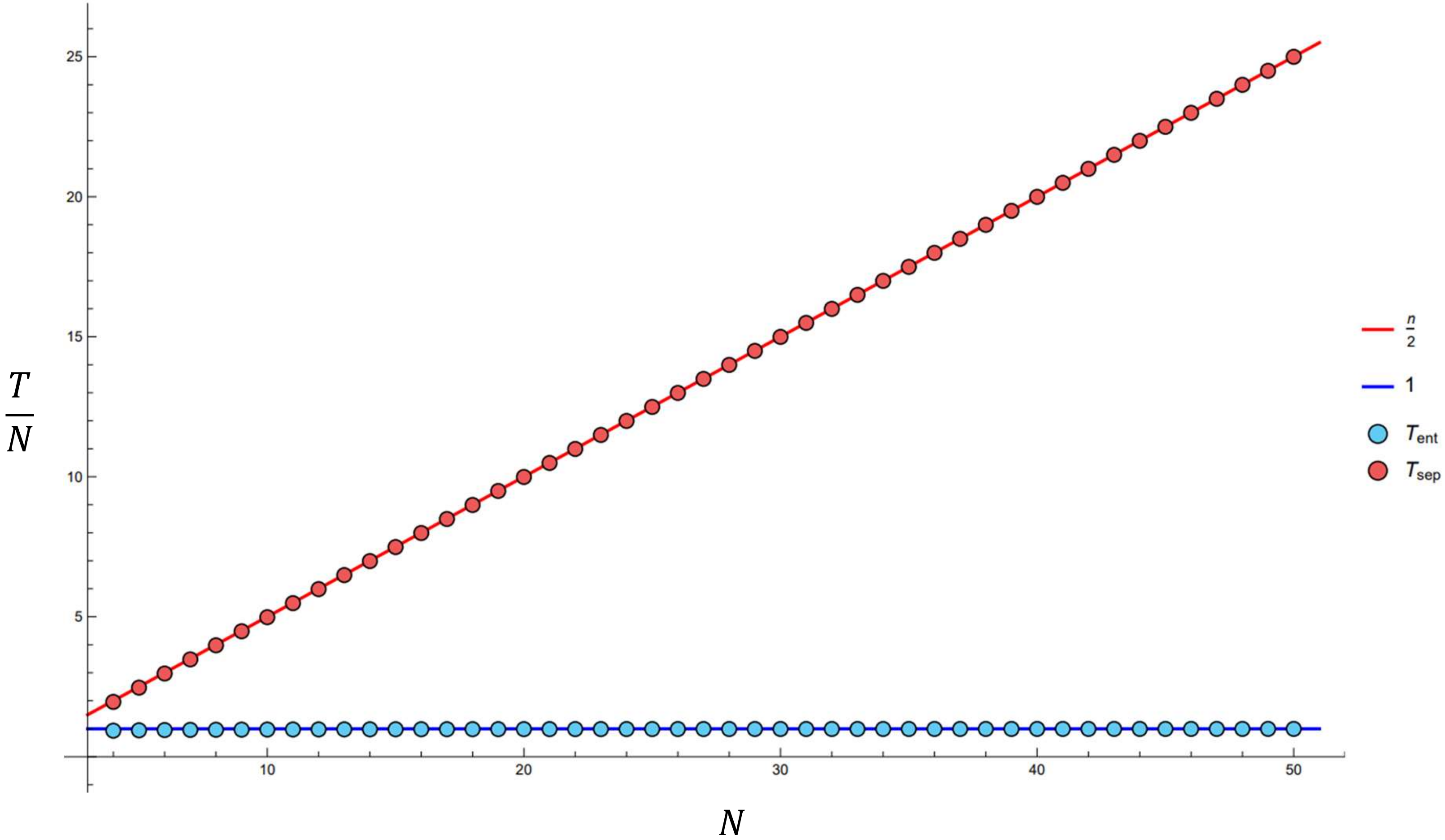


p_1 : double excitation

p_2 : single excitation

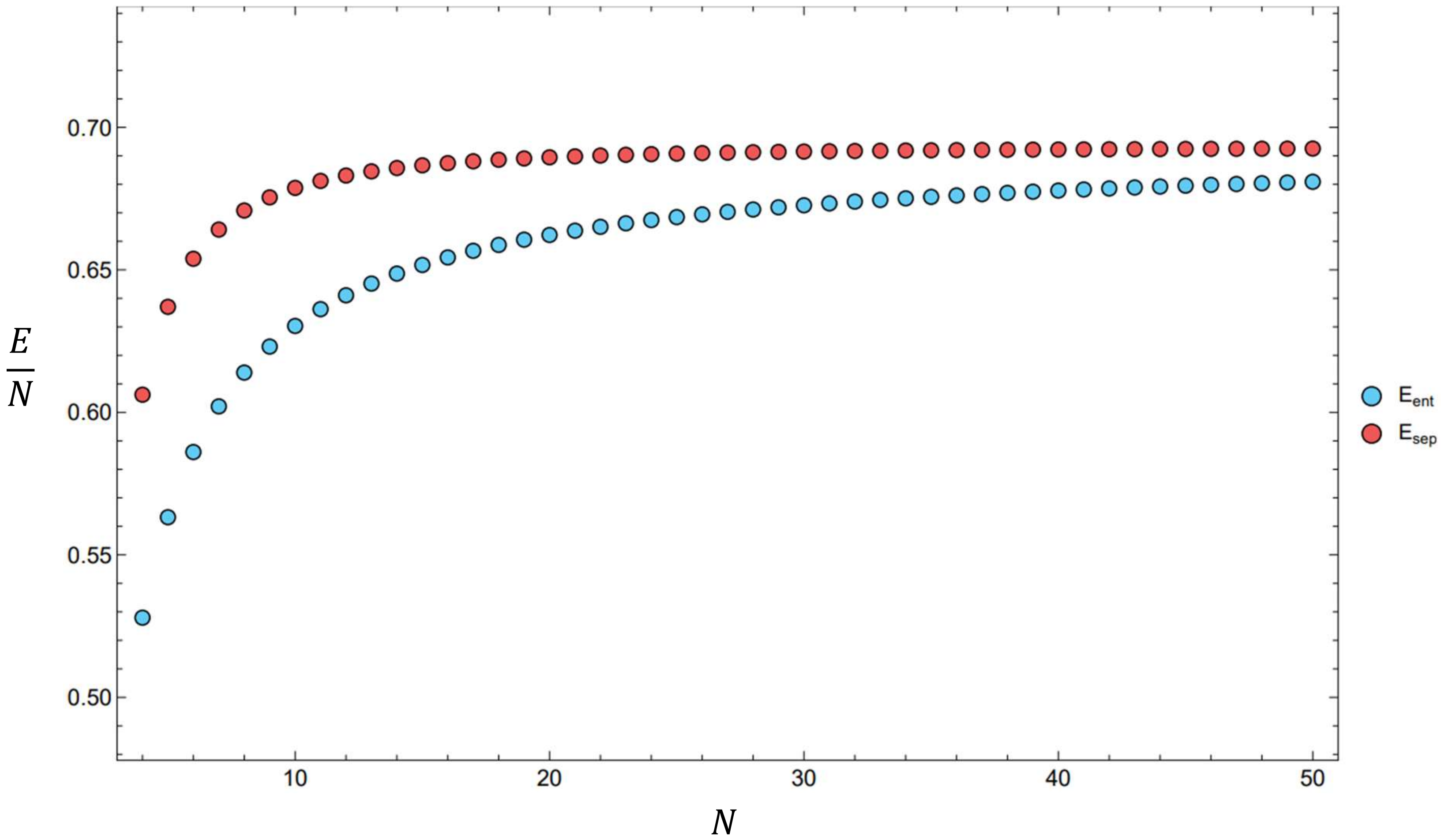
p_3 : no excitation

Results for the XXX model: $\frac{T}{N}$



Results for the XXX model: $\frac{E}{N}$

$$E = \langle H \rangle / T + \log(Z)$$



Other preliminary results and extensions

With this mechanism we can also obtain ...

- Bounds for other models
  - XX model: $H_{\mathrm{XX}} = g\left(S_x^2 + S_y^2\right)$
  - XXZ model: $H_{\mathrm{XXZ}} = g\left(S_x^2 + S_y^2\right) + hS_z^2$

  $$\frac{T_{\mathrm{ent}}}{N} = c_1, \frac{T_{\mathrm{sep}}}{N} \sim c_2 N$$

  - Models undergoing some dynamics, e. g. $H(\tau) = g\left(S_x^2 + S_y^2\right) + h(\tau)S_z^2$
- Bounds for other entanglement monotones
  - Similar results for GR
- Generalizations
  - Entropy formulation
  - More general method to obtain lower bound
  - Mixed states

Outlook

We are also looking at

- Bounds for more complicated models (e. g. XYZ) and entanglement measures (relative entropy, geometric measure of entanglement)
- Extensions to non-Hermitian observables:

$$(N-1)\sum_{k\in I}\left(\tilde{\Delta}\mathrm{J}_k(q_k)\right)^2-\sum_{k\notin I}\left\langle\widetilde{\mathrm{J}_k^2}(q_k)\right\rangle+N(N-1)\mathrm{j}^2\geq 0$$

$$\mathrm{J}_k^{(m)}(q_k)=\sum_{n=1}^{k_m}e^{-iq_kn}\mathrm{j}_k^{(n)},\qquad \left\langle\widetilde{\mathrm{J}_k^2}(q_k)\right\rangle:=\langle\mathrm{J}_k^2(q_k)\rangle-\sum_n\left\langle\left(\mathrm{j}_k^{(n)}\right)^2\right\rangle=\sum_{n\neq m}e^{iq_k(n-m)}\langle\,\mathrm{j}_k^{(n)}\mathrm{j}_k^{(m)}\rangle$$

Summary: Bounds on entanglement monotones

$$\varepsilon_M = \max\{0, -\min_{W \in M} \mathrm{tr}\,(W\varrho)\}$$

For a thermal state $\varrho = \frac{e^{-H/T}}{Z}$ we can find

- A lower bound based on an entanglement witness

$$\varepsilon_M(\varrho) \geq \max\{0, -\langle W \rangle\}$$

- An upper bound based on a separable ansatz state

$$\varepsilon_M(\varrho) \leq \varepsilon_M(\sigma)$$

$$\sigma = \sum_k p_k \left(|\uparrow\uparrow\uparrow\uparrow \cdots\rangle\langle\uparrow\uparrow\uparrow\uparrow \cdots| \otimes \cdots \otimes |\downarrow\downarrow\uparrow\uparrow \cdots\rangle\langle\downarrow\downarrow\uparrow\uparrow \cdots|\right)$$