# Geometry and dynamics of non-Hermitian Hamiltonians

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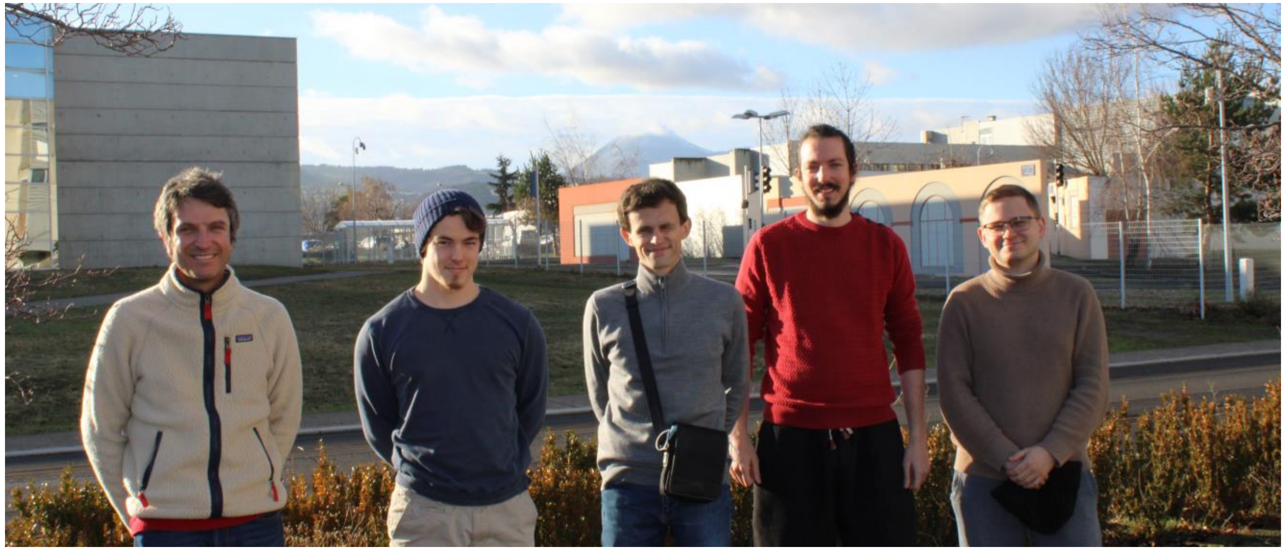
# Acknowledgments



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# Outline

- Introduction
  - Non-Hermitian Hamiltonians
  - Quantum geometric tensor
- Quantum geometry and dynamics of the ring of exceptional points
  - RR/LR quantum geometric tensor
  - Trivial/non-trivial non-Hermitian drift
- Conclusion

# Non-Hermitian Hamiltonians

For non-Hermitian Hamiltonians $H \neq H^\dagger$, left and right eigenvectors are different

$H|\psi\rangle = E|\psi\rangle$ and $\langle\psi|H^\dagger = E^*\langle\psi|$

$H^\dagger|\phi\rangle = E^*|\phi\rangle$ and $\langle\phi|H = E\langle\phi|$

with $\langle\psi_n|\psi_{n'}\rangle \neq \delta_{nn'}$

$\langle\phi_n|\psi_{n'}\rangle = \delta_{nn'}$ → **Biorthogonal quantum mechanics**

D. C. Brody, *J. Phys. A: Math. Theor.* **47**, 035305 (2014)

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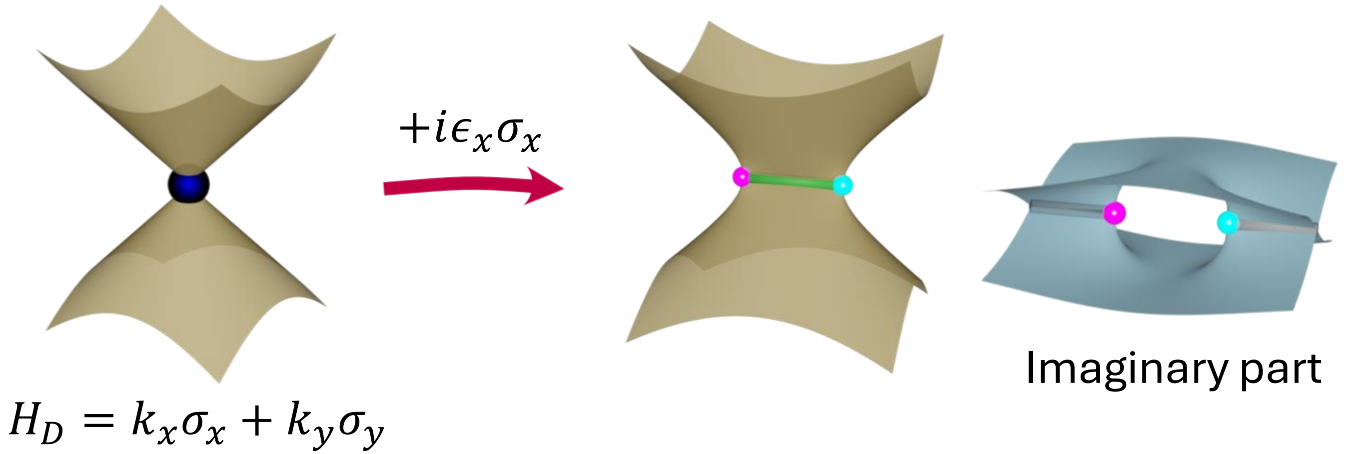
$$H = \varepsilon_0 I + (\boldsymbol{\varepsilon} + i\boldsymbol{\gamma}) \cdot \boldsymbol{\sigma} \quad (\varepsilon, \gamma \in \mathbb{R})$$

$H = H_r + iH_i$ with $H_r^\dagger = H_r$ and $H_i^\dagger = H_i$

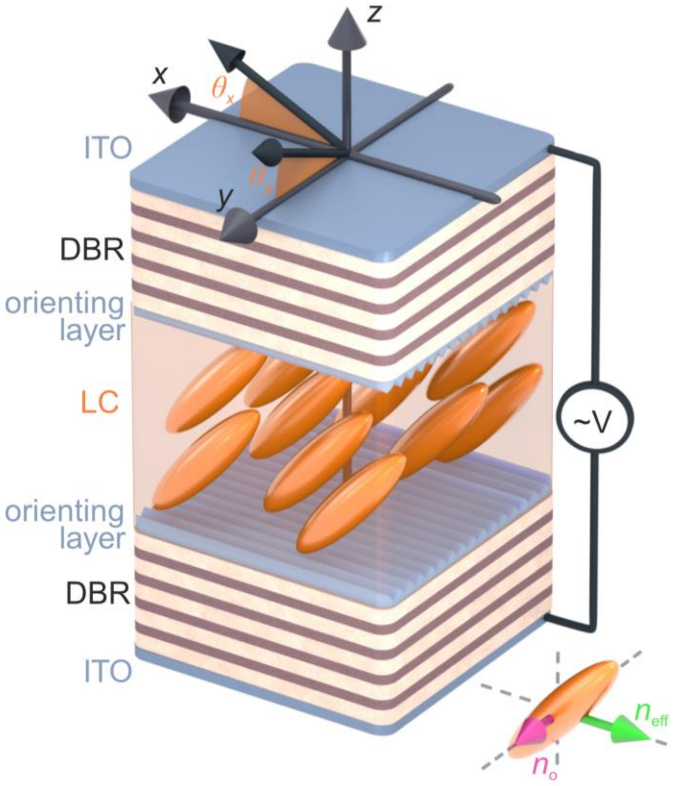
$$\frac{\partial\rho}{\partial t} = \frac{i}{\hbar}([\rho, H_r] - \{\rho, \Gamma_i\})$$




1 Dirac point (Hermitian) 2 exceptional points (non-Hermitian)

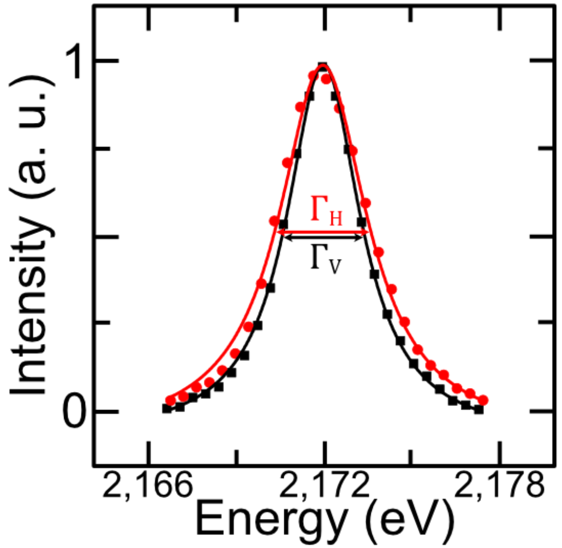
Review: E. Bergholtz *et al.*, *Rev. Mod. Phys.* **93**, 015005 (2021)

# Non-Hermiticity in photonics

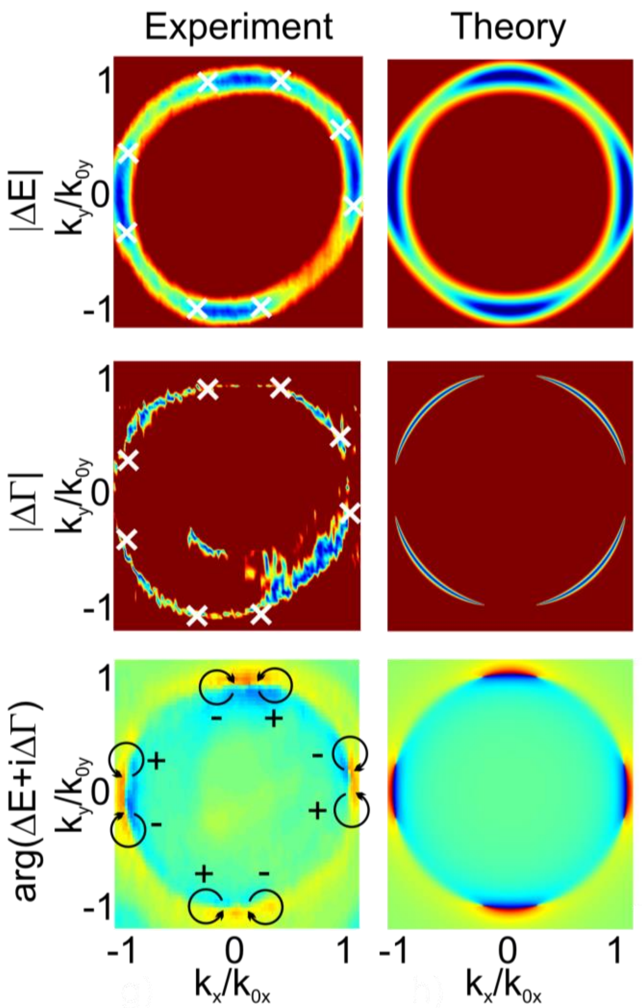



$$H_{\boldsymbol{k}}^{real} = \begin{pmatrix} \frac{E_H^{N+2}+E_V^N}{2} + \frac{\hbar^2 k_x^2}{2m_x} + \frac{\hbar^2 k_y^2}{2m_y} & \Delta - \beta' k^2 - \beta(k_x - ik_y)^2 \\ \Delta - \beta' k^2 - \beta(k_x + ik_y)^2 & \frac{E_H^{N+2}+E_V^N}{2} + \frac{\hbar^2 k_x^2}{2m_x} + \frac{\hbar^2 k_y^2}{2m_y} \end{pmatrix}$$

Circular polarisation basis




⇒ Polarization-dependent losses
$\leftrightarrow +i\epsilon_x\sigma_x$
⇒ Exceptional points!




M. Krol, IS *et al.*, *Nat. Commun.* **13**, 5340 (2022)

# Quantum geometric tensor

A quantum system lives on a Kähler manifold

- Complex
- Riemannian → Kähler metric
- Symplectic → Kähler form

*Commun. Math. Phys.* **76**, 289 (1980)

Quantum geometric tensor $T_{jk}^{n} = g_{jk}^{n} + i\Omega_{\mathrm{jk}}^{\mathrm{n}}/2$

$$T_{jk}^{n} = \left\langle \frac{\partial\psi_n}{\partial\lambda_j} \middle| \frac{\partial\psi_n}{\partial\lambda_k} \right\rangle - \left\langle \frac{\partial\psi_n}{\partial\lambda_j} \middle| \psi_n \right\rangle \left\langle \psi_n \middle| \frac{\partial\psi_n}{\partial\lambda_k} \right\rangle$$

Kähler metric: Fubini-Study metric

- Fubini–Study metric = Kähler metric on $\mathbf{CP}^n$ ($\mathbf{CP}^1$ = Bloch sphere)
- Quantum distance:

$$ds^2 = g_{jk} d\lambda_j d\lambda_k = 1 - |\langle\psi(\lambda)|\psi(\lambda+\delta\lambda)\rangle|^2$$

Kähler form: Berry curvature

- Integrated, gives the Chern number (topology)
- Influence the dynamics: G. Sundaram and Q. Niu, *Phys. Rev. B* **59**, 14915 (1999)

Recent developments

- Explains superfluidity of flat bands *Phys. Rev. Lett.* **117**, 045303 (2016)
- Can be measured by spectroscopy (in photonics) *Nature* **578**, 381 (2020)
- Fully geometrical quantum equation of motion *Phys. Rev. B* **104**, 134312 (2021)

# Quantum geometric tensor for non-Hermitian Hamiltonians

L/R L/R L/R

Biorthogonal (L/R) quantum geometric tensor: $\tilde{T}_{jk}^{n} = \left\langle \frac{\partial \phi_n}{\partial \lambda_j} \middle| \frac{\partial \psi_n}{\partial \lambda_k} \right\rangle - \left\langle \frac{\partial \phi_n}{\partial \lambda_j} \middle| \psi_n \right\rangle \left\langle \phi_n \middle| \frac{\partial \psi_n}{\partial \lambda_k} \right\rangle$

Benefits

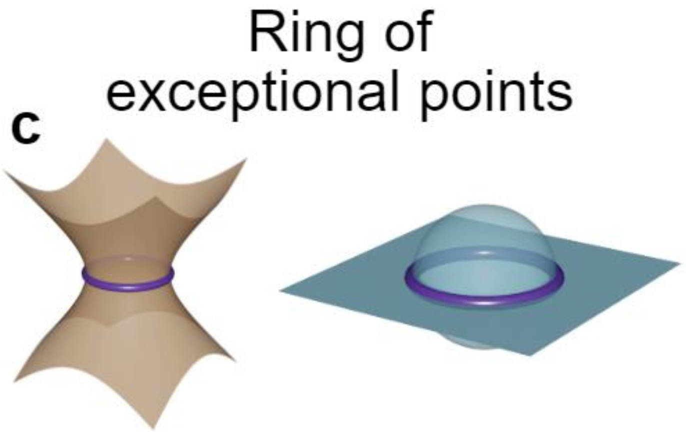
- Mathematically sound
- Identifies all phase transitions

But several caveats:

- The metric becomes pseudo-Riemannian (no distance)
- The Berry curvature becomes ill-defined (4 ≠)

# A ring of exceptional points

$$H = \underbrace{k_x\sigma_x + k_y\sigma_y}_{\text{Dirac point}} \underbrace{+\, i\gamma\sigma_z}_{\text{Non-Hermitian term}} = \begin{pmatrix} i\gamma & ke^{-i\theta} \\ ke^{i\theta} & -i\gamma \end{pmatrix}$$

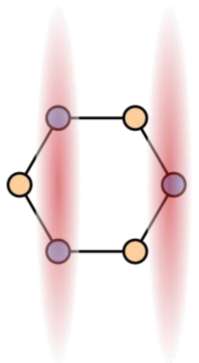


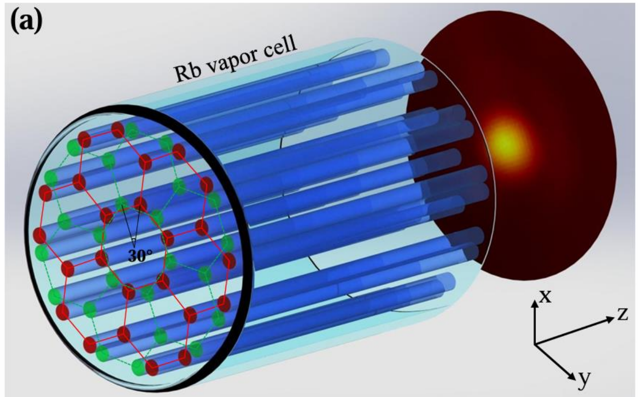

Real part Imaginary part

$$E_{\pm} = \pm\sqrt{k^2 - \gamma^2}$$

In a Rb vapor cell experiment:

Blue sites are pumped
$\Rightarrow$ Less losses $\Rightarrow +i\gamma\sigma_z$



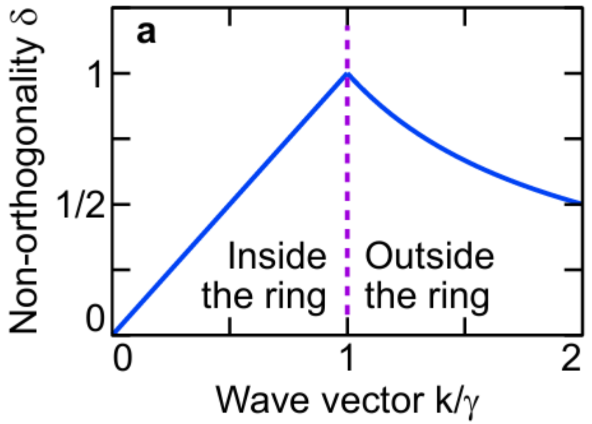



Z. Zhang *et al.*, *Phys. Rev. Lett.* **132**, 263801 (2024)

# Quantum geometric tensor of a ring of exceptional points

$$T^{\pm}_{k,\theta} = \frac{1}{4}\begin{pmatrix} \frac{\delta+(1-\delta)\Theta_{\text{out}}}{|E_\pm|^2} & \frac{-\delta}{kE^*_\pm} \\ \frac{-\delta}{kE_\pm} & \frac{\delta+(1-\delta)\Theta_{\text{in}}}{k^2} \end{pmatrix}$$

- Normal (RR) QGTensor
- Inside and outside must be glued through $\delta$
- Last term does not diverge
- No negative metric



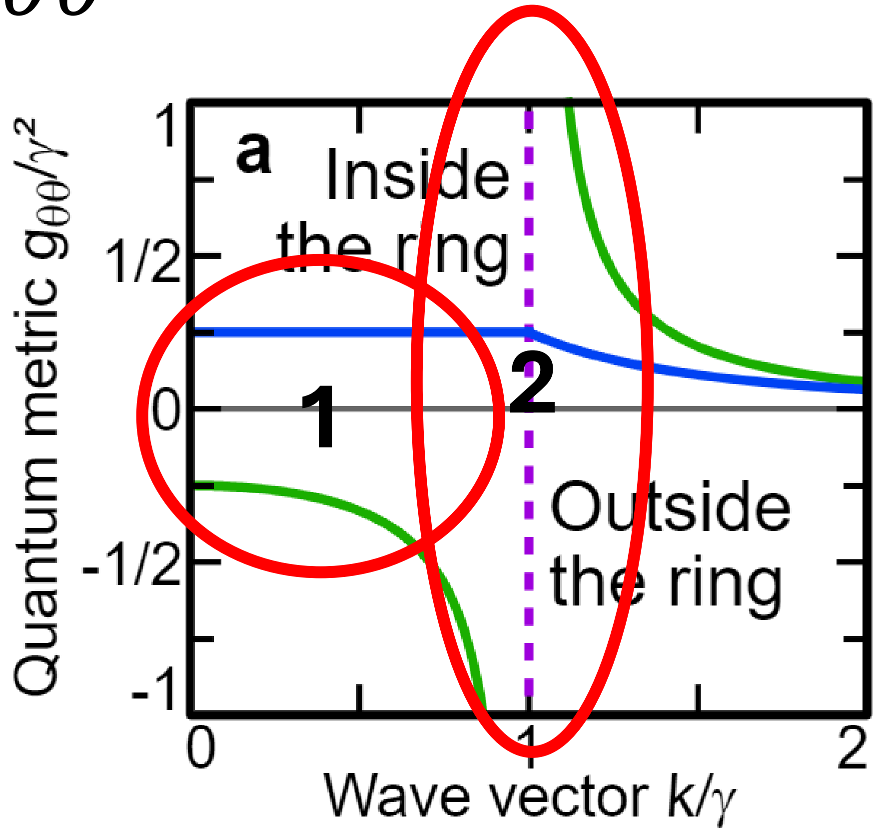

$$\tilde{T}^{\pm}_{k,\theta} = \frac{1}{4}\begin{pmatrix} \frac{-\gamma^2}{|E_\pm|^4} & \frac{\gamma}{|E_\pm|^2 E^*_\pm} \\ \frac{\gamma}{|E|^2 E_\pm} & \frac{1}{E^2_\pm} \end{pmatrix}$$

- Biorthogonal (LR) QGTensor
- Naturally valid inside and outside
- All terms diverge at RingEP
- Some negative metric

# Focus on $g_{\theta\theta}$

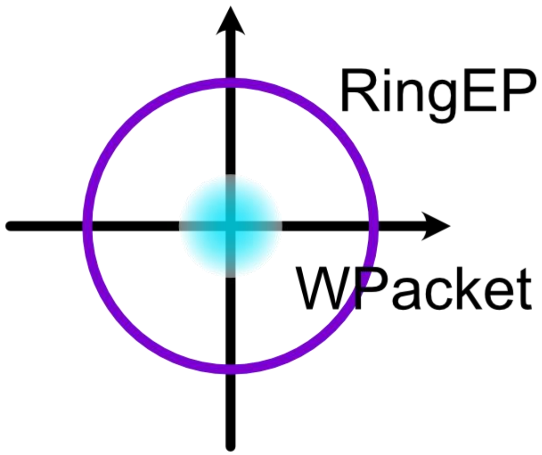
RR: $g_{\theta\theta} = \frac{1}{4}$ inside, $g_{\theta\theta} = \frac{1}{4}\frac{\gamma^2}{k^2}$ outside $(k > \gamma)$

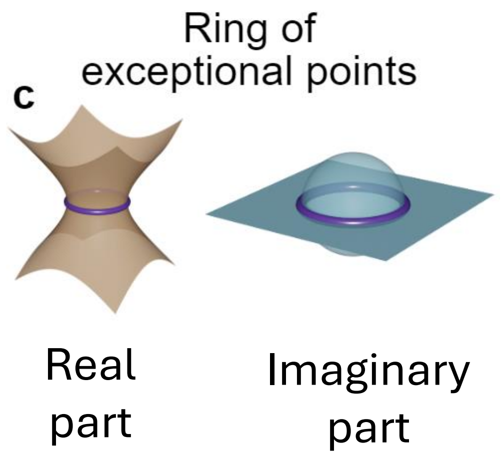
LR: $g_{\theta\theta} = \frac{1}{4(k^2-\gamma^2)}$




Which one is valid?
Do they make different predictions? → Look at the dynamics

# Dynamics inside the ring

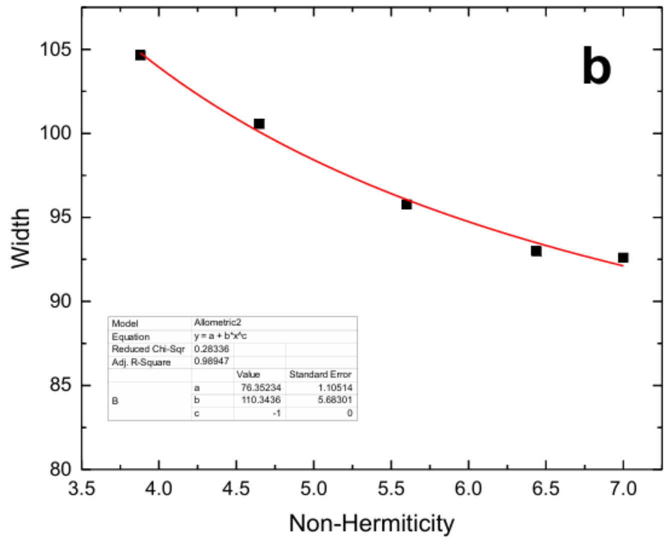



## Shrinking of the wave packet:

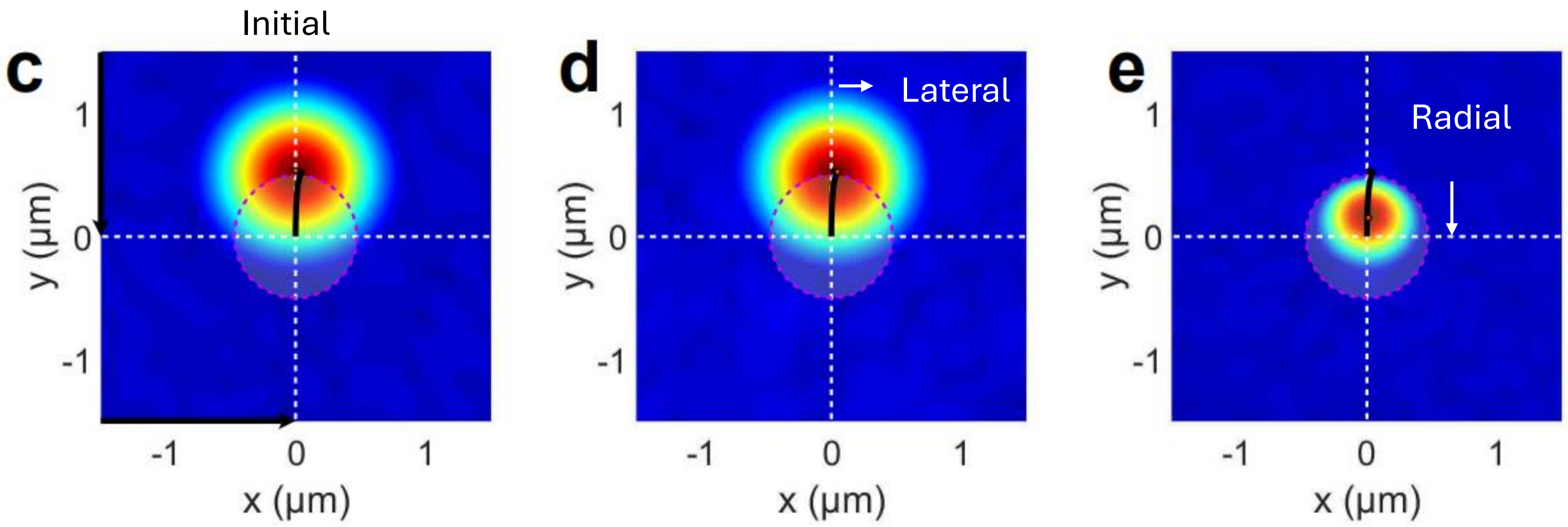
- Modes at $k = 0$ survive the longest

$$\sigma_k(t) \approx \sigma_{k,0} - 2\sqrt{g_{kk}}\sigma_{k,0}^3 t$$




➡ Radial RR metric does not play a role

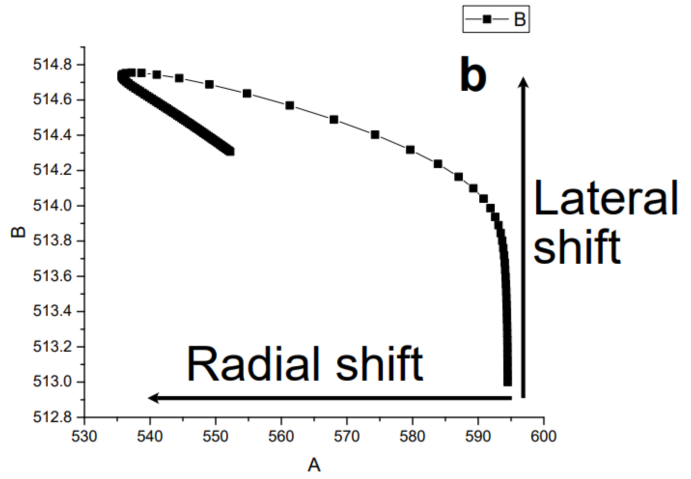
# Dynamics close to the ring



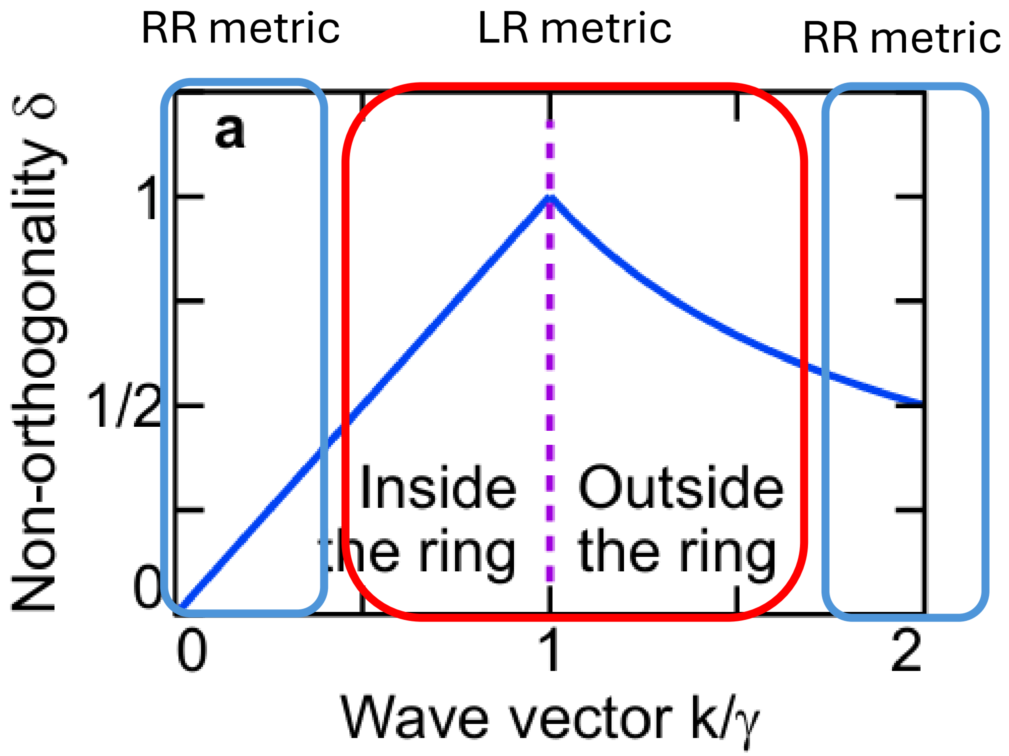

Radial shift: ‘trivial’

Lateral shift: non-trivial

- Small but measurable
- Can be explained by the LR metric (not by RR)



# Non-orthogonality to make a choice




RR metric: distance well-defined

LR metric: accurate predictions

# Conclusion

- Non-Hermitian Hamiltonians are useful
- Their quantum geometry is rich and unexplored

# Perspectives

- Finish this work!
- Understand better the biorthogonal QGTensor (with Pawel Störck)
- Entangled states? → Fubini-Study metric on $\mathbf{CP}^3$ (two qubits)