# Quantum speed limit for perturbed open systems

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Benjamin Yadin, Satoya Imai, Otfried Gühne
University of Siegen

*Geometry of quantum dynamics*
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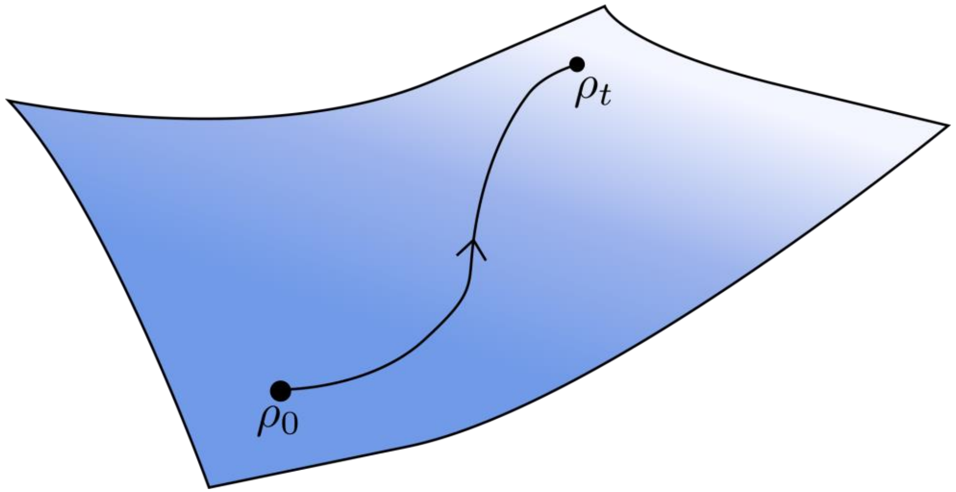
# Quantum speed limit

An uncertainty relation for time?

$$(\Delta H)^2 = \langle H^2 \rangle - \langle H \rangle^2$$

Heisenberg: $\Delta x \Delta p \geq \frac{\hbar}{2}$

Mandelstam-Tamm: $t\,\Delta H \geq \frac{\pi\hbar}{2}$




minimal time to reach an orthogonal state,
evolving under Hamiltonian $H$

Mandelstam & Tamm, J Phys (USSR) IX, 249 (1945)

**Applications:**

Deffner & Campbell, J Phys A 50, 453001 (2017)

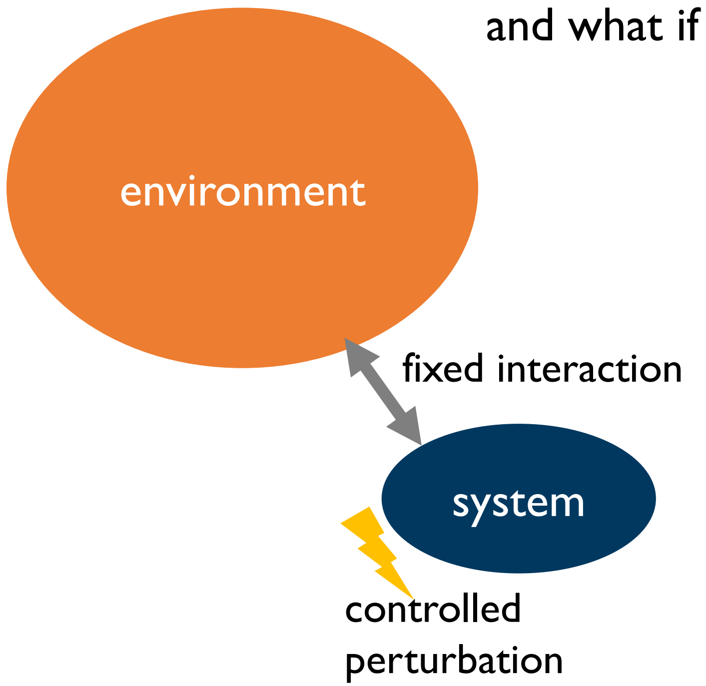
quantum control, information processing speed, quantum heat engines, many-body dynamics

# Perturbations are everywhere

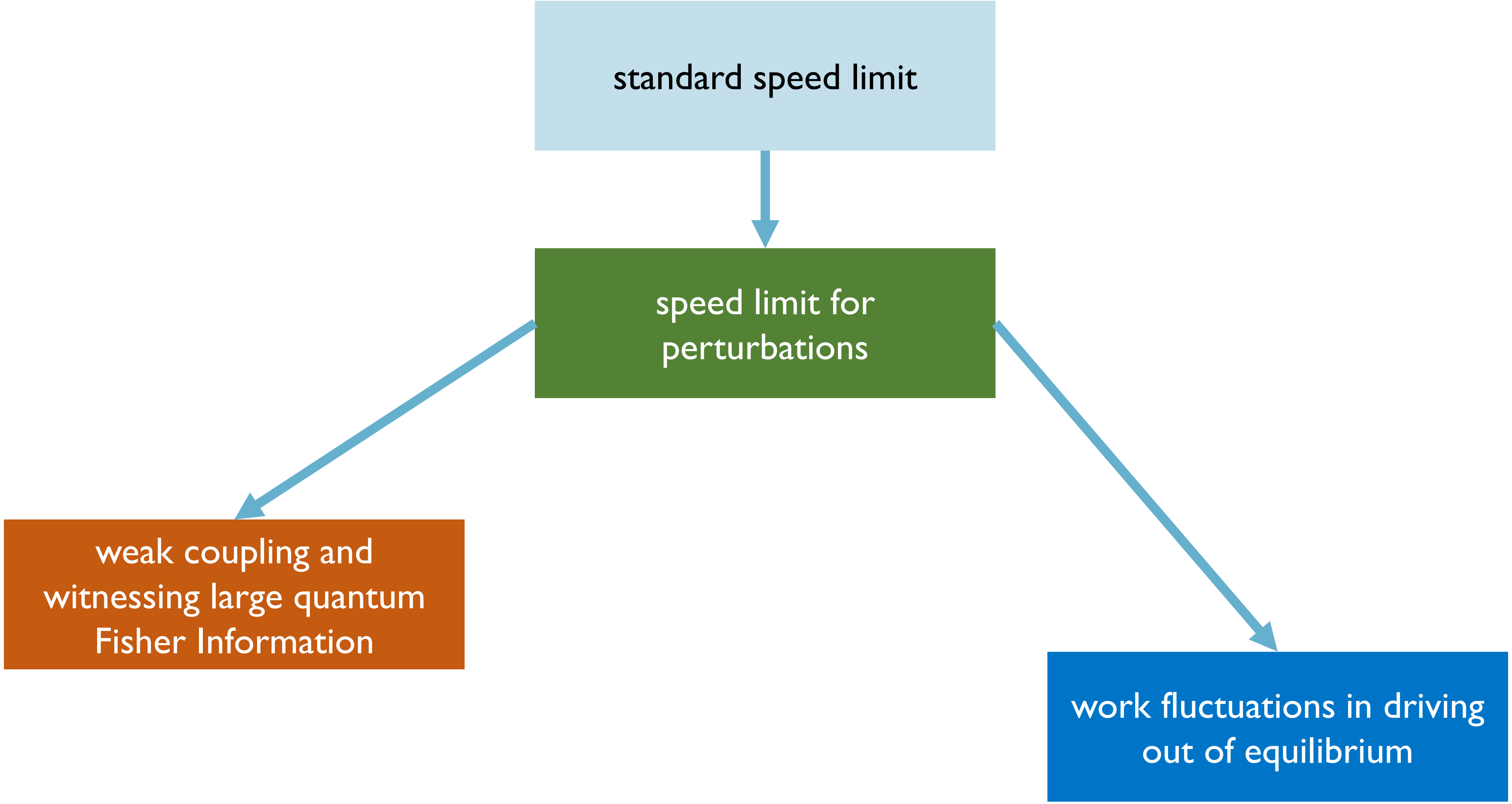
- Perturbation theory, linear response
- Non-equilibrium dynamics, quantum quenches
- Quantum info: metrology, channel discrimination

**How fast can a system respond to a given perturbation?**

and what if it's coupled to an environment?

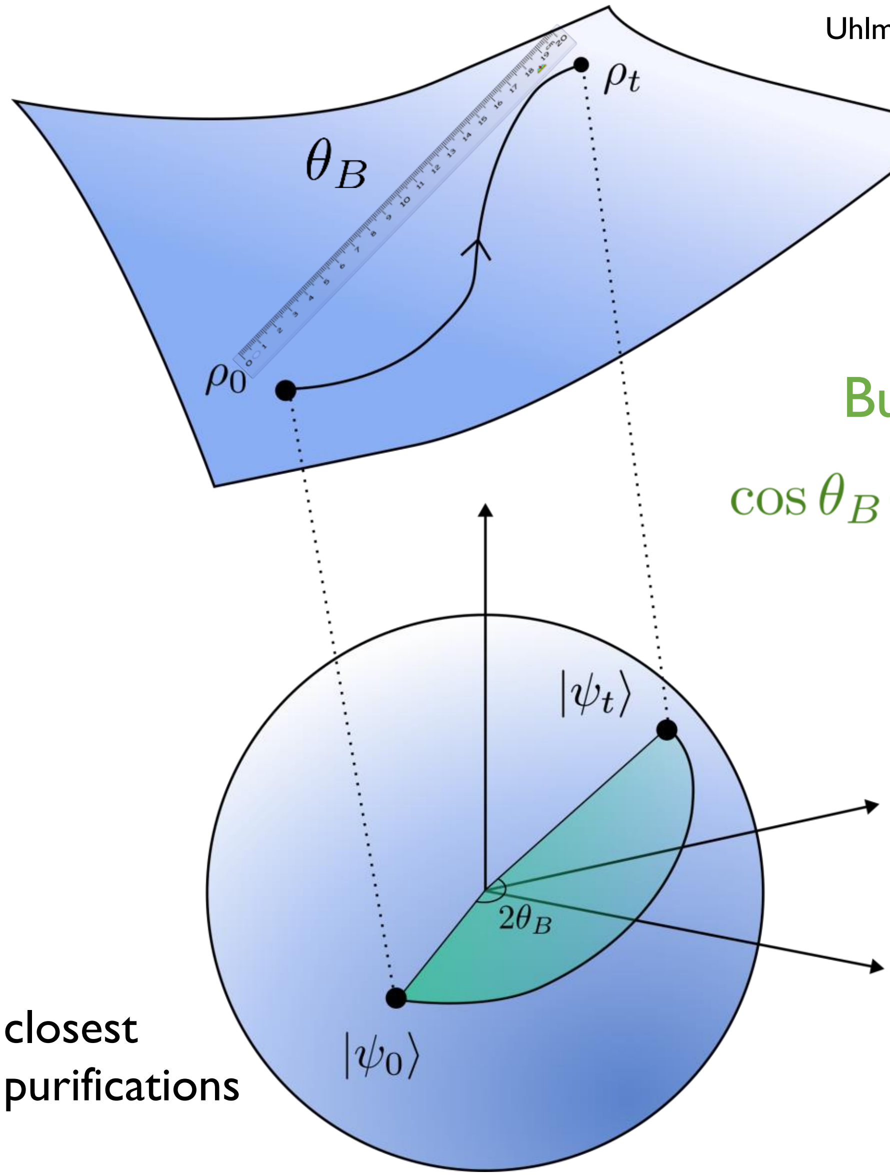


# Outline


standard speed limit
speed limit for perturbations
weak coupling and witnessing large quantum Fisher Information
work fluctuations in driving out of equilibrium

# Uhlmann speed limit

Uhlmann, Phys Lett A 161, 329 (1992)




closest purifications

$$\theta_B(\rho_0, \rho_t) \leq \frac{1}{2}\int_0^t \mathrm{d}s \sqrt{\mathcal{F}(\rho_s, \mathcal{H}_s)}$$

Bures angle

$$\cos\theta_B(\rho_0, \rho_t) = F(\rho_0, \rho_t) = \mathrm{tr}\,|\sqrt{\rho_0}\sqrt{\rho_t}|$$

generator of Lindblad master equation

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \mathcal{L}_t(\rho_t)$$

$$\mathcal{H}_t(\rho) = -i[H_t, \rho]$$

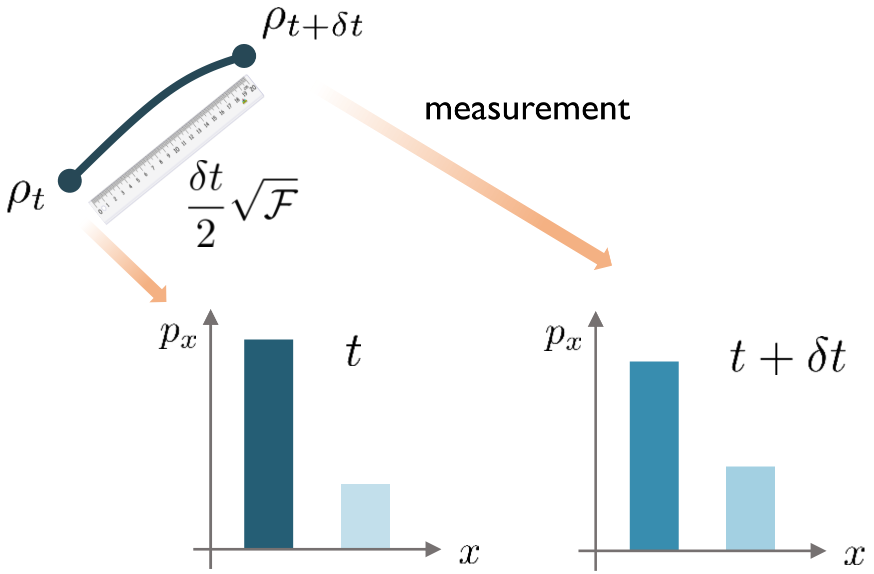
quantum Fisher information

$$\mathcal{F}(\rho, \mathcal{L}) = 2\sum_{i,j}\frac{|\langle i|\mathcal{L}[\rho]|j\rangle|^2}{p_i + p_j} \qquad \left(\rho = \sum_i p_i\,|i\rangle\langle i|\right)$$

# Quantum Fisher information

- Infinitesimal "Bures metric" $$\theta_B(\rho_t, \rho_{t+\delta t})^2 = \frac{1}{4}\mathcal{F}(\rho_t, \mathcal{L}_t)\delta t^2 + \dots$$

- Measure of squared "**statistical speed**"




- In metrology: lower-bounds the uncertainty for estimating elapsed time

  Quantum Cramér-Rao bound

$$\Delta t_{\text{est}} \geq \frac{1}{\sqrt{\mathcal{F}}}$$

Tóth & Apellaniz, J Phys A 47, 424006 (2014)

- Can be used to witness quantum properties: entanglement, optical non-classicality + more…

# Uhlmann speed limit

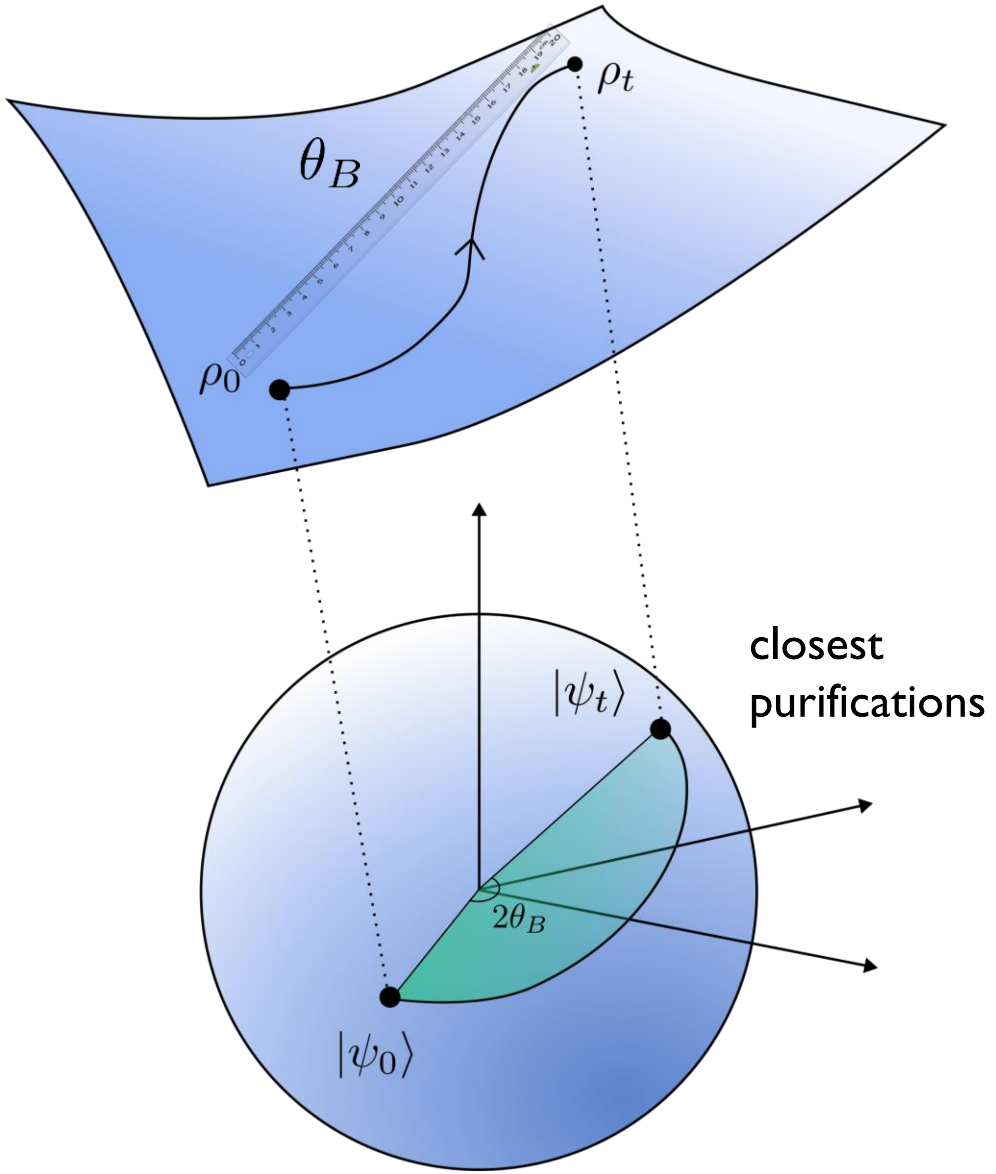
$$\theta_B(\rho_0, \rho_t) \leq \frac{1}{2} \int_0^t \mathrm{d}s \sqrt{\mathcal{F}(\rho_s, \mathcal{H}_s)}$$

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \mathcal{H}_t(\rho_t) = -i[H_t, \rho_t]$$

Fixed Hamiltonian recovers Mandelstam-Tamm

$$\mathcal{F}(\rho, \mathcal{H}) \leq 4(\Delta H)^2$$

implies MT bound: $$\frac{\pi}{2} \leq t\Delta H$$



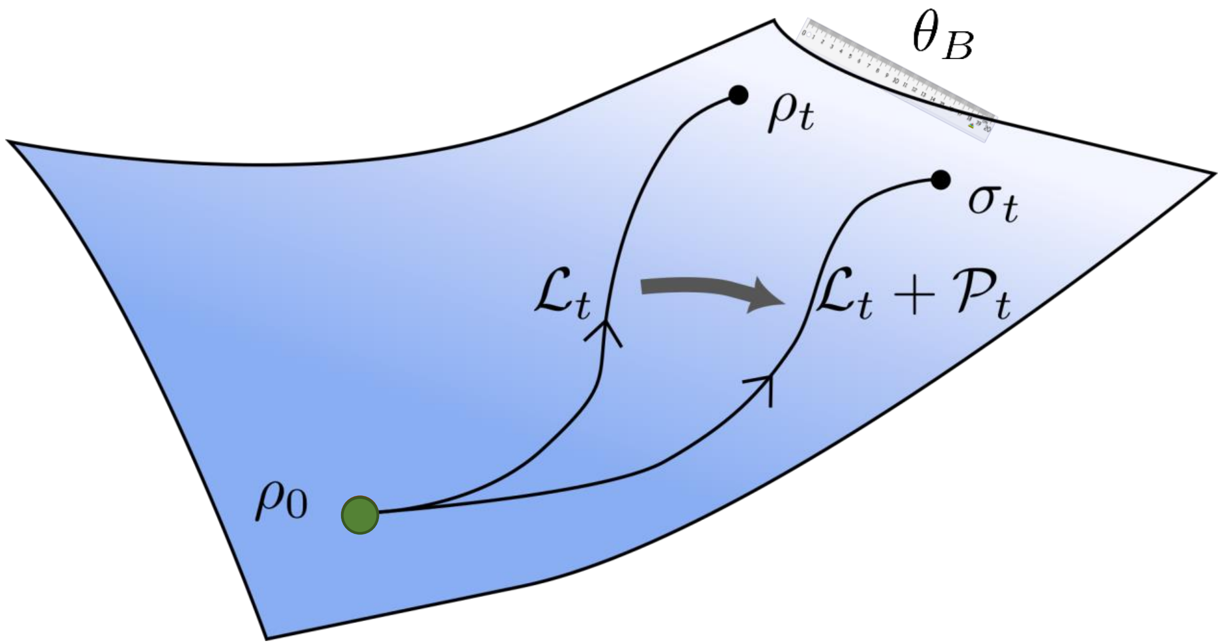
# Perturbation speed limit

# Speed limit for perturbed open systems

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \mathcal{L}_t(\rho_t)$$ unperturbed

$$\frac{\mathrm{d}\sigma_t}{\mathrm{d}t} = \mathcal{L}_t(\sigma_t) + \mathcal{P}_t(\sigma_t)$$ perturbed

$$\sigma_0 = \rho_0$$



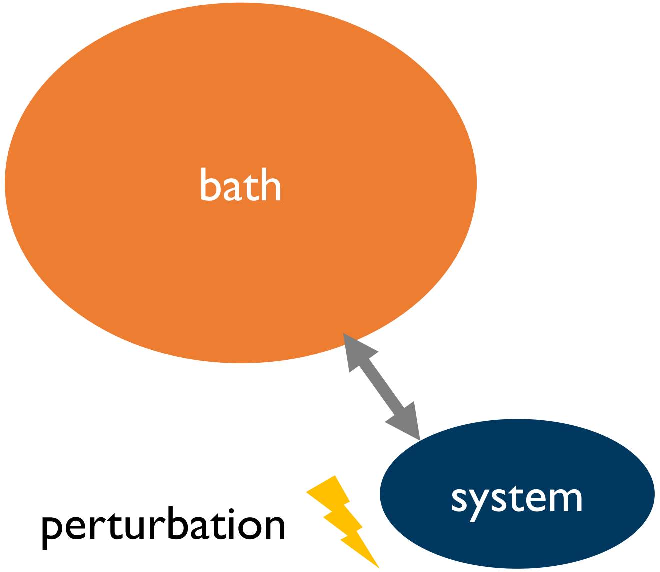

$$\theta_B(\rho_t, \sigma_t) \leq \frac{1}{2} \int_0^t \mathrm{d}s \sqrt{\mathcal{F}(\sigma_s, \mathcal{P}_s)}$$

**main result**

Proof ingredients:

- Triangle inequality
- QFI as a metric
- contractivity under channels ("data processing")

# Speed limit with weak coupling




$v\mathcal{V} = -iv[V, \cdot]$

Interested in the speed of response to a perturbing Hamiltonian $V$…

…but we:

- can't switch off the noise
- want to assume as little about the noise as possible

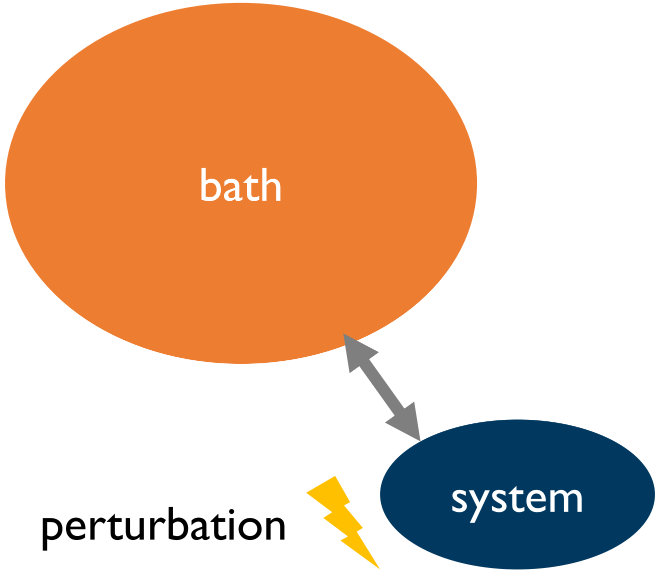
Characteristic time-scales:

- bath correlation $\tau_B$
- bare system evolution $\tau_S$
- system relaxation $\tau_R$
- perturbation $\tau_V \sim 1/v$

weak coupling + secular approximation: $\tau_R \gg \tau_B, \tau_S$

weak perturbation: $\tau_V \gg \tau_B, \tau_S$

# Speed limit with weak coupling




$$v\mathcal{V} = -iv[V, \cdot]$$

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \mathcal{L}(\rho_t)$$

$$\frac{\mathrm{d}\sigma_t}{\mathrm{d}t} = [\mathcal{L} + v\mathcal{V} + \mathcal{L}']\,(\sigma_t)$$

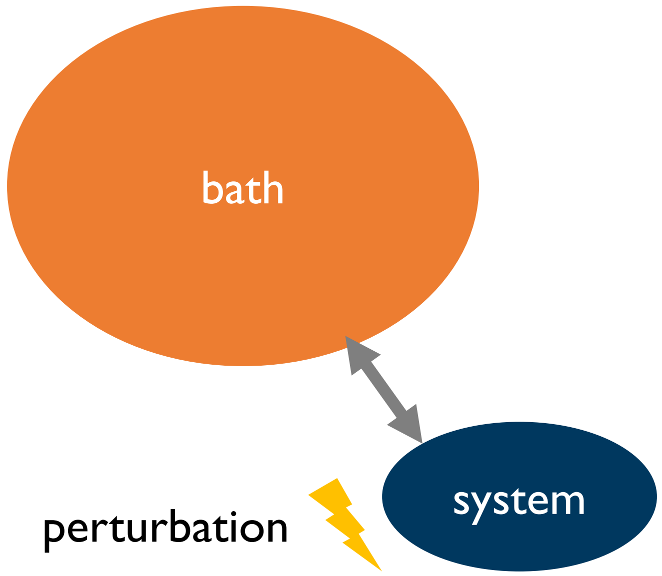
"annoying perturbation" from altered jump operators & Bohr frequencies

we take it to first order: $\mathcal{L}' = v\mathcal{L}^{(1)}$

Error in neglecting the annoying part is determined by $\epsilon := \max_{\psi} \left\| \mathcal{L}^{(1)}(\psi) \right\|$

- can be calculated exactly given the model
- otherwise estimated from timescales $\epsilon = \mathcal{O}\left(\frac{\tau_S}{\tau_R}\right) + \mathcal{O}\left(\frac{\tau_B}{\tau_R}\right)$

# Speed limit with weak coupling




$$v\mathcal{V} = -iv[V, \cdot]$$

**result:**

$$\theta_B(\rho_t, \sigma_t) \leq \left[ \frac{1}{2} \int_0^t \mathrm{d}s \sqrt{\mathcal{F}(\sigma_s, v\mathcal{V})} \right] + \delta(t)$$

$$|\delta(t)| \leq \frac{4\sqrt{2}}{3} \|V\| \, \epsilon^{\frac{1}{2}} (vt)^{\frac{3}{2}} + \epsilon vt$$

**error can be bounded**

$\epsilon$ is guaranteed to be small by assumptions of weak-coupling master equation

$$\epsilon = \mathcal{O}\left(\frac{\tau_S}{\tau_R}\right) + \mathcal{O}\left(\frac{\tau_B}{\tau_R}\right)$$

secular approximation

Born-Markov approximation

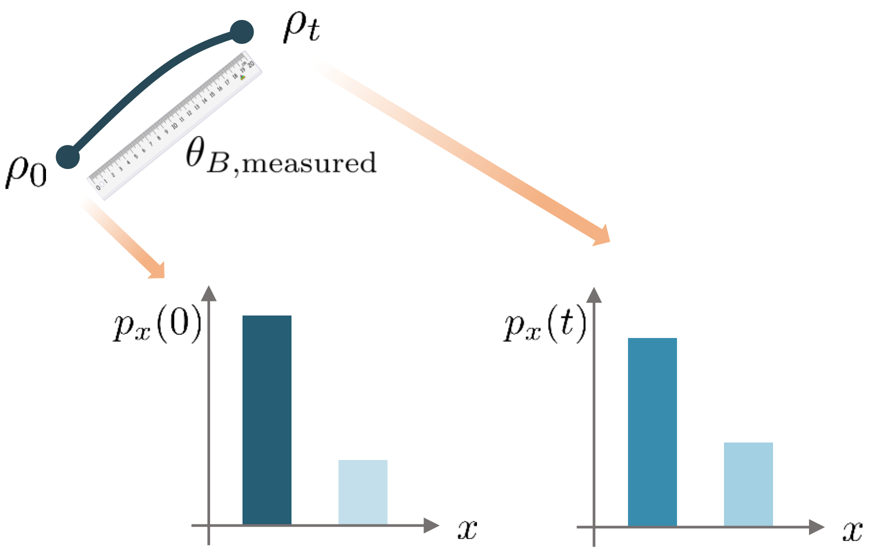
# Witnessing QFI

# Witnessing large QFI

A common method to experimentally lower-bound QFI, **assuming negligible decoherence**:

e.g. in BECs: Strobel et al., *Science* 345, 424 (2014)

- evolve under $vV$ for known time $t$
- collect measurement statistics for initial state and for time $t$
- compute a distance between the distributions



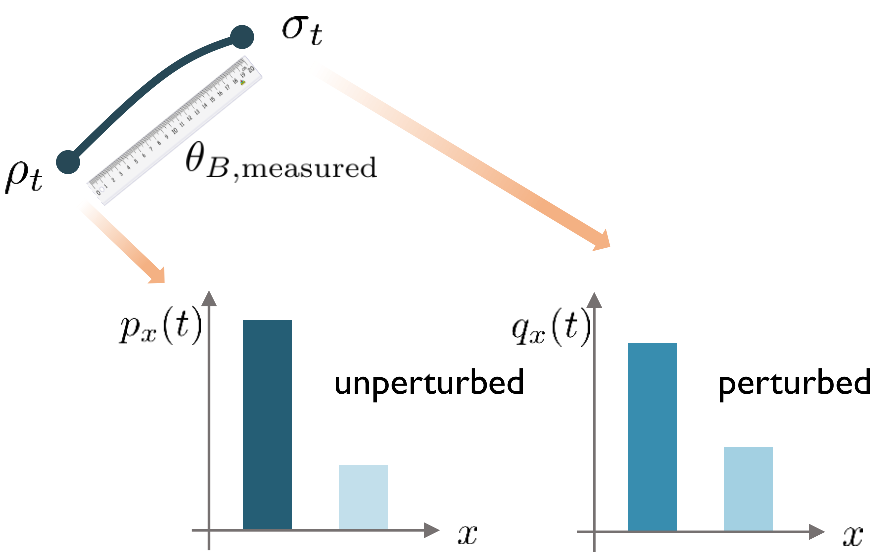

$$\theta_{B,\mathrm{measured}} = \arccos\left[\sum_x \sqrt{p_x(0)p_x(t)}\right]$$

$$\sqrt{\mathcal{F}(\rho,\mathcal{V})} \geq \frac{2\theta_{B,\mathrm{measured}}}{vt}$$

# Witnessing large QFI

A different protocol **allowing for decoherence**:

- evolve under either unperturbed or perturbed dynamics for known time $t$
- collect measurement statistics in both cases at time $t$
- compute a distance between the distributions



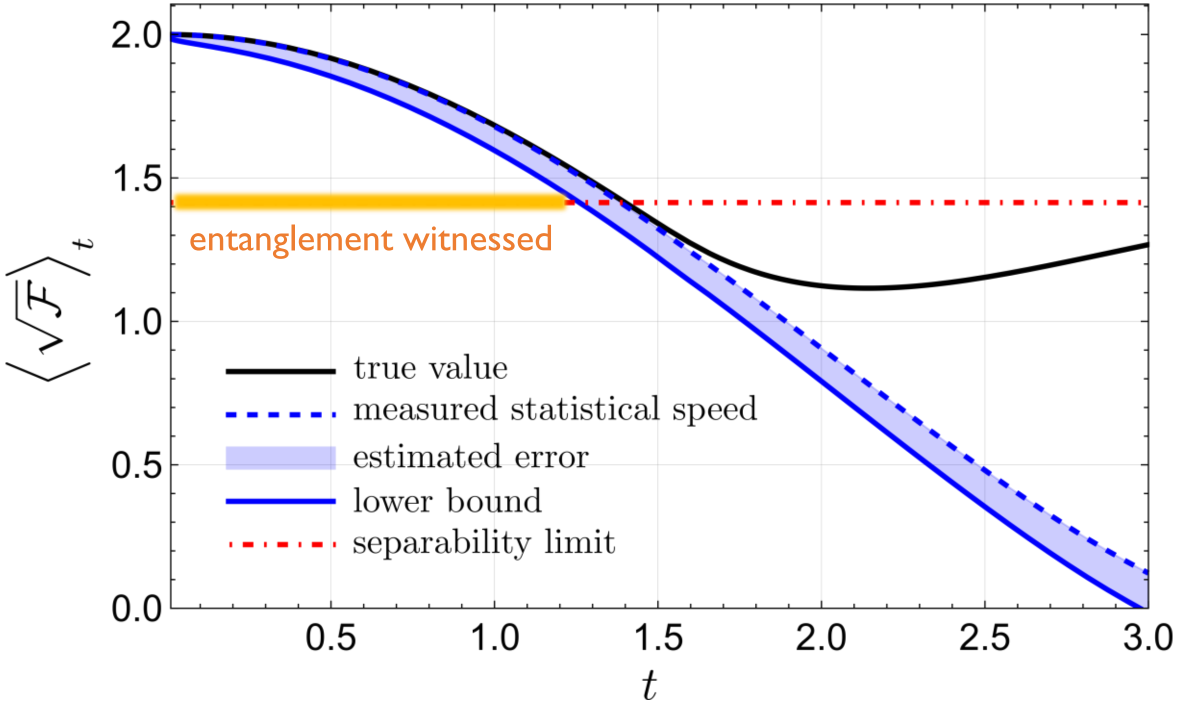

$$\theta_{B,\text{measured}} = \arccos\left[\sum_x \sqrt{p_x(t) q_x(t)}\right]$$

$$\left\langle \sqrt{\mathcal{F}(\sigma_s, \mathcal{V})} \right\rangle_t \geq \frac{2\theta_{B,\text{measured}}}{vt} - \frac{2\delta(t)}{vt}$$

time-averaged speed — error

# Witnessing large QFI - example

$$\langle\sqrt{\mathcal{F}(\sigma_s,\mathcal{V})}\rangle_t \geq \frac{2\theta_{B,\text{measured}}}{vt} - \text{error}$$




2 qubits, each with $H = h\sigma_z$, dephasing with strength $g$

perturbation: $V = (\sigma_x \otimes I + I \otimes \sigma_x)/2$

error parameter bounded by $\epsilon \leq 4g/h$

To witness entanglement:

$\mathcal{F}(\sigma_s,\mathcal{V}) > 2 \Rightarrow$ entanglement

Tóth, PRA (2012)
Hyllus et al., PRA (2012)

start in $|00\rangle + |11\rangle$

measure in Bell basis at time $t$

# Quantum work fluctuations

# Fluctuation-dissipation relation

- Starting with Einstein and Brownian motion:

  near thermal equilibrium, fluctuations and dissipation must be related
  (e.g. diffusion constant and friction)

- In classical stochastic thermodynamics (slow but finite driving):

$$\Delta W^2 = 2k_B T W_{\mathrm{diss}}$$

variance of work
done on system

mean dissipated work:

$$W_{\mathrm{diss}} = \langle W \rangle - \Delta F$$

Jarzynski, PRL 78, 2690 (1997)

# Quantum FDR

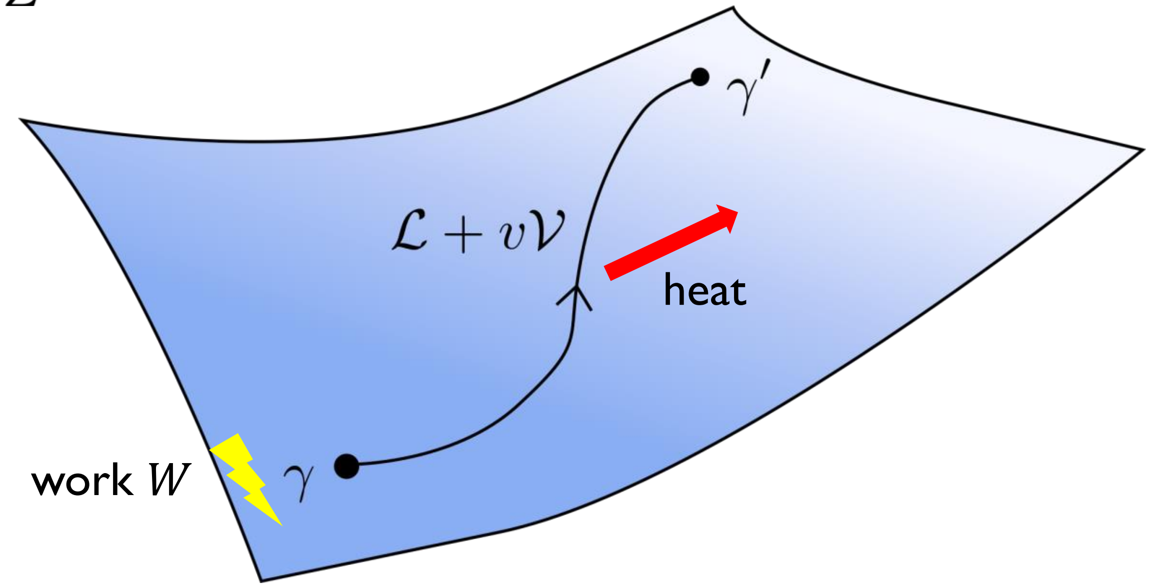
- System in contact with a thermal environment at temperature $T$
- Quench $H \to H' = H + vV$ at $t = 0$ to kick it out of the thermal state $\rho_0 = \gamma = \dfrac{e^{-\beta H}}{Z}$
- System approaches the new thermal state $\gamma' = \dfrac{e^{-\beta H'}}{Z'}$

**modified fluctuation-dissipation relation:**

$$\Delta W^2 = 2k_B T W_{\text{diss}} + \Delta W_Q^2$$

Miller et al., PRL 123 230603 (2019)

quantum correction, vanishes when $[H, V] = 0$



# Quantum work fluctuations

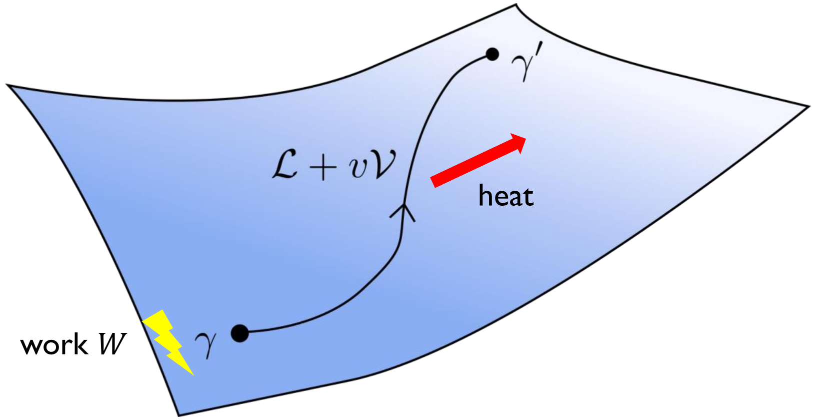
modified FDR:

$$\Delta W^2 = 2k_B T W_{\text{diss}} + \Delta W_Q^2$$

$$\theta_B(\gamma, \rho_t) \leq \sqrt{3} t \Delta W_Q + \delta(t)$$

(related to QFI)

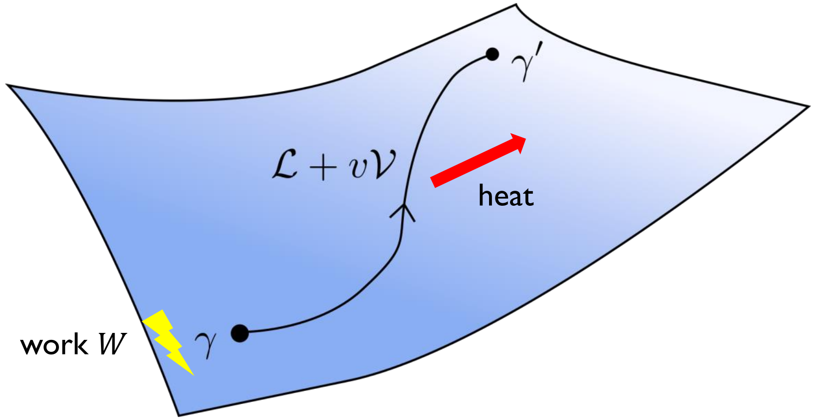
$$\Delta W_Q^2 = \frac{v^2}{2} \int_0^1 \mathrm{d}k \; \mathrm{tr}\left([\gamma^k, V][V, \gamma^{1-k}]\right)$$




**Quantum work fluctuations are needed for fast departure from equilibrium**

# Classical versus quantum driving

$$\theta_B(\gamma, \rho_t) \leq \sqrt{3} t \Delta W_Q + \delta(t)$$




$\delta(t)$ dominates in the classical case, when $[H, V] = 0$

Quantum driving regime: when the quantum term dominates

$$\frac{\Delta W_Q}{\|vV\|} \gg \max\left\{\sqrt{\epsilon v t \|V\|},\ \epsilon\right\}$$

(fast "spiralling" path)

# Summary

- Quantum Fisher Information bounds **speed of response** to perturbations

- Versatile speed limit, holds approximately in **weak coupling**

  – error can be estimated from timescales

- Relates **quantum work fluctuations** to departure from equilibrium

Thanks for your attention!

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with:

Satoya Imai

Otfried Gühne

Universität Siegen

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