Nonclassicality, Entanglement and Positive Polynomials

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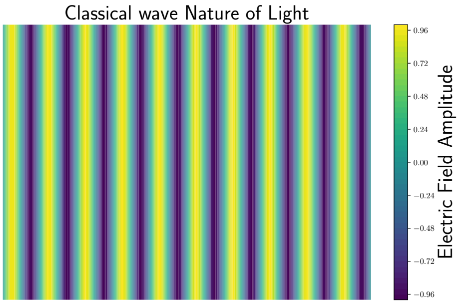
Classical light

Classically, a light mode is described by an electromagnetic wave.



In QM, the pure wave-like behaviour is observed in the time evolution of coherent states $|\alpha\rangle$ parametrized by $\alpha \in \mathbb{C}$.

Reminder: Coherent states

Some facts about coherent states $|\alpha\rangle := e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$:

- form an overcomplete basis
- eigenvectors of the annihilation operator

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

- for $\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$, one has

$$\langle\alpha|\, {:}\hat{x}^a \hat{p}^b{:}\, |\alpha\rangle = \mathrm{Re}(\alpha)^a \mathrm{Im}(\alpha)^b$$

$:\cdot:$ is "normal ordering"

Glauber 1963

Nonclassical light

Light mode described by state ρ of quantum harmonic oscillator

$$\rho = \int_{\mathbb{C}} d\alpha \, P(\alpha) \, |\alpha\rangle \, \langle\alpha|$$

$|\alpha\rangle$: coherent state, P: Glauber function

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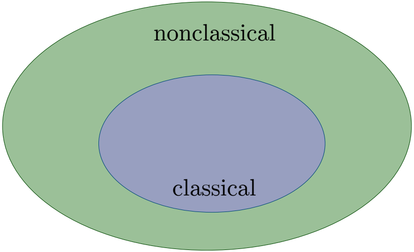
State ρ is **classical** if $P(\alpha) \geq 0$ for all $\alpha \in \mathbb{C}$, otherwise **nonclassical**



Question: How to detect nonclassicality without full reconstruction of P?

Nonclassicality witness

Observable W is called **nonclassicality witness** if $\langle\alpha| W |\alpha\rangle \geq 0$ for all $\alpha \in \mathbb{C}$, as

$$\mathrm{Tr}(W\rho) < 0 \Rightarrow \rho \text{ nonclassical}$$

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Witness by polynomial:

$$W_f := \sum_{a,b} f_{ab} \, {:}\hat{x}^a \hat{p}^b{:} \text{ is witness}$$

$$\Longleftrightarrow$$

$$f(x,p) = \sum_{a,b} f_{ab} \, x^a p^b \geq 0 \text{ for all } (x,p) \in \mathbb{R}^2$$

Witnesses of **degree** $D = \deg(f)$ requires the measurement of moment data $\langle{:}\hat{x}^a \hat{p}^b{:}\rangle_\rho$ with $a + b \leq D$.

Korbicz et al (2005), Shchukin et al (2005)

Sum of squares hierarchy

Polynomial g of degree $D/2 \Rightarrow f = g^2$ is positive polynomial of degree D.

Theorem (Korbicz et al , 2005)

A state ρ is classical if and only if $\langle W_{g^2} \rangle_\rho \geq 0$ for all polynomials g.

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Hierarchy of optimisation problems:

$$\begin{aligned} &\text{given } \rho \\ &t_D(\rho) := \min_f \mathrm{Tr}(W_f \rho) \\ &\text{s.t. } f \in \mathrm{SOS}_D \end{aligned}$$

One has $t_2 \geq t_4 \geq t_6 \geq \ldots$ and if $t_D(\rho) < 0$, ρ is nonclassical.

SOS as a semidefinite program

Consider a polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$
We can rewrite f as

$$f(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} \boxed{M_{00}} & \boxed{M_{01}} & \boxed{M_{02}} \\ \boxed{M_{01}} & \boxed{M_{11}} & \boxed{M_{12}} \\ \boxed{M_{02}} & \boxed{M_{12}} & \boxed{M_{22}} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

with constraints:

$$\boxed{M_{00} = e}, \boxed{2M_{01} = d}, \boxed{M_{11} + 2M_{02} = c}, \boxed{2M_{12} = b}, \boxed{M_{22} = a} \tag{1}$$

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SOS as a semidefinite program

f is a sum of squares **if and only if** there exists a **positive semidefinite** matrix M that fulfills the constraints (1).

Positive polynomials

Is every non-negative polynomial SOS?

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No! (Hilbert, 1888)

Only if the polynomial is:

1. univariate, e.g.,

$$f(x) = x^{10} + 2x^6 + 2$$

2. quadratic, e.g.,

$$f(x_1, \ldots, x_n) = 2x_n^2 + x_1x_2 - 5x_3 + \ldots$$

3. bivariate quartic, e.g.,

$$f(x, y) = x^4 + 3x^2y - xy^2 + \ldots$$

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$\Rightarrow$ If $D \geq 6$, the data $\langle :\hat{x}^a \hat{p}^b: \rangle_\rho$ can possess more detection power than what is visible from SOS_D.

Reznick's hierarchy

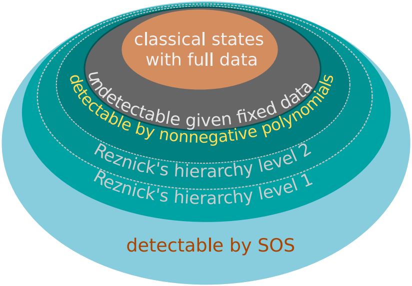
Characterisation of positive polynomials is generally a hard problem, but there is a systematic characterisation

Theorem (Reznick, 1995)

If $f(\underline{x})$ is a positive polynomial, then there exists a $l \in \mathbb{N}$ such that

$$(1 + \|\underline{x}\|^2)^l f(\underline{x}) \in \text{SOS}$$

Given the data $\langle{:}\hat{x}^a \hat{p}^b{:}\rangle_\rho$ with $a + b \leq D$:

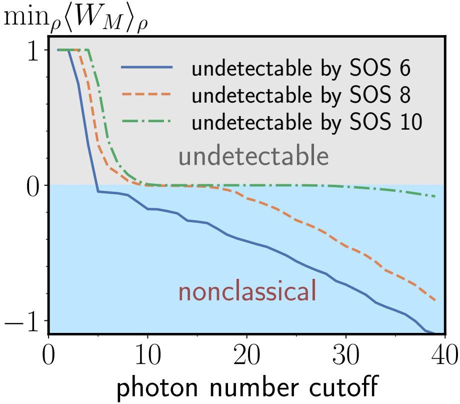


Example: Motzkin polynomial

Consider the Motzkin polynomial

$$f(x,p) = 1 + x^4p^2 + x^2p^4 - 3x^2p^2$$

Motzkin (1967)



$\Rightarrow$ Non-SOS positive polynomials provide a detection advantage

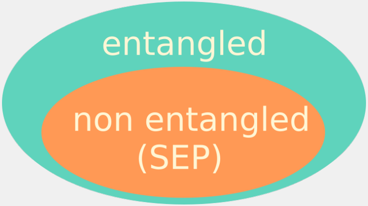
Entanglement

Entanglement and entanglement witnesses

State ρ^{AB} is **separable**, if

$$\rho^{AB} = \sum_{\lambda} p_\lambda \left|x_\lambda\right\rangle \left\langle x_\lambda\right| \otimes \left|y_\lambda\right\rangle \left\langle y_\lambda\right|$$

with $p_\lambda \geq 0$. Otherwise **entangled**!



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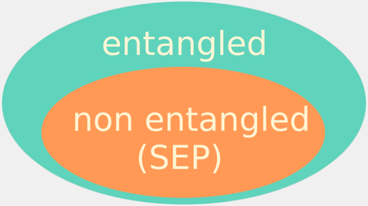
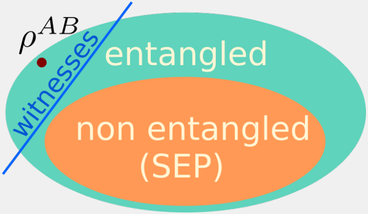


Observable W is an **entanglement witness**, if

$$\langle x, y| W |x, y\rangle \geq 0 \text{ for all } |x, y\rangle .$$



Entanglement witnesses and positive polynomials

Entanglement witnesses W correspond to positive polynomials $p_W(x, y)$ via

$$p_W(x, y) = \langle x, y| \, W \, |x, y\rangle = \sum_{ijkl} W_{ij,kl} \bar{x}_i x_j \bar{y}_k y_l$$

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Reznick's hierarchy for entanglement detection

For $b \in \mathbb{N}$:

$$\begin{aligned} &\text{given } \rho \\ &l_b(\rho) = \min_W \operatorname{Tr}(W\rho) \\ &\text{s.t.} \quad (\|x\|^2 + \|y\|^2)^b p_W(x, y) \in \text{SOS} \\ &\qquad\quad \operatorname{Tr}(W) = d \end{aligned}$$

$b = 0$ exactly corresponds to the PPT criterion

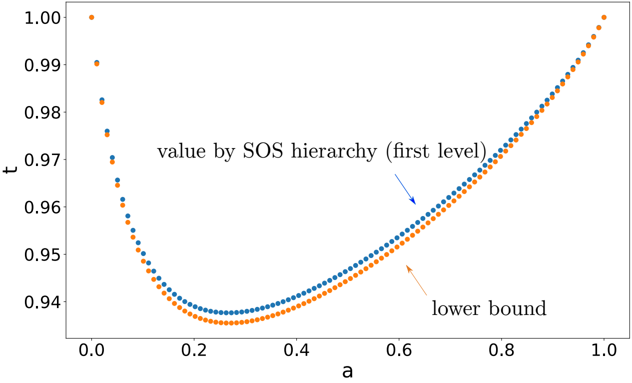
Doherty et al (2004), Bohnet-Waldraff et al (2017), Fang, Fawzi (2020)

Example: Detection of bound entanglement

Horodecki state:

$$\rho(a,t) = t\rho(a) + \frac{1-t}{9}\mathbb{1}$$

$\max t$ s.t. $\rho(a,t)$ is separable:



Horodecki(1997), TO, Yu, Gühne, Nguyen (2022)

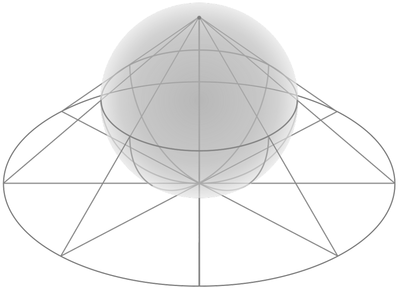
Bosonic entanglement and stereographic projection

Separable bosonic states of k qubits

$$\rho = \int_{S^2} d\underline{x}\, P(\underline{x})(|\underline{x}\rangle\langle\underline{x}|)^{\otimes k}$$
$$= \int_{\mathbb{C}} d\beta\, P(\underline{x}(\beta))(|\underline{x}(\beta)\rangle\langle\underline{x}(\beta)|)^{\otimes k}$$

with $\underline{x}(\beta) = \frac{1}{1+|\beta|^2}(2\mathrm{Re}(\beta), 2\mathrm{Im}(\beta), 1-|\beta|^2)$

Stereographic projection:

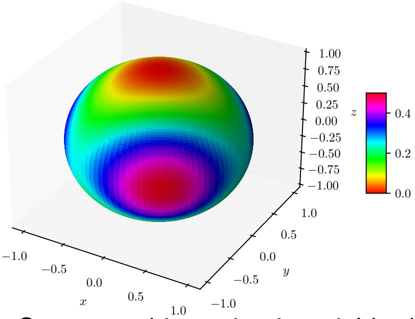


Example: Fidelity witnesses

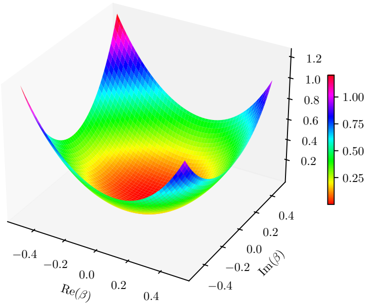
Consider the fidelity witness of the three qubit GHZ state

$$W = \mathbb{1}/2 - |\mathrm{GHZ}\rangle \langle \mathrm{GHZ}|$$

Witness polynomial $p_W(\underline{x})$:



Projected polynomial $\widetilde{p}_W(\beta)$:



$\Rightarrow$ Stereographic projection yields the nonclassicality witness

$$\widetilde{W} = -\hat{a}^3 - \hat{a}^{\dagger 3} + 3\hat{a}^\dagger \hat{a} + 3\hat{a}^{\dagger 2}\hat{a}^2$$

Conclusion and Outlook

- Method for optimal usage of experimental data for nonclassicality detection
- Demonstration of importance of positive polynomials
- Connection of nonclassicality and entanglement by the stereographic projection

Outlook: Study the role of the stereographic projection for entanglement detection in general non-symmetric systems

arXiv:2403.09807

Thank you!