Rotational covariance restricts available quantum states

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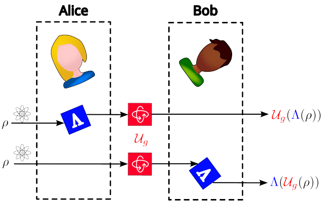


Motivation

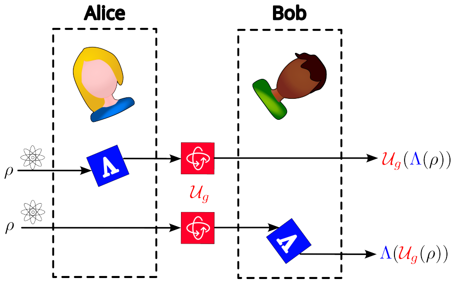
Spin states: direction in 3D space

$$|\psi\rangle = \sum_{j,m} \psi_{j,m} |j, m\rangle$$

SU(2)-**covariant operations:**

$$\Lambda \circ \mathcal{U}_g = \mathcal{U}_g \circ \Lambda$$

for all $g \in \mathrm{SU}(2)$



Which transformations $\rho \mapsto \sigma$ are realizable with SU(2)-**covariant operations?**

SU(2)-covariant operations

- Examples for SU(2)-covariant channels:
  $$\Lambda(\rho) = \rho \text{ and } \Lambda(\rho) = |0,0\rangle\langle 0,0|$$
- Cannot increase directional information
- Known parametrization of Kraus operators

$|\psi\rangle \mapsto |0,0\rangle$ ✓

$|\uparrow\rangle \mapsto |\uparrow\rangle|\uparrow\rangle$ ✗

Limited results for the possibility of the trafo $|\psi\rangle \mapsto |\phi\rangle$:

- All spins in one direction: $\mathcal{H} = \text{span}(\{|j,j\rangle\}_{j=0,\frac{1}{2},1\ldots})$
- Eigenstates of J_z

: a general characterization of possible trafos

Gour and Spekkens. *New Journal of Physics*, 10(3):033023, 2008.

Szymański. *Numerical ranges and geometry in quantum information*. PhD thesis, 2023.

Possible strategies

Perfect + deterministic transformation $\rho \mapsto \sigma$ often impossible

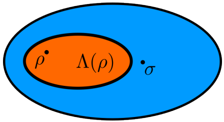
$\Rightarrow$ Drop one of the requirements

Imperfect + deterministic

Minimize the distance

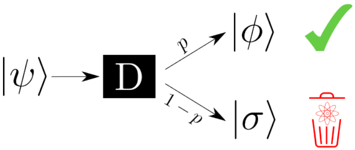
$$D(\Lambda(\rho), \sigma)$$

over all SU(2)-covariant channels Λ



Perfect + stochastic

Maximize the postselection rate p for the transformation $|\psi\rangle \mapsto |\phi\rangle$



Minimum distance

Convex set of reachable states
$\{\Lambda(\rho) \mid \Lambda \text{ SU(2)-covariant channel}\}$

Minimizing

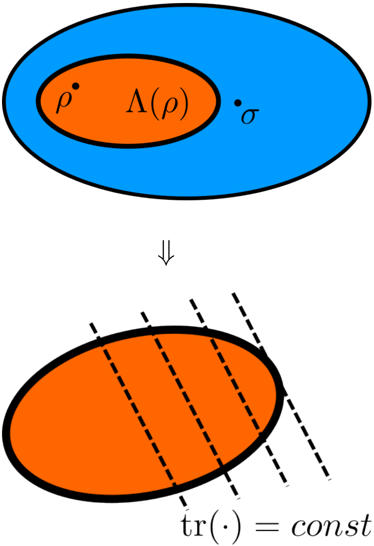
$$D(\Lambda(\rho),\sigma) = \sqrt{2(1-\mathcal{F}(\Lambda(\rho),\sigma))}$$

$\Leftrightarrow$ maximizing the fidelity

$$\mathcal{F}(\Lambda(\rho),\sigma) = \mathrm{tr}\left(\sqrt{\sqrt{\sigma}\Lambda(\rho)\sqrt{\sigma}}\right)$$

Goal: Semidefinite program (SDP)

- Efficiently solvable
- Guaranteed precision of the solution



Minimum distance

SU(2)-**covariant channels**

Gour and Spekkens:

$$\Lambda_{\{\vec{f}_J\}_J}(\rho) = \sum_{J,M} K_{J,M}(\vec{f}_J)\rho K^{\dagger}_{J,M}(\vec{f}_J)$$

$\Rightarrow$ parametrized Kraus operators

Fidelity SDP

Fidelity between two states ρ, σ:

$$\begin{aligned} \max_X \quad & \frac{1}{2}\operatorname{tr}\left(X + X^{\dagger}\right), \\ \text{s.t.} \quad & \begin{pmatrix} \rho & X \\ X^{\dagger} & \sigma \end{pmatrix} \succcurlyeq 0, \\ & X \in \mathbb{C}^{n\times n}. \end{aligned}$$

Gour and Spekkens. *New Journal of Physics*, 10(3):033023, 2008.

Watrous. *Chicago Journal of Theoretical Computer Science*, 2013.

Killoran, *Entanglement quantification and quantum benchmarking of optical communication devices.* Ph.D. thesis, 2012.

Minimum distance

Combine optimization over channels Λ with fidelity SDP

Proposition

The maximum fidelity $\max_\Lambda \mathcal{F}(\Lambda(\rho), \sigma)$ can be obtained as the solution of a SDP.

$$
\begin{aligned}
\max_{X,\{F_J\}_J} \quad & \frac{1}{2}\operatorname{tr}\left(X + X^\dagger\right) & \\
\text{s.t.} \quad & \begin{pmatrix} \sigma & X \\ X^\dagger & \Lambda_{\{F_J\}_J}(\rho) \end{pmatrix} \geq 0 & \text{(for fidelity)} \\
& \xi(\{F_J\}_J) = 0 & \text{(normalization)} \\
& F_J \geq 0 \ \forall J & \text{(positivity)} \\
& X \in \mathbb{C}^{n \times n} &
\end{aligned}
$$

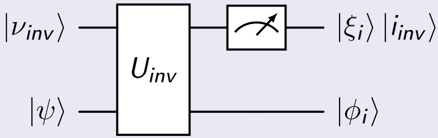
$\Rightarrow$ **solution for any pair of states, i.e. noisy input states**

Group-covariant transformations

Known results

Theory built on
characteristic functions

$$\chi_\psi(g) = \langle\psi| U(g) |\psi\rangle , \quad g \in \mathrm{SU}(2)$$



$$\Rightarrow \quad |\phi\rangle \otimes |\xi\rangle \text{ or } |\sigma\rangle$$

Problem: $\chi_\psi(\vec{r}) = \langle\psi| e^{i(r_x J_x + r_y J_y + r_z J_z)} |\psi\rangle$ impractical

$\Rightarrow$ **Need for an easy-to-handle representation of** SU(2)

Marvian. *Symmetry, Asymmetry and Quantum Information*. PhD thesis, 2012.

Polynomial SU(2) representation

Fundamental representation

$$U_{u,v} = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}, \ \det\left[U_{u,v}\right] = |u|^2 + |v|^2 = 1$$

Homogeneous bivariate polynomials of degree $2j$

$$f\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \sum_{m=-j}^{j} c_m z_1^{j+m} z_2^{j-m}$$

transform as

$$f\left[U_{u,v}^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \sum_{m=-j}^{j} c'_m z_1^{j+m} z_2^{j-m}$$

B. C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Springer, 2003.

Polynomial SU(2) representation

Representations for spin-j

$$U_{u,v}^{(j=\frac{1}{2})} = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}, \quad U_{u,v}^{(j=1)} = \begin{pmatrix} u^2 & -\sqrt{2}uv^* & v^{*2} \\ \sqrt{2}uv & uu^* - vv^* & -\sqrt{2}u^*v^* \\ v^2 & \sqrt{2}u^*v & u^{*2} \end{pmatrix}, \ldots$$

Characteristic functions become polynomials

$$\chi_\psi(u,v) = \langle\psi| U_{u,v} |\psi\rangle = \sum_{m',m\in\{-j,\ldots,j\}} U_{m',m}^{(j)} \psi_{m'}^* \psi_m$$

$\Rightarrow$ **We know how to deal with polynomials** 

Maximum postselection rate

Characteristic function theory + polynomials + SDP:

Proposition

The maximum postselection rate for the SU(2)-covariant trafo $|\psi\rangle \mapsto |\phi\rangle$ is given by

$$\max_{\xi,\sigma} \quad \mathrm{tr}(\xi)$$

$$\text{s.t.} \quad \chi_\psi = \chi_\phi \chi_\xi + \chi_\sigma \qquad \text{SU(2)-covariance}$$

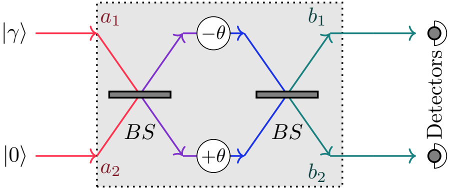
$$\xi, \sigma \geq 0 \qquad \text{states in the Stinespring dilation}$$

$\Rightarrow$ **solves the interconversion problem for generic pure states** $|\psi\rangle, |\phi\rangle$

Application to phase estimation

Goal: Estimate the phase applied to two light modes

Purple box: U(1) operation
Input state: $|\psi\rangle = |\gamma\rangle\,|0\rangle$ (coherent state $\otimes$ vacuum state)



$\Rightarrow$ **Improve input state** U(1)**-covariantly:**
pre-processing = post-processing

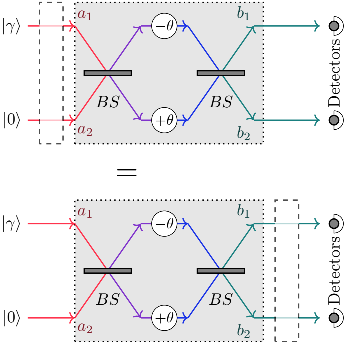
Tool: stochastic covariant transformations

Higher resolution

$$\Delta\theta_\phi \approx 0.74\Delta\theta_\psi$$

for the post-measurement state

$$|\phi\rangle = |\gamma(1-\epsilon)\rangle\left(\cos\tau\,|0\rangle - \sin\tau\,|2\rangle\right)$$



Summary

Characterization of state transformations under rotational covariance:

Minimum distance between $\Lambda(\rho)$ and a target state σ using SU(2)-covariant channels as a semidefinite program

Maximum probability for $|\psi\rangle \mapsto |\phi\rangle$ using SU(2)-covariant channels as a semidefinite program

Phase-estimation improvement with stochastic U(1)-covariant operations

Thank you for the attention!

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