arXiv:2310.00612
arXiv:2408.10323

Felix Huber, U. Bordeaux

Uncertainty relations and coding bounds from state polynomial optimization

joint works with:
Moisés Morán & Gerard Munné, Jagiellonian U. Kraków
Andrew Nemec, Texas A & M

Quantum uncertainty relations

Heisenberg uncertainty relation

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2\pi}$$

Pauli observable uncertainty

$$\Delta X + \Delta Y + \Delta Z \geq 2$$

where

$$\Delta \hat{O} = \langle \hat{O}^2 \rangle_\varrho - \langle \hat{O} \rangle_\varrho^2$$

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- Non-commuting operators cannot be simultaneously measured.
- How to find such relations systematically?

Generalizing the Pauli uncertainty

- Operators $\{A_i\}_{i=1}^n$ and $\chi_{ij} \in \{0, 1\}$ such that

$$A_i A_j = (-1)^{\chi_{ij}} A_j A_i \,, \quad A_i = A_i^\dagger \,, \quad A_i^2 = \mathbb{1}$$

- Encode relations in graph:

$$i \sim j \quad \text{if} \quad A_i A_j = -A_j A_i$$

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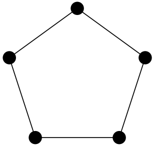
Example

$\mathcal{A}_1 = \{XZIII, IXZII, IIXZI, IIIXZ, ZIIIX\}$
$\mathcal{A}_2 = \{XIX, ZXI, IZX, ZZZ, ZIX\}$



- Same anti-commutativity graph
- Same uncertainty relation

$$\sum_{A\in\mathcal{A}_1} \Delta A = \sum_{A'\in\mathcal{A}_2} \Delta A' \quad \geq |\mathcal{A}| - \sum_{A\in\mathcal{A}} \langle A\rangle_\varrho^2 = n - \beta$$

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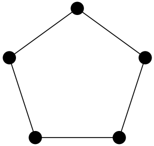

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(reversible Clifford circuit connects sets $\mathcal{A}_1$ and $\mathcal{A}_2$)

Aim: determine β

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Aim: determine

$$\beta = \sup_{\varrho, \mathcal{H}, A_i} \quad \sum_{i=1}^n \langle A_i \rangle_\varrho^2$$

Then $\sum_i \Delta A_i \geq n - \beta$ is a tight additive uncertainty relation.

- To find β, use NPA hierarchy

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Navascués-Pironio-Acín hierarchy / quantum Lasserre

- Goal: find largest eigenvalue of a non-commutative expression under constraints, e.g. max $\mathrm{tr}(P\varrho)$, P a polynomial.
- Applications: Bell inequalities, bounds on ground state energy

Key idea: index a moment matrix with monomials in operators $(X_1, \ldots, X_n)$ up to some degree ℓ, $M(P, Q) = \langle P^\dagger Q \rangle$

Example

Two operators A, B

$$M = \begin{array}{c} \\ \mathbb{1} \\ A \\ B \\ A^2 \\ AB \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbb{1} & A & B & A^2 & AB & \cdots \end{array} \\ \left[\begin{array}{cccccc} \langle \mathbb{1} \rangle & \langle A \rangle & \langle B \rangle & \langle A^2 \rangle & \langle AB \rangle & \\ & \langle A^\dagger A \rangle & \langle A^\dagger B \rangle & \langle A^\dagger A^2 \rangle & \langle A^\dagger AB \rangle & \\ & & \langle B^\dagger B \rangle & \langle B^\dagger A^2 \rangle & \langle B^\dagger AB \rangle & \\ & & & \langle (A^2)^\dagger A^2 \rangle & \langle (A^2)^\dagger AB \rangle & \\ & & & & \langle (AB)^\dagger AB \rangle & \\ & & & & & \ddots \end{array}\right] \end{array} \succeq 0$$

Navascués-Pironio-Acín hierarchy (II)

$$M \quad = \quad \begin{array}{c} \\ \mathbb{1} \\ A \\ B \\ A^2 \\ AB \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbb{1} & A & B & A^2 & AB & \dots \end{array} \\ \left[\begin{array}{cccccc} \langle \mathbb{1} \rangle & \langle A \rangle & \langle B \rangle & \langle A^2 \rangle & \langle AB \rangle & \\ & \langle A^\dagger A \rangle & \langle A^\dagger B \rangle & \langle A^\dagger A^2 \rangle & \langle A^\dagger AB \rangle & \\ & & \langle B^\dagger B \rangle & \langle B^\dagger A^2 \rangle & \langle B^\dagger AB \rangle & \\ & & & \langle (A^2)^\dagger A^2 \rangle & \langle (A^2)^\dagger AB \rangle & \\ & & & & \langle (AB)^\dagger AB \rangle & \\ & & & & & \ddots \end{array}\right] \end{array} \succeq 0 \, ,$$

- Apply constraints from observable relations:
  $M(P, Q) = M(R, S)$ if $P^\dagger Q = R \dagger S$
- Objective function is linear combination of entries.
- Maximize over all matrices $M \succeq 0$ satisfying the constraints.

Problem: $\beta = \sum_{i=1}^{n} \langle A_i \rangle^2$ is quadratic in the entries!

DPS/symmetric extension hierarchy

Idea: Use the Doherty-Parrilo-Spedalieri / symmetric extension hierarchy to get quadratic terms.

Quantum de Finetti theorem: Given k-partite ϱ_k. If for all $n \in \mathbb{N}$, there exists ϱ_n such that $\pi \varrho_n \pi^{-1} = \varrho_n$ for all $\pi \in S_n$ and

$$\mathrm{tr}_{n \setminus k}(\varrho_n) = \varrho_k$$

then

$$\varrho_k = \int \varrho^{\otimes k} d\mu(\varrho)$$

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- Approximate $M^{\otimes 2}$ by a hierarchy of $\mathrm{tr}_{n \setminus 2}(M_n)$

$$M_n(P, Q; E, F; \dots; K, L) = \langle P^\dagger Q \rangle \langle E^\dagger F \rangle \cdots \langle K^\dagger L \rangle$$

- Convergerging sequence of upper bounds on $\beta = \sum_i \langle A_i \rangle^2$.

"Scalar extension" Pozas et al 2017, "State polynomial optimization" Klep et al 2023

In practise: relaxations

Useful relaxation: Index moment matrix with entries $\langle A_i^\dagger \rangle A_i$,
$\Gamma_{ij} = \langle A_i^\dagger \rangle \langle A_j \rangle \langle A_i A_j^\dagger \rangle$

$$\Gamma \quad = \quad \begin{array}{c} \\ \mathbb{1} \\ \langle A_1 \rangle A_1^\dagger \\ \vdots \\ \langle A_n \rangle A_n^\dagger \end{array} \begin{array}{c} \begin{array}{cccc} \mathbb{1} & \langle A_1 \rangle A_1^\dagger & \ldots & \langle A_n \rangle \rangle A_n \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right] \end{array} \succeq 0\,,$$

Properties:

- $\Gamma_{00} = \langle \mathbb{1} \rangle = 1$
- $\Gamma_{ii} = \Gamma_{i0} = \langle A_i^\dagger \rangle \langle A_i \rangle \langle A_i A_i^\dagger \rangle = \langle A_i^\dagger \rangle \langle A_i \rangle$
- max $\sum_{i=1}^{n} \Gamma_{ii}$ approximates β
- If Γ has these properties, then so has $\mathrm{Re}(\Gamma) = (\Gamma + \Gamma^T)/2$

Theta body

Optimization is over set

$$\mathrm{TH}(G) = \left\{ \mathrm{diag}(M) \,\middle|\, \begin{pmatrix} 1 & x^T \\ x & M \end{pmatrix} \succeq 0 \,, x_i = M_{ii} \ \forall i \,, M_{ij} = 0 \text{ if } i \sim j \right\}.$$

with $i \sim j$ if $A_i A_j = -A_j A_i$

- $TH(G)$ is also known as the theta body of G.
- Maximum over sum of $\mathrm{TH}(G)$ is the Lovász theta number.

This gives the bound

$$\alpha(G) \leq \beta \leq \vartheta(G)$$

where the lower bound is the independence number α.

Hastings/O'Donnel 2021, De Gois et al 2022

Stronger relaxations

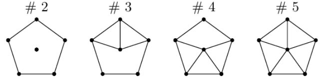
Index by k products of $\langle A_i^\dagger \rangle A_i$

- Efficiently computable bounds

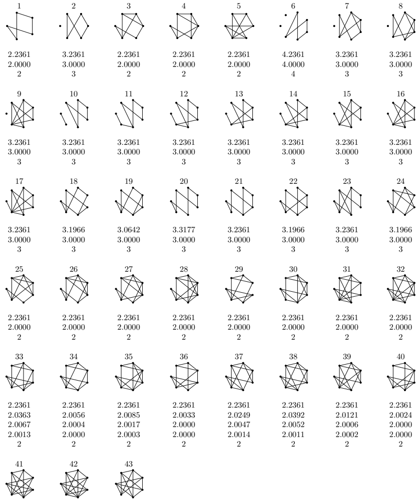
$$\alpha \leq \beta \leq \vartheta_k \leq \cdots \leq \vartheta_1$$

where α is the independence number of a graph.

- Generalizes to arbitrary operators (qudits…)



	# 2	# 3	# 4	# 5
ϑ_1	3.2361	2.2361	2.2361	2.2361
ϑ_2	3.0000	2.0000	2.0000	2.0000
α	3	2	2	2



Graph	ϑ_1	ϑ_2	ϑ_3	ϑ_7	α
1	2.2361	2.0000			2
2	3.2361	3.0000			3
3	2.2361	2.0000			2
4	2.2361	2.0000			2
5	2.2361	2.0000			2
6	4.2361	4.0000			4
7	3.2361	3.0000			3
8	3.2361	3.0000			3
9	3.2361	3.0000			3
10	3.2361	3.0000			3
11	3.2361	3.0000			3
12	3.2361	3.0000			3
13	3.2361	3.0000			3
14	3.2361	3.0000			3
15	3.2361	3.0000			3
16	3.2361	3.0000			3
17	3.2361	3.0000			3
18	3.1966	3.0000			3
19	3.0642	3.0000			3
20	3.3177	3.0000			3
21	3.2361	3.0000			3
22	3.1966	3.0000			3
23	3.2361	3.0000			3
24	3.1966	3.0000			3
25	2.2361	2.0000			2
26	2.2361	2.0000			2
27	2.2361	2.0000			2
28	2.2361	2.0000			2
29	2.2361	2.0000			2
30	2.2361	2.0000			2
31	2.2361	2.0000			2
32	2.2361	2.0000			2
33	2.2361	2.0363	2.0067	2.0013	2
34	2.2361	2.0056	2.0004	2.0000	2
35	2.2361	2.0085	2.0017	2.0003	2
36	2.2361	2.0033	2.0000	2.0000	2
37	2.2361	2.0249	2.0047	2.0014	2
38	2.2361	2.0392	2.0052	2.0011	2
39	2.2361	2.0121	2.0006	2.0002	2
40	2.2361	2.0024	2.0000	2.0000	2
41	2.2361	2.0910	2.0076	2.0024	2
42	2.2361	2.0000	2.0000	2.0000	2
43	2.1099	2.0950	2.0938	2.0938	2

Moisés Morán, FH, PRL 132, 200202 (2024)

Outlook

- When is $\alpha = \beta$?
  See Xu/Schwonnek/Winter PRX Quantum 5 (2), 020318 (2024)

- Position-momentum uncertainty also holds classically.
  A theory for distinguishing classical from quantum uncertainty / moment inequalities?

  For example, classically for all measures μ

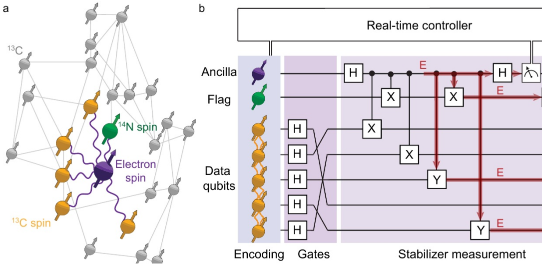
$$\left(\int x^4 y^2 \, \mathrm{d}\mu \right) \left(\int x^2 y^4 \, \mathrm{d}\mu \right) - \left(\int x^2 y^2 \, \mathrm{d}\mu \right)^3 \geq 0$$

  Klep et al. 2024

  What is the quantum bound?

- Other interesting non-linear expressions in QIT?
  $\longrightarrow$ Quantum Error correcting codes :)

Quantum computing



Blatt laboratory, Abobeih et al. 2022

- Information is physical (atoms, photons, electric charges . . .)
- Quantum physics is noisy / quantum information is fragile
- Quantum information cannot be treated classically (state collapse, no-cloning, continuity of states & errors)

. . . we need a way to protect quantum states from noise

Quantum error-correcting codes

- A quantum code encodes quantum information (e.g. a qubit) redundantly. The encoded state can be recovered after being affected by noise.
- Quantum codes form the backbone of quantum computers: allow for fault-tolerant processing of quantum information.

Example

$$\alpha |0\rangle + \beta |1\rangle \quad \mapsto \quad \alpha |000\rangle + \beta |111\rangle$$

- circumvents no-cloning.
- discretization of errors.
  (detect single flip by measurement of ZZI, ZIZ, IZZ)
- syndrome measurement collapses state onto code subspace.

Conditions for quantum error correction

- Code is a subspace $(\mathbb{C}^2)^{\otimes n}$, represented by a projector Π.
- Noise acts as a quantum channel on a state ϱ

$$\mathcal{N}(\varrho) = \sum_{E_\mu \in N} E_\mu(\varrho) E_\mu^\dagger$$

where $\sum_\mu E_\mu^\dagger E_\mu = \mathbb{1}$.

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- Think of this as a type of sphere packing: equivalent to

$$\langle \phi | E_\mu E_\nu | \psi \rangle = c_{\mu\nu} \langle \phi | \psi \rangle$$

  for all $E_\mu, E_\nu \in N$. and $|\phi\rangle, |\psi\rangle \in \mathrm{ran}(\Pi)$.

Error basis

Our goal: correct all tensor-product errors of at most weight δ.

The *Pauli matrices* form a basis for complex 2×2 matrices.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Tensor-product basis
  $\mathcal{E}_n = \left\{ E_\alpha = e_{\alpha_1} \otimes \ldots \otimes e_n \,|\, e_i \in \{I, X, Y, Z\} \right\}$
- Weight $\mathrm{wt}(E_\alpha)$ is the number of coordinates where E_α acts non-trivially. E.g. $\mathrm{wt}(IXIZZ) = 3$

Fundamental question in (quantum) coding theory

A $((n, K, \delta))$ quantum code is then a projector Π on $(\mathbb{C}^2)^{\otimes n}$ of rank K, such that

$$\Pi E_a^\dagger E_b \Pi = c_{ab} \Pi$$

for all $E_a, E_b \in \mathcal{E}_n$ with $\mathrm{wt}(E_a^\dagger E_b) < \delta$.

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- A code is *pure*, if $c_{ab} \propto \mathrm{tr}(E_a E_b)$ for $1 < \mathrm{wt}(E_a^\dagger E_b) < \delta$. (maximally mixed marginals)
- A code is *self-dual* if $K = 1$ and the code is pure.

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Question

For a given block length n and distance d, what is the maximal size K for quantum codes $((n, K, d))_D$ on n quDits?

- Linear programming bounds
- Analytical bounds (q. Singleton)
- SDP bounds. . . ?

Contributions

Result (A)

A quantum code with parameters $((n, K, \delta))_2$ *exists if and only if a certain SDP hierarchy is feasible at every level.*

Result (B)

The Lovász number bounds the existence of self-dual quantum codes. A symmetrization recovers the quantum Delsarte bound.

Result (C)

There is an SDP of size $O(n^4)$ *based on the Terwilliger algebra. Codes with parameters* $((7, 1, 4))_2$, $((8, 9, 3))_2$, *and* $((10, 5, 4))_2$ *do not exist.*

An attempt: Stabilizer codes

- Stabilizer group $S = \langle g_1, \dots, g_{n-k} \rangle$, $-\mathbb{1} \notin S$, g_i generators.
- Projector onto the code subspace

$$\Pi = \frac{1}{2^{n-k}} \sum_{i=1}^{n-k} (\mathbb{1} + g_i) = \frac{1}{2^{n-k}} \sum_{s \in S} s$$

- Simplest case: graph states $g_i = X_i \bigotimes_{j \in N(i)} Z_j$
- Commutative group: $[g_i, g_j] = 0$ for all g_i, g_j.

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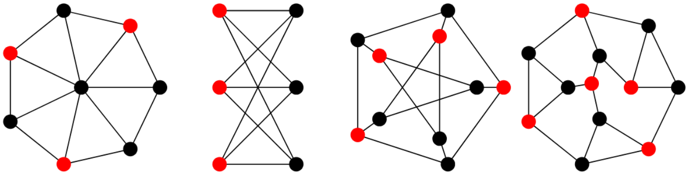
Is there a stabilizer code with $K = 1$ with distance δ
(i.e. a $\delta - 1$-uniform graph state)?

- Need maximal commuting subgroup of Pauli group.
- If $E_a E_b = -E_b E_a$, then not both can be in S.
- If $1 < \mathrm{wt}(E_a) < \delta$, then E_a is not in S.

Independence number

Independence number (maximum size of disconnected set)

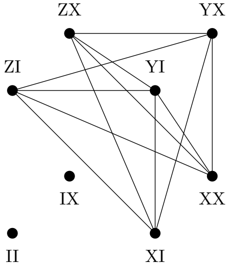
$$\alpha \quad = \quad \max_{H \subset V} \quad |H| \quad \text{s.t.} \quad (i,j) \notin E \quad \text{for all} \quad i,j \in H$$



- It is known that $\alpha(G) \leq \theta(G)$
- ϑ is efficiently computable, α is not

Lovász bound for stabilizer codes

- Index by "Pauli cube":
  $\mathcal{E}_{n,\delta} = \{E_a \in \mathcal{E}_n \mid \mathrm{wt}(E_a) \geq \delta\}$
- Confusability graph:
  $a \sim b$ if $E_a E_b = -E_b E_a$.
- If a self-dual stabilizer code with distance δ exists, then there is an independent set of size $\alpha = 2^n$
- As a consequence: $2^n \leq 1 + \vartheta(G)$.



Lovász bound for all quantum codes

Key idea (c.f. uncertainty relations)

- Write $\langle E_a \rangle$ for $\mathrm{tr}(E_a \varrho)$ with $\varrho = \Pi/K$.
- Construct moment matrix $\Gamma_{ab} = \langle E_a^\dagger \rangle \langle E_b \rangle \langle E_a E_b^\dagger \rangle$ for $E_\alpha \in \mathcal{E}_n$,

$$\Gamma = \begin{array}{c} \mathbb{1} \\ \langle E_1 \rangle E_1^\dagger \\ \vdots \\ \langle E_N \rangle E_N^\dagger \end{array} \overset{\begin{array}{cccc} \mathbb{1} & \langle E_1 \rangle E_1^\dagger & \cdots & \langle E_N \rangle E_N^\dagger \end{array}}{\begin{bmatrix} 1 & \Gamma_{01} & \cdots & \Gamma_{0N} \\ & \Gamma_{11} & \dots & \Gamma_{1N} \\ & & \ddots & \vdots \\ & & & \Gamma_{NN} \end{bmatrix}} \succeq 0\,,$$

- For two qubits, one would index with $\langle II \rangle II, \langle IX \rangle IX, \dots, \langle YZ \rangle YZ, \langle ZZ \rangle ZZ$.

Lovász bound for all quantum codes (II)

$$\Gamma = \begin{array}{c} \mathbb{1} \\ \langle E_1^\dagger \rangle E_1 \\ \vdots \\ \langle E_N^\dagger \rangle E_N \end{array} \overset{\begin{array}{cccc} \mathbb{1} & \langle E_1 \rangle E_1^\dagger & \cdots & \langle E_N \rangle E_N^\dagger \end{array}}{\begin{bmatrix} 1 & \Gamma_{01} & \cdots & \Gamma_{0N} \\ & \Gamma_{11} & \dots & \Gamma_{1N} \\ & & \ddots & \vdots \\ & & & \Gamma_{NN} \end{bmatrix}} \succeq 0 \,,$$

$$N = 4^n - 1$$

Consider $K = 1$. Then:

- $\Gamma_{00} = \langle \mathbb{1} \rangle = 1$
- $\Gamma_{ab} = \Gamma_{a0} = \langle E_a \rangle^2$
- $\sum_{a=0}^{N} \Gamma_{aa} = 2^n$, corresponding to $\mathrm{tr}(\varrho^2) = 1$.

Note: If Γ is a valid moment matrix, then so is $(\Gamma + \Gamma^T)/2$.
Impose extra condition:

- $\Gamma_{ab} = 0 \quad \text{if} \quad E_a E_b = -E_b E_a.$

Lovász bound for all quantum codes (III)

This corresponds to a hyperplane in the theta body

$$\mathrm{TH}(G) = \left\{ \mathrm{diag}(M) \,\middle|\, \begin{pmatrix} 1 & x^T \\ x & M \end{pmatrix} \succeq 0 \,, x_a = M_{aa} \; \forall a \,, M_{ab} = 0 \text{ if } a \sim b \right\}.$$

where the *quantum confusability graph* has $\mathcal{E}_n \backslash \mathbb{1}$ as vertices and

$$a \sim b \quad \text{if} \quad \begin{cases} 0 < \mathrm{wt}(E_a E_b) < \delta & \text{or} \\ E_a E_b = -E_b E_a & \end{cases}$$

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Lovász bound on self-dual quantum codes

If a $((n, 1, \delta))$ code exists, then
b) $TH(G)$ contains an element with $2^n = 1 + \sum_{a=1}^{N} M_{aa}$ and
a) $2^n \leq \vartheta(G) + 1$

- Already excludes the $((4, 1, 3))$ code / four-qubit AME state.

Symmetry-reduction

This scales badly: Pauli cube has 4^n elements!

- Average Γ over all row and column permutations which keep triples of weights $i = \mathrm{wt}(E_a)$, $j = \mathrm{wt}(E_b)$, and $k = \mathrm{wt}(E_a^\dagger E_b)$ invariant.
- The resulting matrix

$$\tilde{\Gamma} = \sum_{\pi \in \mathrm{Aut}_0} \pi \Gamma \pi^{-1}$$

  can be block-diagonalized with the Terwilliger algebra.
- This results in an SDP of size $O(n^4)$.

Gijswijt, Schrijver, Tanaka, J. Comb. Theory, A 113, 8, 2006, 1719-1731

$\longrightarrow$ Efficiently computable Lovasz bounds!

Complete SDP hierarchy for code existence

1. Formulate the Knill-Laflamme conditions $\Pi E_a E_b \Pi = c_{ab}\Pi$ as

$$K \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=j}} \mathrm{tr}(E\varrho E^\dagger \varrho) = \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=j}} \mathrm{tr}(E\varrho)\,\mathrm{tr}(E^\dagger \varrho)$$

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2. State polynomial optimization: Consider non-commutative letters $\{x_i\}$. Form words $w = x_{j_1} \dots x_{j_k}$. Associate expectations $\langle w \rangle$ behaving as $v\langle w \rangle = \langle w \rangle v$ and $\langle v\langle w \rangle\rangle = \langle v \rangle\langle w \rangle$. State monomials have the form $w_{i_1}\langle w_{i_2}\rangle \dots \langle w_{i_m}\rangle$. Use Positivstellensatz for positive state polynomials / corresponding moment hierarchy. (Note: Γ from above is an intermediate level!) Klep et al. 2023

- This recovers RHS in above condition. LHS…?

Complete SDP hierarchy (II)

3. Use the quantum MacWilliams identity

$$B(x,y) = A\left(\frac{x+3y}{2}, \frac{x-y}{2}\right),$$

where $A(x,y) = \sum_{j=0}^{n} A_j(\Pi) x^{n-j} y^j$ and likewise for $B(x,y)$.

4. Hierarchy is dimension-free: restrict to qubits by characterization of quasi-Clifford algebras with generator relations $\alpha_i \alpha_j = (-1)^{\chi_{ij}} \alpha_j \alpha_i$, $\chi_{ij} \in \{0,1\}$ and $\alpha_i^2 = 1$.

Gastineau-Hills 1982

5. Impose that $\varrho = \Pi/K$: use swap-like constraints,

$$\operatorname{tr}(\varrho^m) = \operatorname{tr}\left((1,2,\ldots,m)\varrho^{\otimes m}\right) = \frac{1}{K^{m-1}}$$

expanded in Pauli matrices.

Applications

- Averaging the Lovász bound over distance-preserving automorphism leads to the quantum Delsarte bound,

$$\begin{aligned} \eta = \max \quad & \sum_{j=0}^{n} A_j\,, \\ \text{subject to} \quad & A_0 = 1\,, \quad A_j \geq 0 \;\; \text{with equality for} \;\; 1 < j < \delta\,, \\ & \sum_{i=0}^{n} K_j(i) A_i \geq 0 \quad \text{for } j = 0, \ldots, n\,. \end{aligned}$$

  If $\eta < 2^n$, then code does not exist.

- Hierarchy with $O(n^4)$ scaling: Average over distance and zero-preserving autormorphisms. Symmetry-reduce using the 4-ary Terwilliger algebra. Gijswijt, Schrijver, Tanaka 2006

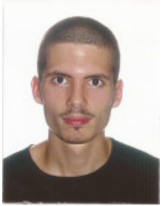
- Infeasibility certificates for $((7, 1, 4))$, $((8, 9, 3))$, $((10, 5, 4))$ codes.

Contributions

- Complete hierarchies of SDP bounds for uncertainty relations and the existence of quantum codes.
- Quantum analogies of the classical Lovász and Delsarte bounds.
- Numerically practical relaxations.
- Flexibility of applications, formally dimension-free: extensions to qudit codes & more general confusability graphs possible.

M. Morán, FH, Uncertainty relations from state polynomial optimization, `arXiv:2408.10323`

G. Munné, A. Nemec, FH, SDP bounds on quantum codes, `arXiv:2310.00612`, PRL 132, 200202 (2024)


Support: Foundation for Polish Science TEAM-NET POIR.04.04.00-00-17C1/18-00 (AB, GM, FH) and QLCI (AN)

Outlook

- Other applications for non-linear expressions in expectations?
- A theory for classical vs quantum moments?
- More general settings: quantum capacity of a graph?
- Rational certificates for exact non-existence proof.