Reachability, Accessibility, and Observability in Quantum Systems Theory

Aspects of a Unified Lie Framework

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joint work with

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Questions by the Quantum Engineer

... To Be Answered by the Mathematician

To which extent can a quantum dynamical system be

- controlled (closed) or accessed (open) ?
- simulated ?
- observed, sensed or tomographied ?

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To which extent can a quantum dynamical system be

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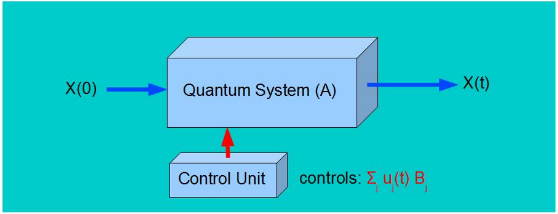
What can one infer just from its

- Hamiltonian (or Kossakowski-Lindblad) generators?
- ... and their symmetries ?

Systems Theory



See controlled Schrödinger eq. $|\dot\psi(t)\rangle = -i(H_0 + \sum_j u_j(t) H_j)|\psi(t)\rangle$ as

- bilinear control system: $\dot x(t) = (A + \sum_j u_j(t) B_j) x(t)$ with $x(0) = x_0$

Algebraic Characterisation

system algebra $\mathfrak{k} := \langle A, B_j \,|\, j = 1, 2, ..., m \rangle_{\text{Lie}}$

symmetries 1°: $\mathfrak{k}' := \{ s \in \mathfrak{gl}(N) \,|\, [s, A]=0=[s, B_j],\ \forall j \}$

See controlled Liouville eqn. $|\dot\rho(t)\rangle = -i\,(\widehat{H}_0 + \sum_j u_j(t)\widehat{H}_j)\,|\rho(t)\rangle$ as

- bilinear control system: $\dot X(t) = (\mathrm{ad}_A + \sum_j u_j(t)\mathrm{ad}_{B_j})\,X(t)$ w. $X(0) = X_0$

Algebraic Characterisation

system algebra 1 $\quad \mathfrak{k} := \langle A, B_j \,|\, j = 1, 2, ..., m\rangle_{\mathrm{Lie}}$

Reachable Set of States ($\mathfrak{k}$ compact)

$$\mathrm{Reach}(\rho_0) = \{K\rho_0 K^\dagger \,|\, K \in \langle \exp \mathfrak{k}\rangle\} =: \mathcal{O}_{\mathbf{K}}(\rho_0) = \mathrm{Ad}_{\mathbf{K}}(\rho_0)$$

NB: $\mathrm{ad}_A(X) = [A, X] \mathrel{\widehat{=}} \widehat{A}\,\mathrm{vec}(X) := (\mathbb{1} \otimes A - A^\top \otimes \mathbb{1})\,\mathrm{vec}(X)$ and $\mathrm{Ad}_{\mathbf{K}}(\cdot) := \mathbf{K}(\cdot)\mathbf{K}^\dagger = (\exp \mathrm{ad}_{\mathfrak{k}})(\cdot)$

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- bilinear control system: $\dot{X}(t) = (\mathrm{ad}_A + \sum_j u_j(t)\mathrm{ad}_{B_j})\,X(t)$ w. $X(0) = X_0$

Algebraic Characterisation

system algebra 1 $\quad \mathfrak{k} := \langle A, B_j \,|\, j = 1, 2, \ldots, m\rangle_{\text{Lie}}$

system algebra 2: $\quad \mathrm{ad}_{\mathfrak{k}} := \langle \mathrm{ad}_A, \mathrm{ad}_{B_j} \,|\, j = 1, 2, \ldots, m\rangle_{\text{Lie}}$

symmetries 2°: $\quad \mathrm{ad}'_{\mathfrak{k}} := \{S \in \mathfrak{gl}(N^2) \,|\, [S, \mathrm{ad}_A]=0=[S, \mathrm{ad}_{B_j}],\ \forall j\}$

NB: Notation $\mathrm{ad}_A \mathrel{\hat{=}} \widehat{A} := (\mathbb{1} \otimes A - A^\top \otimes \mathbb{1})$

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Algebraic Characterisation

system algebra 1 $\quad \mathfrak{k} := \langle A, B_j \,|\, j = 1, 2, \ldots, m\rangle_{\mathrm{Lie}}$ $\quad = \mathfrak{su}(n)$ (universal)

system algebra 2: $\quad \mathrm{ad}_{\mathfrak{k}} := \langle \mathrm{ad}_A, \mathrm{ad}_{B_j} \,|\, j = 1, 2, \ldots, m\rangle_{\mathrm{Lie}}$

symmetries 2°: $\quad \mathrm{ad}'_{\mathfrak{k}} := \{S \in \mathfrak{gl}(N^2) \,|\, [S, \mathrm{ad}_A] = 0 = [S, \mathrm{ad}_{B_j}],\ \forall j\}$ $\quad = 2$-dim. (triv.)

NB: Notation $\mathrm{ad}_A \hat{=} \widehat{A} := (\mathbb{1} \otimes A - A^\top \otimes \mathbb{1})$

Cor. 22 in JMP **52** (2011), 113510

Theorem details
fast

Theorem (universality by trivial ad-symmetries)

Let $\{H_\nu \,|\, \nu = d; 1, 2, \dots, m\}$ *be drift and control Hamiltonians of control system* Σ *with irreducible simple system algebra* $\mathfrak{k}$.

Then Σ *is fully controllable, i.e.* $\mathfrak{k} = \mathfrak{su}(2^n)$, *if and only if*

- *the joint commutant to* $\mathrm{ad}_\mathfrak{k}$ *is two-dimensional, i.e.*

  $$\mathrm{ad}_\mathfrak{k}{}' = \mathrm{span}\{\mathbb{1}^{\otimes 2}, |\mathbb{1}\rangle\langle\mathbb{1}|\}.$$

Symmetry vs. Controllability

Single Symmetry Condition

JMP **52** 113510 (2011), OSID **24** 1740019 (2017)

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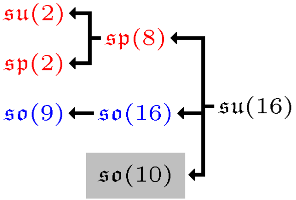

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- CARTAN subalgebras: $\mathfrak{so}(N)$ and $\mathfrak{usp}(\frac{N}{2})$ have intertwiners S,S
  $H_\nu S + S H_\nu^t = 0$ with $S\bar{S} = \mathbb{1}$ resp. $S\bar{S} = -\mathbb{1}$
- intertwiners add symmetry to $\mathrm{ad}_{\mathfrak{so},\mathfrak{usp}}$: $[\mathrm{ad}_{H_\nu}, |S\rangle\langle S|^{\Gamma_1}] = 0, \; \forall \nu$

  NB: $|S\rangle\langle S|^{\Gamma_1} = K(S \otimes S^\dagger)$

Observability Result

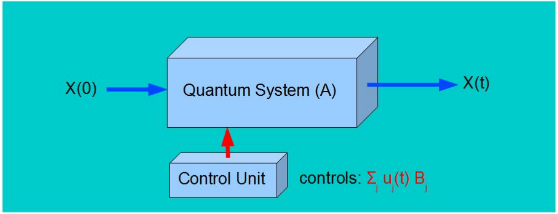
Let the system algebra $\mathfrak{k}_\Sigma$ of (Σ) be an irreducible subalgebra of $\mathfrak{su}(N)$. Then the bilinear control system (Σ) is observable by $C = C^\dagger$ if and only if

- *the joint commutant to $\mathrm{ad}_\mathfrak{k}$ and $P_{\tilde{c}} := |\widetilde{C}\rangle\langle\widetilde{C}|$ is two-dimensional, i.e.*
- $\dim \left((\{P_{\tilde{c}}\} \cup \{i\,\mathrm{ad}_{H_\nu} \,|\, \nu = 0, 1, \ldots, m\})' \right) = 2.$

NB well known: a fully controllable system is always observable, but an observable system need not be fully controllable.

Notation: $\tilde{C}$ traceless part of C and $\mathrm{ad}_H \hat{=} \mathbf{1} \otimes H - H^\top \otimes \mathbf{1}$.

Systems Theory



Consider observed *linear* control system

- control part: $\dot{x}(t) = Ax(t) + Bv$
- observation part: $y(t) = Cx(t)$

Conditions for Full Controllability and Observability (cp. cyclic vectors)

- controllable $\Leftrightarrow$ $\operatorname{rank}\,[B, AB, A^2B, \dots, A^{N-1}B] = N$
- observable $\Leftrightarrow$ $\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} = N$

Take observed *(bi)linear* system with constant control

[Elliott (2008), Sec. 5.3.2; Grasselli & Isidori (1977)]

- control part: $|\dot{X}(t)\rangle = (\widehat{A} + u\widehat{B})|X(t)\rangle =: \widehat{A}_u|X(t)\rangle$
- observation part: $y(t) = \langle C|X(t)\rangle$

Observability Condition (cp. cyclic vectors)

- observable $\Leftarrow \exists u \in \mathbb{R}$ s.th. $\text{rank} \begin{bmatrix} \langle C| \\ \langle C|\widehat{A}_u \\ \vdots \\ \langle C|\widehat{A}_u^{N^2-1} \end{bmatrix} = N^2$

NB: $A, B, C, X \in \mathbb{C}^{N\times N}$ and $\widehat{A} := \mathbb{1} \otimes A - A^\top \otimes \mathbb{1}$ acting on $|X\rangle \equiv \text{vec}\, X$ (or likewise on $\langle C|$).

Consider observed N-level bilinear control system (Σ)

$$\begin{aligned} \dot\rho(t) &= -i(\hat H_0 + \Sigma_j u_j(t)\hat H_j)\,\rho(t) \quad \rho(0)\equiv\rho_0 \quad \text{and} \\ y(t) &= \operatorname{tr}\{C\,\rho(t)\} \end{aligned}$$

with system algebra $\mathfrak{k}_\Sigma := \langle (iH_0, iH_j \,|\, j=1,\dots,m\rangle_{\text{Lie}} \subseteq \mathfrak{su}(N)$

Definition (variation of D'Alessandro (2003) and (2008))

Consider the system (Σ) observed by C. W.r.t. the system algebra $\mathfrak{k}_\Sigma$, its

- observability space can be defined as

  $\mathcal{O}_\Sigma(C) := \operatorname{span}_{\mathbb{R}}\{\operatorname{ad}^\nu_{\mathfrak{k}}(i\tilde C)\,|\,\nu=0,1,2,...\}$

  with $\operatorname{ad}^\nu_{\mathfrak{k}}(i\tilde C) := \{[k_1,[k_2,...[k_\nu, i\tilde C]...]]\,|\,k_i \in \{iH_0, iH_1,\dots,iH_m\}\}$

write $\tilde C$ for traceless part of C

NB: $\mathcal{O}_\Sigma(C)$ comprises the orbit $\mathcal{O}_{\mathbf{K}_\Sigma}(i\tilde C) := \mathbf{K}(i\tilde C)\mathbf{K}^\dagger = \exp \operatorname{ad}_{\mathfrak{k}}(i\tilde C) = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\operatorname{ad}^\nu_{\mathfrak{k}}(i\tilde C) \subset \mathcal{O}_\Sigma(C)$.

Consider observed N-level bilinear control system (Σ)

$$\begin{aligned} \dot\rho(t) &= -i(\hat H_0 + \Sigma_j u_j(t)\hat H_j)\,\rho(t) \quad \rho(0) \equiv \rho_0 \quad \text{and} \\ y(t) &= \mathrm{tr}\{C\,\rho(t)\} \end{aligned}$$

with system algebra $\mathfrak{k}_\Sigma := \langle (iH_0, iH_j \,|\, j = 1, \dots, m\rangle_{\text{Lie}} \subseteq \mathfrak{su}(N)$

Definition (D'Alessandro (2003) and (2008))

The system (Σ) is observable by C iff for any pair $\tilde\rho_1, \tilde\rho_2$ of states the equality

$$\mathrm{tr}\{C\,\tilde\rho_1(t)\} = \mathrm{tr}\{C\,\tilde\rho_2(t)\} \quad \forall t \in \mathbb{R} \text{ and joint controls } u_j(t)$$

implies $\tilde\rho_1 = \tilde\rho_2$, which is the case if and only if

$$\mathcal{O}_\Sigma(C) \overset{\text{iso}}{=} \mathfrak{su}(N)\ .$$

write $\tilde\rho$ for traceless part of ρ

NB: $\mathfrak{su}(N)$ comprises all (skew)hermitian matrices and thus is informationally complete.

A B

Example (symmetric observability space)

$\mathfrak{k}_\Sigma = \langle i\sigma_{x1}, i\sigma_{y1}, i\sigma_{1x}, i\sigma_{1y}\rangle_{\text{Lie}}$ and observable $i\tilde{C} = i\sigma_{zz} \Rightarrow \mathcal{O}_\Sigma(C) = \mathfrak{p}_\Sigma$ with

$$\begin{aligned} \mathfrak{k}_\Sigma &= \big(\mathfrak{su}(2)\otimes \mathbb{1} + \mathbb{1}\otimes\mathfrak{su}(2)\big) \stackrel{\text{iso}}{=} \big(\mathfrak{su}(2)\oplus\mathfrak{su}(2)\big) \stackrel{\text{iso}}{=} \mathfrak{so}(4) \\ \mathfrak{p}_\Sigma &= i\langle \sigma_{xx}, \sigma_{xy}, \sigma_{xz}; \sigma_{yx}, \sigma_{yy}, \sigma_{yz}; \sigma_{zx}, \sigma_{zy}, \sigma_{zz}\rangle \end{aligned}$$

NB: $\mathfrak{so}(4)$ semisimple and $\mathfrak{so}(N)$ simple for $N \geq 5$.

pro memoria: $\mathfrak{g}$ (semi)simple Lie algebra. CARTAN decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ with $\mathfrak{p} = \mathfrak{k}^\perp \cap \mathfrak{g}$ as well as $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$ and $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$.

A B

Example (full observability space)

$\mathfrak{k}_\Sigma = \langle i\sigma_{x1}, i\sigma_{y1}, i\sigma_{1x}, i\sigma_{1y}\rangle_{\text{Lie}}$ and observable $i\tilde{C} = i(\sigma_{zz} + \sigma_{z1} + \sigma_{1z})$

$$\begin{aligned} \mathfrak{k}_\Sigma &= (\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \overset{\text{iso}}{=} \mathfrak{so}(4) \\ \mathfrak{p}_\Sigma &= i\langle \sigma_{xx}, \sigma_{xy}, \sigma_{xz}; \sigma_{yx}, \sigma_{yy}, \sigma_{yz}; \sigma_{zx}, \sigma_{zy}, \sigma_{zz}\rangle \\ \mathcal{O}_\Sigma(C) &= \mathfrak{su}(4) = \mathfrak{k}_\Sigma \oplus \mathfrak{p}_\Sigma \end{aligned}$$

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Observability of Closed Systems

Observed Bilinear Control Systems: CARTAN Control Algebra

Let (Σ) be a bilinear control syst. observed by C with simple system algebra $\mathfrak{k}_\Sigma := \langle (iH_0, iH_j \,|\, j = 1, \dots, m \rangle_{\text{Lie}} \subseteq \mathfrak{g} \equiv \mathfrak{su}(N)$, observability space $\mathcal{O}_\Sigma(C) := \text{span}_\mathbb{R}\{\text{ad}_\mathfrak{k}^\nu(i\tilde{C}) \,|\, \nu = 0, 1, 2, \dots\}$.

Theorem (Structure of Observability Space)

- $\mathfrak{k}_\Sigma$ *is in a* CARTAN *subalgebra of su(N) iff there is a unitary S with* $|S\rangle \in \ker(\mathbb{1} \otimes H_j + H_j \otimes \mathbb{1})$ *jointly for all j.*
- $\mathfrak{k}_\Sigma \subseteq \mathfrak{so}(N)$ *iff* $S\bar{S} = +\mathbb{1}$ *or* $\mathfrak{k}_\Sigma \subseteq \mathfrak{usp}(\frac{N}{2})$ *iff* $S\bar{S} = -\mathbb{1}$.
- *Then for the observability space one has (with same S)*

$$\mathcal{O}_\Sigma(C) = \begin{cases} \text{Lie algebra } \mathfrak{k} \subseteq \mathfrak{k}_\Sigma & \text{for } |S\rangle \in \ker(\mathbb{1} \otimes \tilde{C} + \tilde{C} \otimes \mathbb{1}) \\ \text{symmetric space } \mathfrak{p} \subseteq \mathfrak{p}_\Sigma & \text{for } |S\rangle \in \ker(\mathbb{1} \otimes \tilde{C} - \tilde{C} \otimes \mathbb{1}) \\ \text{linear space } \mathfrak{l} \subseteq \mathfrak{su}(N) & \text{for } i\tilde{C} = i\tilde{C}_\mathfrak{k} + i\tilde{C}_\mathfrak{p},\ \tilde{C}_{\mathfrak{k},\mathfrak{p}} \neq 0. \end{cases}$$

- $\mathfrak{l} = \mathcal{O}_\Sigma(\tilde{C}_\mathfrak{k}) \oplus \mathcal{O}_\Sigma(\tilde{C}_\mathfrak{p}) = \mathfrak{k}_\Sigma \oplus \mathfrak{p}_\Sigma = \mathfrak{su}(N)$

- System algebra $\mathfrak{k}_\Sigma$ is simple irred. CARTAN subalg of $\mathfrak{su}(N)$

Theorem (Observability in Simple CARTAN System Algebras)

A bilinear control system (Σ) *with* $\mathfrak{k}_\Sigma \subseteq \mathfrak{so}(N)$ *or* $\mathfrak{k}_\Sigma \subseteq \mathfrak{usp}(\frac{N}{2})$ *is observable by* C *if and only if the observable* $\tilde{C} = \tilde{C}_\mathfrak{k} + \tilde{C}_\mathfrak{p}$ *has non-vanishing components in* $\mathfrak{k}_\Sigma$ *and* $\mathfrak{p}_\Sigma$*, i.e.* $\tilde{C}_\mathfrak{k} \neq 0 \neq \tilde{C}_\mathfrak{p}$.

So for the unique vec. 'Obata unitary' $|S\rangle \in \ker(\mathbb{1} \otimes H_j + H_j \otimes \mathbb{1})|_{\forall j}$ *one has to have*

- $|S\rangle \notin \ker(\mathbb{1} \otimes \tilde{C} \pm \tilde{C} \otimes \mathbb{1})$ *for both choices of signs;*

hence also $|S\rangle\langle S| \notin (\mathbb{1} \otimes \tilde{C} \pm \tilde{C} \otimes \mathbb{1})'$.

Consider observed N-level bilinear control system (Σ)

$$\begin{aligned} \dot\rho(t) &= -i(\hat H_0 + \Sigma_j u_j(t)\hat H_j)\,\rho(t) \quad \rho(0) \equiv \rho_0 \quad \text{and} \\ y(t) &= \operatorname{tr}\{C\,\rho(t)\} \end{aligned}$$

irred. simple system algebra $\mathfrak{k}_\Sigma := \langle (iH_0, iH_j \,|\, j = 1, \dots, m\rangle_{\text{Lie}} \subseteq \mathfrak{g} \equiv \mathfrak{su}(N)$

observability space $\mathcal{O}_\Sigma(C) := \operatorname{span}_{\mathbb{R}}\{\operatorname{ad}^{\nu}_{\mathfrak{k}}(i\tilde C) \,|\, \nu = 0, 1, 2, \dots\}$

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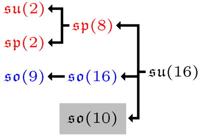
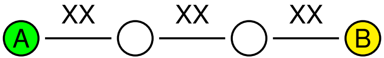
Structure of Observability Space (by $\mathrm{ad}_{\mathfrak{k}_\Sigma}$-Invariant Subspaces)

$$\mathcal{O}_\Sigma(C) \subseteq \begin{cases} \textit{Lie algebra } \mathfrak{k}_\Sigma & \textit{for } (i\tilde C) \in \mathfrak{k}_\Sigma \\ \textit{orthocomplement } \mathfrak{m}_\Sigma & \textit{for } (i\tilde C) \in (\mathfrak{k}_\Sigma^\perp \cap \mathfrak{g}) \equiv \mathfrak{m}_\Sigma \\ \textit{lin. space } \mathfrak{l} \subseteq \mathfrak{g} = \mathfrak{k}_\Sigma \oplus \mathfrak{m}_\Sigma & \textit{for } i\tilde C = i\tilde C_{\mathfrak{k}_\Sigma} + i\tilde C_{\mathfrak{m}_\Sigma} \end{cases}$$

$$\mathfrak{l} = \mathcal{O}_\Sigma(C_{\mathfrak{k}}) \oplus \mathcal{O}_\Sigma(C_{\mathfrak{m}}) \subseteq \mathfrak{g} \equiv \mathfrak{su}(N)$$

in $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} := \mathfrak{k}^\perp \cap \mathfrak{g}$ and $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{m}] \subseteq \mathfrak{m}$ take $\mathfrak{k}$ and $\mathfrak{m}$ as '$\mathrm{ad}_{\mathfrak{k}}$-invariant subspaces' of $\mathfrak{g}$





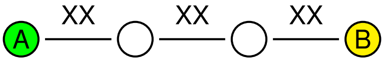
Example ($\mathrm{ad}_{\mathfrak{k}}$-invariant observability space $\mathfrak{m}$)

$\mathfrak{k}_\Sigma = \langle ix111, iy111, iXX, i111x, i111y \rangle_{\mathrm{Lie}}$, ($XX = xx11 + yy11 + 1xx1 + 1yy1 + 11xx + 11yy$)

observable $i\tilde{C} = ixxx1 \Rightarrow \mathcal{O}_\Sigma(C) = \mathfrak{m}_\Sigma \perp \mathfrak{k}_\Sigma$ with

$$\begin{aligned} \mathfrak{k}_\Sigma &= \mathfrak{so}(10) \quad \text{(45 dimensional)} \\ \mathcal{O}_\Sigma(C) &= \mathfrak{m}_\Sigma := \mathfrak{so}(10)^\perp \cap \mathfrak{su}(16) \quad \text{(210 dimensional)} \end{aligned}$$

NB: $\mathfrak{so}(10) \subset \mathfrak{su}(16)$ not CARTAN-type (FROBENIUS-SCHUR ind. $\imath := \int_g \chi(g^2)d\mu = \pm 1$ for real or quat. type), but complex ($\imath = 0$)



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$\mathfrak{k}_\Sigma = \langle ix111, iy111, iXX, i111x, i111y \rangle_{\text{Lie}}$, $(XX = xx11 + yy11 + 1xx1 + 1yy1 + 11xx + 11yy)$

observable $i\tilde{C} = i(xxx1 + 1z11)$

$$\mathfrak{k}_\Sigma = \mathfrak{so}(10) \quad \text{(45 dimensional)}$$

$$\mathfrak{m}_\Sigma = \mathfrak{so}(10)^\perp \cap \mathfrak{su}(16) \quad \text{(210 dimensional)}$$

$$\mathcal{O}_\Sigma(C) = \mathfrak{su}(16) = \mathfrak{k}_\Sigma \oplus \mathfrak{m}_\Sigma = \mathfrak{g}_\Sigma$$

NB: $\mathfrak{so}(10) \subset \mathfrak{su}(16)$ not of CARTAN-type, but complex $(\imath = 0)$

in $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} := \mathfrak{k}^\perp \cap \mathfrak{g}$ and $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$ take $\mathfrak{k}$ and $\mathfrak{m}$ as 'ad$_\mathfrak{k}$-invariant subspaces' of $\mathfrak{g}$

Recall for projections onto invariant subspaces:

Elementary Fact (e.g. S. Roman, *Advanced Linear Algebra* (2008), Thm. 2.24)

Let $F : V \to V$ be a linear map. $W \subsetneq V$ is F-invariant subspace if $F(W) \subseteq W$. Let P be the projector from V onto W. Then

- *W is F-invariant if and only if $PFP = FP$ and*
- *W and $W^\perp$ (in V) are both F-invariant if and only if $FP = PF$.*

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- *W and $W^\perp$ (in V) are both F-invariant if and only if $FP = PF$.*

Corollary (Commutant for Invariant Subspace)

In $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{m}$ by $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$ both $\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{k}^\perp$ are $\mathrm{ad}_\mathfrak{k}$-invariant.

Hence the projectors $P_\mathfrak{k}, P_\mathfrak{m}$ are in the commutant of $\mathrm{ad}_\mathfrak{k}$: $\{P_\mathfrak{k}, P_\mathfrak{m}\} \subset (\mathrm{ad}_\mathfrak{k})'$.

Observability of Closed Systems

Projectors onto Invariant Subspaces

Corollary (Commutant for Invariant Subspace)

In $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{m}$ by $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$ both $\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{k}^\perp$ are $\mathrm{ad}_\mathfrak{k}$-invariant.

Hence the projectors $P_\mathfrak{k}, P_\mathfrak{m}$ are in the commutant of $\mathrm{ad}_\mathfrak{k}$: $\{P_\mathfrak{k}, P_\mathfrak{m}\} \subset (\mathrm{ad}_\mathfrak{k})'$.

implies

Proposition

Let (Σ) be an bilinear control system with irred. (semi)simple system algebra $\mathfrak{k}_\Sigma = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_\kappa$ and assume $\mathfrak{su}(n) = \mathfrak{k}_\Sigma \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\mu$ with $\mathfrak{m}_j$ all $\mathrm{ad}_{\mathfrak{k}_\Sigma}$-invariant.[1]

Then system (Σ) is observable by $C \in \mathfrak{her}(n)$ iff for its rk-1 projector $P_{\tilde{c}} = |\widetilde{C}\rangle\langle\widetilde{C}|$

- $[P_{\tilde{c}}, P_{\mathfrak{k}_i}]_{i=1}^{\kappa} \neq 0 \neq [P_{\tilde{c}}, P_{\mathfrak{m}_j}]_{j=1}^{\mu}$

[1] So $[\mathrm{ad}_{\mathfrak{k}_\Sigma}, P_{\mathfrak{k}_i}]_{i=1}^{\kappa} = 0 = [\mathrm{ad}_{\mathfrak{k}_\Sigma}, P_{\mathfrak{m}_j}]_{j=1}^{\mu}$

again $\widetilde{C}$ traceless part of C

Corollary (Commutant for Invariant Subspace)

In $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{m}$ by $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{m}] \subseteq \mathfrak{m}$ both $\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{k}^\perp$ are $\mathrm{ad}_\mathfrak{k}$-invariant.

Hence the projectors $P_\mathfrak{k}, P_\mathfrak{m}$ are in the commutant of $\mathrm{ad}_\mathfrak{k}$: $\{P_\mathfrak{k}, P_\mathfrak{m}\} \subset (\mathrm{ad}_\mathfrak{k})'$.

implies

Proposition

Let (Σ) be an bilinear control system with irred. (semi)simple system algebra $\mathfrak{k}_\Sigma = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_\kappa$ and assume $\mathfrak{su}(n) = \mathfrak{k}_\Sigma \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\mu$ with $\mathfrak{m}_j$ all $\mathrm{ad}_{\mathfrak{k}_\Sigma}$-invariant.[1]

Then system (Σ) is observable by $C \in \mathfrak{her}(n)$ iff for its rk-1 projector $P_{\tilde{c}} = |\widetilde{C}\rangle\langle\widetilde{C}|$

- $[P_{\tilde{c}}, P_{\mathfrak{k}_i}]_{i=1}^{\kappa} \neq 0 \neq [P_{\tilde{c}}, P_{\mathfrak{m}_j}]_{j=1}^{\mu}$

Example: $\mathfrak{su}(8) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ with dims $63 = 3+3+3+9+9+9+27$

[1] So $[\mathrm{ad}_{\mathfrak{k}_\Sigma}, P_{\mathfrak{k}_i}]_{i=1}^{\kappa} = 0 = [\mathrm{ad}_{\mathfrak{k}_\Sigma}, P_{\mathfrak{m}_j}]_{j=1}^{\mu}$

again $\tilde{C}$ traceless part of C

Consider observed N-level closed bilinear control system (Σ) on the orbit of ρ_0

$$\dot\rho(t) = -i(\hat H_0 + \Sigma_j u_j(t)\hat H_j)\,\rho(t) \quad \rho(0) \equiv \rho_0 \quad \text{and}$$
$$y(t) = \operatorname{tr}\{C\,\rho(t)\}$$

system algebra $\mathfrak{k}_\Sigma := \langle (i\widehat{H}_0, i\widehat{H}_j \,|\, j = 1,\dots,m\rangle_{\mathrm{Lie}} \subseteq \mathrm{ad}_{\mathfrak{su}(N)}$

Definition (suggestion)

The closed system (Σ) is tomographiable by C w.r.t. ρ_0 iff both

1. the reachable set of ρ_0 under (Σ) has non-empty interior (i.e. is accessible) ,
2. the observability space of $\tilde C$ under (Σ) is informationally complete, i.e. all of $i\,\mathfrak{su}(N)$.

Notation: $\tilde C$ is traceless part.

open systems
fast

Consider observed N-level open Markovian bilinear control system (Σ)

$$\begin{aligned} \dot\rho(t) &= -(i\hat H_0 + i\Sigma_j u_j(t)\hat H_j + \hat\Gamma_{GKSL})\,\rho(t) \quad \rho(0) \equiv \rho_0 \quad \text{and} \\ y(t) &= \operatorname{tr}\{C\,\rho(t)\} \end{aligned}$$

NB: Lie wedge $\mathfrak{w}_\Sigma \subseteq \langle (i\hat H_0 + \hat\Gamma), i\hat H_j\rangle_{\text{Lie}} \subseteq \mathfrak{gl}(N^2, \mathbb{R})$

reachable set $\text{Reach}_\Sigma(\rho_0) := \mathbf{S}_\Sigma(\text{vec}(\rho_0))$, $\mathbf{S}_\Sigma := \overline{\langle \exp \mathfrak{w}_\Sigma \rangle}$.

Definition (suggestion)

The open system (Σ) is tomographiable by C w.r.t. ρ_0 if and only if both

1. (Σ) is accessible w.r.t. ρ_0 , (i.e. $\text{Reach}_\Sigma(\rho_0)$ has non-empty interior)
2. the observability space of $\tilde C$ under (Σ) is informationally complete.

Notation: $\tilde C$ is traceless part.

Map Accessibility of Open Systems

Symmetry Conditions

Let $\dot{X} = -(i\hat{H}_0 + \Gamma_{GKSL} + i\sum_j u_j(t)\hat{H}_j)X(t)$ be a N-level bil. control system Σ with

system algebra $\mathfrak{g}_\Sigma := \langle (i\hat{H}_0 + \Gamma), i\hat{H}_j \,|\, j = 1, \ldots, m\rangle_{\text{Lie}}$.

Corollary

The following two are equivalent:

1. *The unital n-qubit system variant is (map)accessible* $(N = 2^n)$.
2. *The unital system algebra has commutant of dimension 2 and (for $N > 2$) its $\mathfrak{k}$-part exceeds* $\mathrm{ad}_{\mathfrak{su}(N)}$.

Likewise one may conjecture the equivalence of

3. *The non-unital n-qubit system variant is (map)accessible.*
4. *The non-unital system algebra has commutant of dimension 1.*

Let $\dot{X} = -(i\hat{H}_0 + \hat{\Gamma}_{GKSL} + i\sum_j u_j(t)\hat{H}_j)X(t)$ be N-level bilinear control system (Σ) w.

system algebra $\mathfrak{g}_\Sigma := \langle (i\hat{H}_0 + \hat{\Gamma}_{GKSL}), i\hat{H}_j \,|\, j = 1, \dots, m\rangle_{\rm Lie} \subseteq \mathfrak{g}^{LK}$.

Corollary (standard)

The following are equivalent:

1. *The system is map accessible .*
2. *The reachable set* $\mathrm{Reach}(\mathbb{1})$ *is a subsemigroup* $\mathbf{S} \subset \mathbf{G}^{LK}$ *w. non-empty interior* $(\mathbf{G}^{LK} := \langle \exp \mathfrak{g}^{LK}\rangle)$.
3. *The unital system algebra is*

$$\mathfrak{g}_\Sigma \overset{\text{iso}}{=} \begin{cases} \mathfrak{gl}(N^2\text{-}1, \mathbb{R}) = \mathfrak{g}_0^{LK} \text{ or} \\ \mathfrak{so}(N^2\text{-}1) \oplus \mathbb{R}\,. \end{cases}$$

Accessibility of Open Systems

Analogue to Controllability in Closed Systems

Let $\dot{X} = -(i\hat{H}_0 + \hat{\Gamma}_{GKSL} + i\sum_j u_j(t)\hat{H}_j)X(t)$ be N-level bilinear control system (Σ) w.

system algebra $\mathfrak{g}_\Sigma := \langle (i\hat{H}_0 + \hat{\Gamma}_{GKSL}), i\hat{H}_j \,|\, j = 1,\dots,m\rangle_{\text{Lie}} \subseteq \mathfrak{g}^{LK}$.

Corollary (Kurniawan, Dirr, Helmke, IEEE TAC **52**, 1984 (2011))

The following are equivalent:

1. *The unital n-qubit system is map accessible* $(N = 2^n)$.
2. *The reachable set* $\text{Reach}(\mathbb{1})$ *is a subsemigroup* $\mathbf{S} \subset \mathbf{G}_0{}^{LK}$ *w. non-empty interior* $(\mathbf{G}_0^{LK} := \langle \exp \mathfrak{g}_0^{LK} \rangle)$.
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Accessibility of Open Systems

Analogue to Controllability in Closed Systems

Let $\dot{X} = -(i\hat{H}_0 + \hat{\Gamma}_{GKSL} + i\sum_j u_j(t)\hat{H}_j)X(t)$ be 2^n-level bilinear control system (Σ) w.

syst. algs $\mathfrak{g}_\Sigma := \langle (i\hat{H}_0 + \hat{\Gamma}_{GKSL}), i\hat{H}_j \,|\, j = 1, \dots, m\rangle_{\text{Lie}} \subseteq \mathfrak{g}_0^{\text{LK}}$ (unital), $\mathfrak{g}^{\text{LK}}$ (non-unital).

Note

In coherence-vector representation, elements in $\mathfrak{g}_0^{\text{KL}}$ and $\mathfrak{g}^{\text{KL}}$ and their commutants $(\mathfrak{g}_0^{\text{KL}})'$ and $(\mathfrak{g}^{\text{KL}})'$ take the form

$$\mathfrak{g}_0^{\text{KL}} \ni \Gamma_0 = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{and} \quad (\mathfrak{g}_0^{\text{KL}})' \ni \Gamma_0' = \left(\begin{array}{c|c} \alpha\mathbb{1} & 0 \\ \hline 0 & \beta \end{array}\right)$$

$$\mathfrak{g}^{\text{KL}} \ni \Gamma = \left(\begin{array}{c|c} A & a \\ \hline 0 & 0 \end{array}\right) \quad \text{and} \quad (\mathfrak{g}^{\text{KL}})' \ni \Gamma' = \left(\begin{array}{c|c} \gamma\mathbb{1} & 0 \\ \hline 0 & \gamma \end{array}\right),$$

where with $N = 2^n$, $a \in \mathbb{R}^{(N^2-1)}$ and $A \in \mathfrak{gl}(N^2-1, \mathbb{R})$ and $\alpha, \beta, \gamma \in \mathbb{R}$.

NB: observe semidirect-product structure $[(A, a), (B, b)] = ([A, B], Ab - Ba)$.

Let $\dot{X} = -(i\hat{H}_0 + \hat{\Gamma}_{GKSL} + i\sum_j u_j(t)\hat{H}_j)X(t)$ be 2^n-level bilinear control system (Σ) w. syst. algs $\mathfrak{g}_\Sigma := \langle (i\hat{H}_0 + \hat{\Gamma}_{GKSL}), i\hat{H}_j \,|\, j = 1, \dots, m\rangle_{\text{Lie}} \subseteq \mathfrak{g}_0^{\text{LK}}$ (unital), $\mathfrak{g}^{\text{LK}}$ (non-unital).

Embedding

The Lindblad-Kossakowski Lie algebra $\mathfrak{g}^{LK}$ is a semidirect sum

$$\mathfrak{g}^{LK} := \mathfrak{gl}(N^2 - 1, \mathbb{R}) \oplus_s \mathfrak{i}_0 = \mathfrak{g}_0^{LK} \oplus_s \mathfrak{i}_0$$

of the unital part $\mathfrak{g}_0^{LK}$ with the ideal of translation generators $\mathfrak{i}_0 \simeq \mathbb{R}^{N^2-1}$.
It generates a group of affine maps

$$\mathbf{G} := \mathbf{GL}(N^2 - 1, \mathbb{R}) \otimes_s I_0 \supseteq \mathbf{S}$$

embracing the Markovian Lie-semigroup of GKSL-quantum maps $\mathbf{S}$.

NB: The system algebra is also the smallest Lie algebra comprising the Lie wedge: $\mathfrak{g}_\Sigma \supseteq \mathfrak{w}_\Sigma$.

Let $\dot{X} = -(i\hat{H}_0 + \Gamma_{GKSL} + i\sum_j u_j(t)\hat{H}_j)X(t)$ be a N-level bil. control system Σ with

system algebra $\mathfrak{g}_\Sigma := \langle (i\hat{H}_0 + \Gamma), i\hat{H}_j \,|\, j = 1, \dots, m\rangle_{\mathrm{Lie}}$.

Corollary

The following two are equivalent:

1. *The unital n-qubit system variant is (map)accessible* ($N = 2^n$).
2. *The unital system algebra has commutant of dimension 2 and (for $N > 2$) its $\mathfrak{k}$-part exceeds* $\mathrm{ad}_{\mathfrak{su}(N)}$.

Likewise one may conjecture the equivalence of

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1 closed free versus **2** closed observed versus **3** open Markovian

1 closed coherently controlled (cc) systems:
system algebra $\mathfrak{k}_\Sigma := \langle i\,\mathrm{ad}_{H_0}, \dots, i\,\mathrm{ad}_{H_m}\rangle_{\mathrm{Lie}} = \mathrm{ad}_{su(N)} \Leftrightarrow \dim\big((\mathfrak{k}_\Sigma)'\big) = 2$
get full controllability (universality) by irreducibility of $\mathfrak{k}_\Sigma$ (adjoint repr.!)

2 closed systems, cc with observable C:
check joint commutant $(\mathfrak{k}_\Sigma \cup P_C)'$

- get full observability by irreducibility of $\{\mathfrak{k}_\Sigma \cup P_C\}$
- get (C, ρ_0) observable pair by shared $\mathfrak{k}_\Sigma$-invariant support (proj. $P_{\mathfrak{m}}$)

3 open systems, cc with constant (non)unital Markovian noise:
system alg. $\mathfrak{g}_{\Sigma_{[0]}} := \langle (i\,\mathrm{ad}_{H_0} + \widehat{\Gamma}^{[0]}_{GKLS}), \dots, i\,\mathrm{ad}_{H_m}\rangle_{\mathrm{Lie}}$ w. $\dim\big((\mathfrak{g}_{\Sigma_{[0]}})'\big) = 1\,[2]$
get (non)unital map accessibility by irreducibility of $\mathfrak{g}_{\Sigma_0}$ (resp. $\mathfrak{g}_\Sigma$)

Conclusion: Symmetries in Bilinear Control Systems

(2) plus Observable (or fixed ρ_0)

1 closed free versus 2 closed observed versus 3 open Markovian

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get (non)unital map accessibility by irreducibility of $\mathfrak{g}_{\Sigma_0}$ (resp. $\mathfrak{g}_\Sigma$)

Def.: Thermal Operations

(1) couple system ρ_S to bath of $T \geq 0$ (by $\otimes \rho_B^{(T)}$)

(2) energy-conserving evolution (by $U \in \{H_0 \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B\}'$)

(3) project back onto system (by tr_B)

$$\begin{array}{ccc}
\rho_S(0) \otimes \rho_B^{(T)} & \xrightarrow[(2)]{\mathrm{Ad}_U} & \rho_{SB}(U) \\
\iota_B \big\uparrow (1) & & (3) \big\downarrow \mathrm{tr}_B \\
\rho_S(0) & \xrightarrow[\mathsf{TO}(H_0,T)]{} & \rho_S(U)\,.
\end{array}$$

Markovianity Filter (via Lie Wedges L(·))

$$\mathrm{MCPTP} := \overline{\langle \exp\left(\mathsf{L}(\mathrm{CPTP})\right)\rangle}_{\mathrm{SG}} \equiv \overline{\left\langle \exp\left(\mathfrak{w}_{GKSL}\right)\right\rangle}_{\mathrm{SG}}$$

$$\mathrm{MTO}(H_0,T) := \overline{\langle \exp\left(\mathsf{L}(\overline{\mathrm{TO}(H_0,T)})\right)\rangle}_{\mathrm{SG}}$$

$$\mathrm{MEnTO}(H_0,T) := \overline{\langle \exp\left(\mathsf{L}(\mathrm{EnTO}(H_0,T))\right)\rangle}_{\mathrm{SG}}$$

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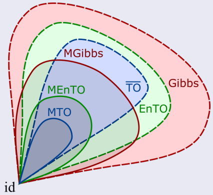

where

$$\mathsf{L}(\mathrm{Gibbs}(H_0,T)) = \left\{ L \in \mathfrak{w}_{GKSL} \mid e^{-H_0/T} \in \ker(L) \right\}$$

$$\mathsf{L}(\mathrm{EnTO}(H_0,T)) = \left\{ L \in \mathsf{L}(\mathrm{Gibbs}(H_0,T)) \mid \mathrm{ad}_{H_0} \in L' \right\}$$

$\mathsf{L}(\overline{\mathrm{TO}(H_0,T)})$ *as in main Thm. of OSID* ***30*** *(2023), 2350005*

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Markovian Thermal Operations

Lie-Semigroup Frame

OSID **30** (2023), 2350005

Starting Point: Thermal Operations

(1) couple system ρ_S to bath of $T \geq 0$ (by $\otimes \rho_B^{(T)}$)

(2) energy-conserving evolution (by $U \in \{H_0 \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B\}'$)

(3) project back onto system (by tr_B)

$$
\begin{array}{ccc}
\rho_S(0) \otimes \rho_B^{(T)} & \xrightarrow[(2)]{\mathrm{Ad}_U} & \rho_{SB}(U) \\
\iota_B \Big\uparrow (1) & & (3) \Big\downarrow \mathrm{tr}_B \\
\rho_S(0) & \xrightarrow[\mathsf{TO}(H_0,T)]{} & \rho_S(U)\,.
\end{array}
$$

Theorem (Lie wedge $\mathsf{L}\big(\overline{\mathsf{TO}(\mathsf{H}_0,\mathsf{T})}\big)$)

Given $H_B \in \mathfrak{her}(m)$, $H_{\rm tot} \in \mathfrak{her}(mn)$ *s.th.* $[H_{\rm tot}, H_0 \otimes \mathbb{1} + \mathbb{1} \otimes H_B] = 0$.

If $\Phi(t)$ *solves* $\dot\Phi(t) = -\big(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\rm tot}\big)\Phi(t)\,, \Phi(0) = \mathrm{id}$ *with any* $H \in \mathfrak{her}(n)$ *s.th.* $[H, H_0] = 0$ *and*

$$\widehat{\Gamma}_{B,\rm tot} := \sum_{j,k=1}^{m} \Big(\tfrac{1}{2}\big(V_{jk}^\dagger V_{jk}(\cdot) + (\cdot)V_{jk}^\dagger V_{jk}\big) - V_{jk}(\cdot)V_{jk}^\dagger \Big)\,,$$

with $V_{jk} = e^{-E_k'/(2T)}\,\mathrm{tr}_{|g_k\rangle\langle g_j|}(H_{\rm tot})$ *where* $\sum_{j=1}^m E_j' |g_j\rangle\langle g_j|$ *is any spectral decomposition of the bath Hamiltonian* H_B,

then $(\Phi(t))_{t\geq 0}$ *is a continuous one-parameter semigroup in* $\overline{\mathsf{TO}(H_0,T)}$ *with* $-\big(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\rm tot}\big)$ *being an element in the Lie wedge* $\mathsf{L}\big(\overline{\mathsf{TO}(\mathsf{H}_0,\mathsf{T})}\big)$.

Theorem (Lie wedge $\mathrm{L}\big(\overline{\mathrm{TO}(\mathrm{H}_0,\mathrm{T})}\big)$)

If $\Phi(t)$ *solves* $\dot\Phi(t) = -\big(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\mathrm{tot}}\big)\Phi(t)\,,\ \Phi(0) = \mathrm{id}$ *where*

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then $(\Phi(t))_{t\geq 0} \subset \overline{\mathrm{TO}(H_0,T)}$ *w.* $-\big(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\mathrm{tot}}\big)$ *in its Lie wedge* $L\big(\overline{\mathrm{TO}(H_0,T)}\big)$.

Proof Idea.

For curve of thermal operation $\gamma(t) : t \mapsto \mathrm{tr}_B\big(e^{-itH_{\mathrm{tot}}}((\cdot)\otimes\rho_B^{(T)})e^{itH_{\mathrm{tot}}}\big)$ with $\gamma(0) = \mathrm{id}$, show

(1) $\dot\gamma(0) \in \mathrm{E}\big(\mathrm{L}(\overline{\mathrm{TO}(H_0,T)})\big)$ with $\overline{\mathrm{TO}(H_0,T)}$ being a compact, convex semigroup,

(2) $\ddot\gamma(0) \in \mathrm{L}\big(\overline{\mathrm{TO}(H_0,T)}\big)$ for $H_B = \sum_{j=1}^m E_j'|g_j\rangle\langle g_j|$ so $e^{-H_B/T} = \sum_{j=1}^m e^{-E_j'/T}|g_j\rangle\langle g_j|$.

Thus altogether one gets $-\big(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\mathrm{tot}}\big) = -i\,\mathrm{ad}_H + \frac{1}{2}\,\mathrm{tr}(e^{-H_B/T})\ddot\gamma(0) \in \mathrm{L}\big(\overline{\mathrm{TO}(H_0,T)}\big)$. □

Theorem (Lie wedge $\mathrm{L}\big(\overline{\mathrm{TO}(\mathrm{H}_0,\mathrm{T})}\big)$)

If $\Phi(t)$ *solves* $\dot\Phi(t) = -\big(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\mathrm{tot}}\big)\Phi(t)\,,\ \Phi(0) = \mathrm{id}$ *where*

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Thus altogether one gets $-(i\,\mathrm{ad}_H + \widehat{\Gamma}_{B,\mathrm{tot}}) = -i\,\mathrm{ad}_H + \frac{1}{2}\,\mathrm{tr}(e^{-H_B/T})\ddot\gamma(0) \in \mathrm{L}(\overline{\mathrm{TO}(H_0,T)})$. □

Definition (via the respective Lie wedges)

$$\mathrm{MTO}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\overline{\mathrm{TO}(H_0,T)})\right)\rangle}_{\mathrm{SG}}$$

$$\mathrm{MEnTO}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\mathrm{EnTO}(H_0,T))\right)\rangle}_{\mathrm{SG}}$$

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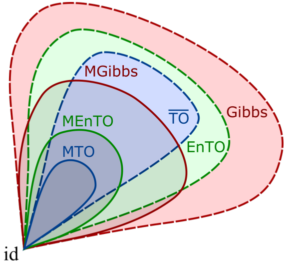

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$$\mathrm{MCPTP} := \overline{\langle \exp\left(\mathsf{L}(\mathrm{CPTP})\right)\rangle}_{\mathrm{SG}}$$

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Markovianity at Large

Lie-Semigroup Frame

ROMP **64** (2009), 93 & OSID **30** (2023), 2350005

Definition (via Lie Wedges)

$$\mathrm{MCPTP} := \overline{\langle \exp\left(\mathrm{L}(\mathrm{CPTP})\right)\rangle}_{\mathrm{SG}} \equiv \overline{\left\langle \exp\left(\mathfrak{w}_{GKSL}\right)\right\rangle}_{\mathrm{SG}}$$

$$\mathrm{MTO}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\overline{\mathrm{TO}(H_0,T)})\right)\rangle}_{\mathrm{SG}}$$

$$\mathrm{MEnTO}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\mathrm{EnTO}(H_0,T))\right)\rangle}_{\mathrm{SG}}$$

$$\mathrm{MGibbs}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\mathrm{Gibbs}(H_0,T))\right)\rangle}_{\mathrm{SG}}$$

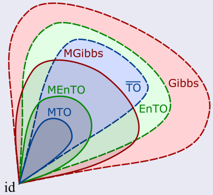
Markovianity Filter (via Lie Wedges L(·))

$$\mathrm{MCPTP} := \overline{\langle \exp\left(\mathrm{L}(\mathrm{CPTP})\right)\rangle}_{\mathrm{SG}} \equiv \overline{\big\langle \exp\left(\mathfrak{w}_{GKSL}\right)\big\rangle}_{\mathrm{SG}}$$

$$\mathrm{MTO}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\overline{\mathrm{TO}(H_0,T)})\right)\rangle}_{\mathrm{SG}}$$

$$\mathrm{MEnTO}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\mathrm{EnTO}(H_0,T))\right)\rangle}_{\mathrm{SG}}$$

$$\mathrm{MGibbs}(H_0,T) := \overline{\langle \exp\left(\mathrm{L}(\mathrm{Gibbs}(H_0,T))\right)\rangle}_{\mathrm{SG}}$$



where

$$\mathrm{L}(\mathrm{Gibbs}(H_0,T)) = \left\{L \in \mathfrak{w}_{GKSL} \mid e^{-H_0/T} \in \ker(L)\right\}$$

$$\mathrm{L}(\mathrm{EnTO}(H_0,T)) = \left\{L \in \mathrm{L}(\mathrm{Gibbs}(H_0,T)) \mid \mathrm{ad}_{H_0} \in L'\right\}$$

$\mathrm{L}(\overline{\mathrm{TO}(H_0,T)})$ *as in main Thm. above*