Norway grants

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Quantum Information at Colliders

Kazuki Sakurai

(University of Warsaw)

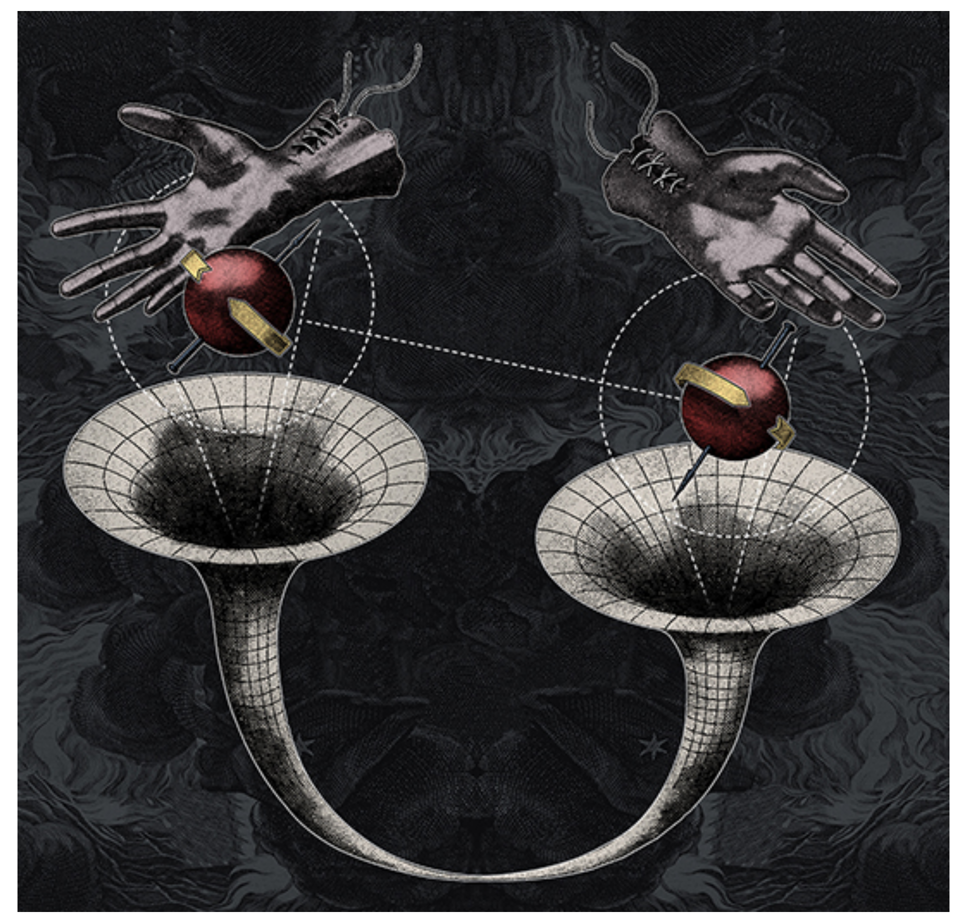
- KS, M. Spannowsky [2310.01477]
- M. Altakach, F. Maltoni, K. Mawatari, P. Lamba, KS, *Phys.Rev.D* 107 (2023) 9, 093002 [2211.10513]

2024/4/8, Seminar @ Siegen

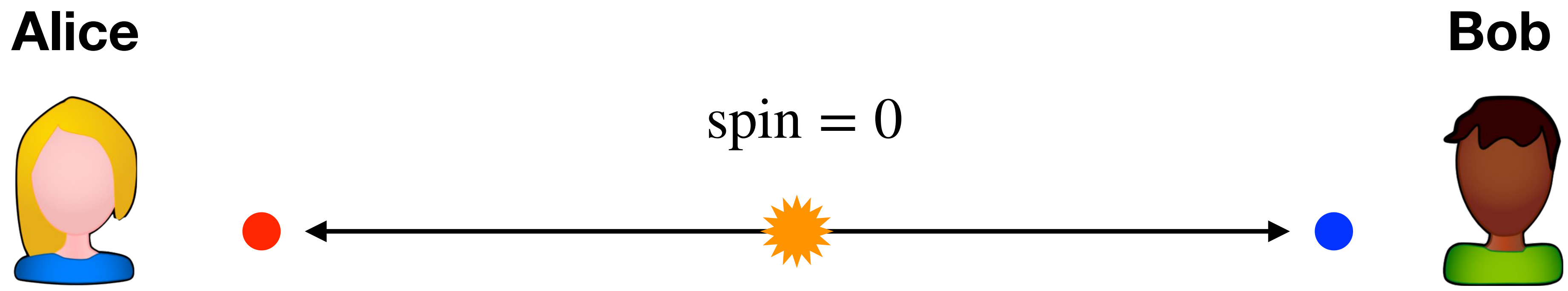
Entanglement and other quantum properties are crucial in:

- developing **quantum technology/devices**
- understanding **QFT** and quantum **gravity**

IBM Q system



Entanglement

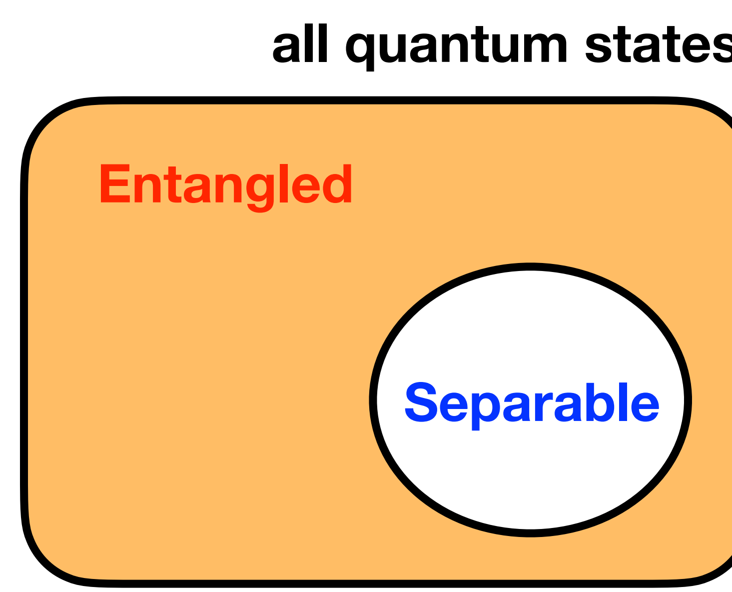


$$|\Psi_{AB}^{(0,0)}\rangle \simeq \overset{\text{Alice}}{|+\rangle} \otimes \overset{\text{Bob}}{|-\rangle} - |-\rangle \otimes |+\rangle$$

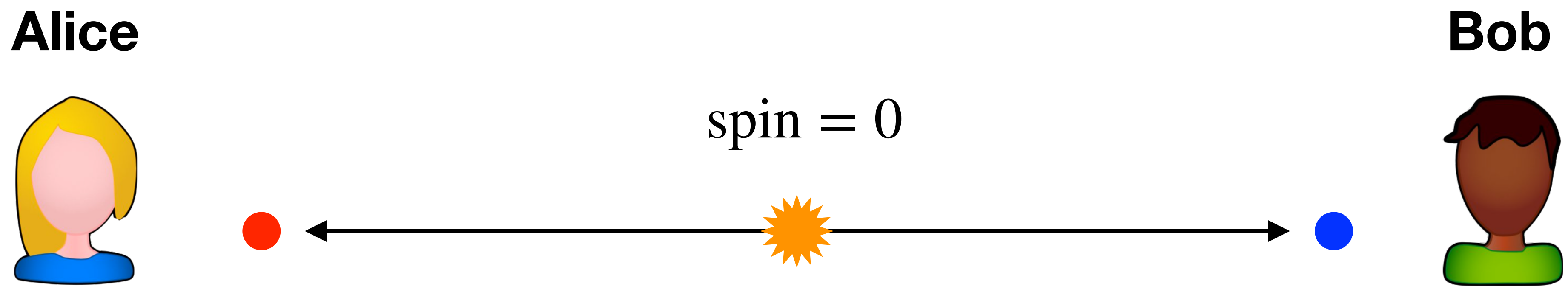
$$\neq |\Psi_A\rangle \otimes |\Psi_B\rangle \quad \longleftarrow \quad \textbf{entangled}$$

$$|\Psi_{AB}^{\text{sep}}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \quad \longleftarrow \quad \textbf{separable}$$

$$\text{Entanglement} = \overline{\text{Separable}}$$



Entanglement



Alice ↘ ↙ Bob

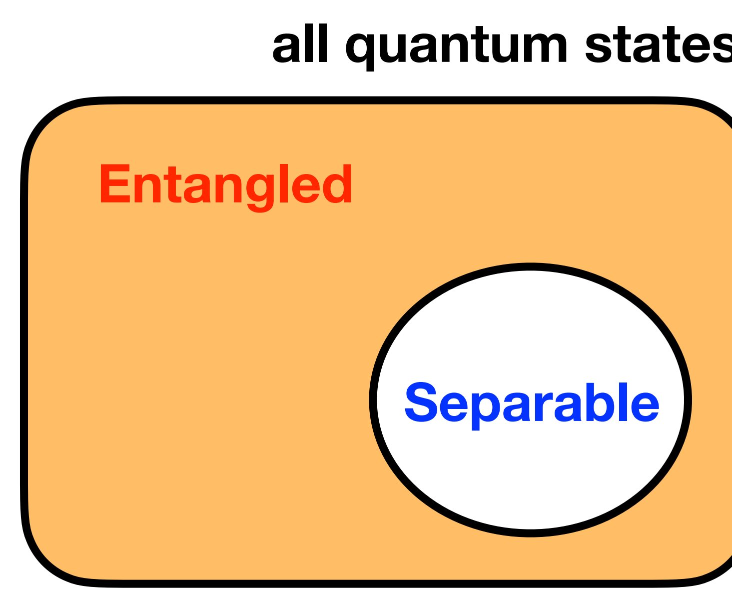
$$|\Psi_{AB}^{(0,0)}\rangle \simeq |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \xrightarrow{\text{Alice measures } s_z \text{ and found } +} |+,-\rangle$$

$$\neq |\Psi_A\rangle \otimes |\Psi_B\rangle \longleftarrow \textbf{entangled}$$

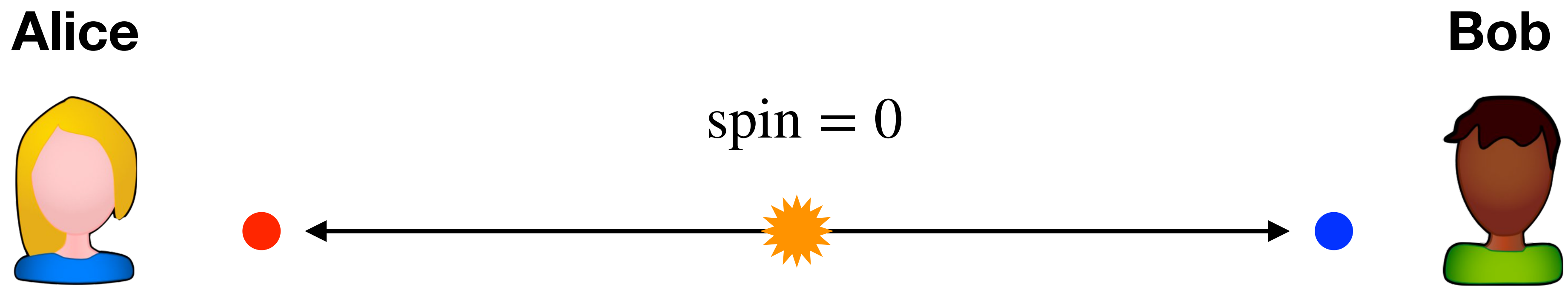
Alice's local measurement changes the global state involving (spacelike separated) Bob's particle.
~ **Nonlocality**

$$|\Psi_{AB}^{\text{sep}}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \longleftarrow \textbf{separable}$$

$$\text{Entanglement} = \overline{\text{Separable}}$$



Entanglement in mixed states



For a classical ensemble $\{ (p_1, |\Psi_1\rangle), (p_2, |\Psi_2\rangle), \cdots \}$, the density operator is defined as

$$\hat{\rho} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \qquad \left(p_i \geq 0, \ \sum_i p_i = 1 \right)$$

For a bipartite system $\mathscr{H}_A \otimes \mathscr{H}_B$

$$\hat{\rho} \neq \sum_i q_i \, \rho_i^A \otimes \rho_i^B \longleftarrow \text{entangled}$$

$$\hat{\rho} = \sum_i q_i \, \rho_i^A \otimes \rho_i^B \longleftarrow \text{separable}$$

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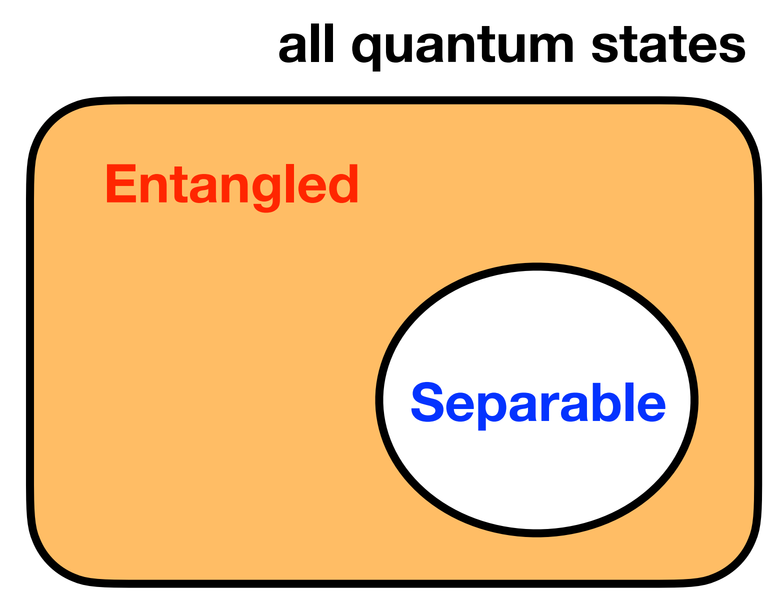
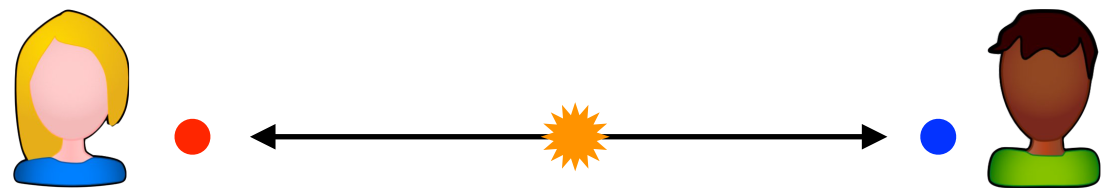


measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

probability for a: $p_A(a\,|\,x)$



$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b\,|\,y)$

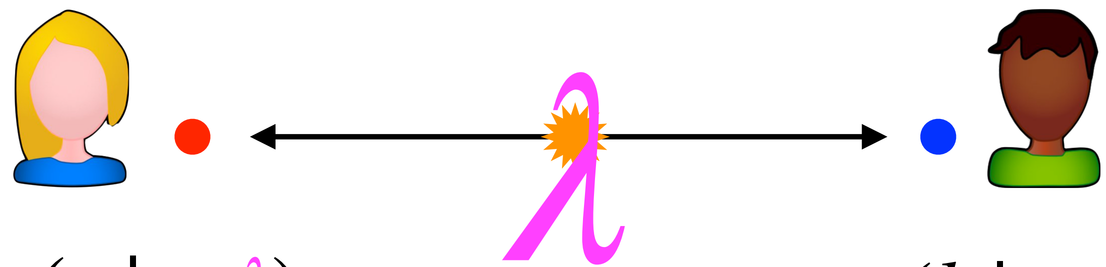
Any data is expressed by the *joint distribution* $p(a, b\,|\,x, y)$

⇒ Models describing the experiment can be classified by possible forms of $p(a, b\,|\,x, y)$

measures axis: $x \in \{\vec{n}\}$ $\quad$ q_λ $\quad$ $y \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$ $\quad$ $b \in \{-1, +1\}$

probability for a: $p_A(a \mid x)$ $\quad$ $p_B(b \mid y)$



$p_A(a \mid x, \lambda)$ $\quad$ $p_B(b \mid y, \lambda)$

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⇒ Models describing the experiment can be classified by possible forms of $p(a, b \mid x, y)$

probability for λ

Local theories

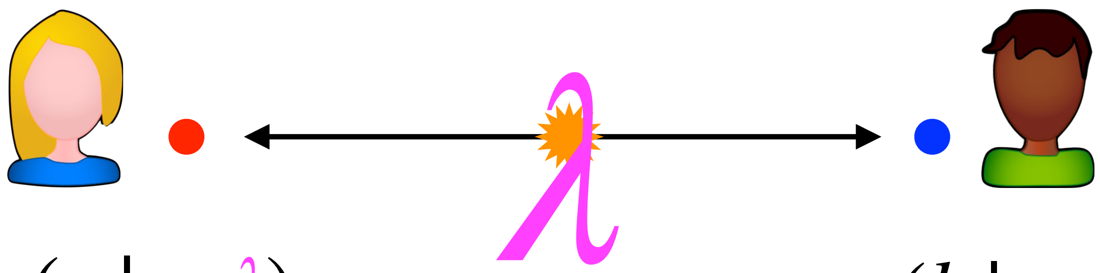
$$p_L(a, b \mid x, y) = \sum_\lambda q_\lambda \cdot p_A(a \mid x, \lambda) \cdot p_B(b \mid y, \lambda)$$

measures axis: $x \in \{\vec{n}\}$ $\qquad$ $y \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$ $\qquad$ $b \in \{-1, +1\}$

probability for a: $p_A(a|x)$ $\qquad$ $p_B(b|y)$

q_λ



$p_A(a|x, \lambda)$ $\qquad$ $p_B(b|y, \lambda)$

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density operator

Quantum Mechanics $\quad p_Q(a, b|x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$

For projective measurement:

$$M_{a|x} = |a_x\rangle\langle a_x|$$

$$\hat{s}_x|a_x\rangle = a_x|a_x\rangle$$

measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

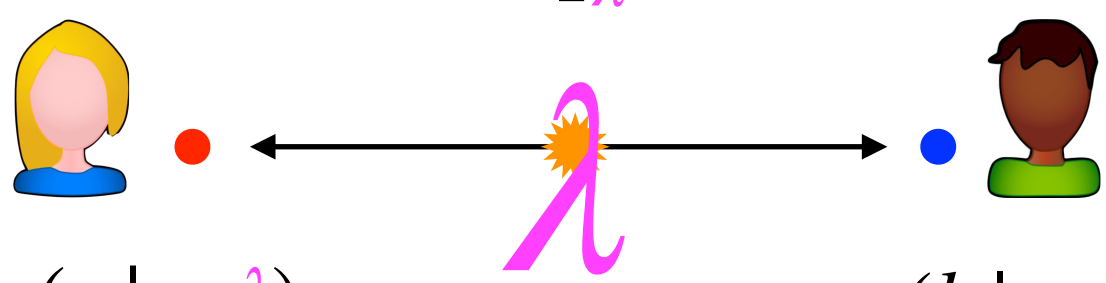
probability for a: $p_A(a|x)$

q_λ

$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b|y)$



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For separable quantum states:

$$\rho_{AB} = \sum_\lambda q_\lambda \cdot \rho_A^\lambda \otimes \rho_B^\lambda \implies p_{Q_{\mathrm{sep}}}(a, b|x, y) = \sum_\lambda q_\lambda \cdot \mathrm{Tr}\left[\rho_A^\lambda M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^\lambda M_{a|x}\right]$$

measures axis: $x \in \{\vec{n}\}$ $\quad q_{\lambda}$ $\quad y \in \{\vec{n}\}$

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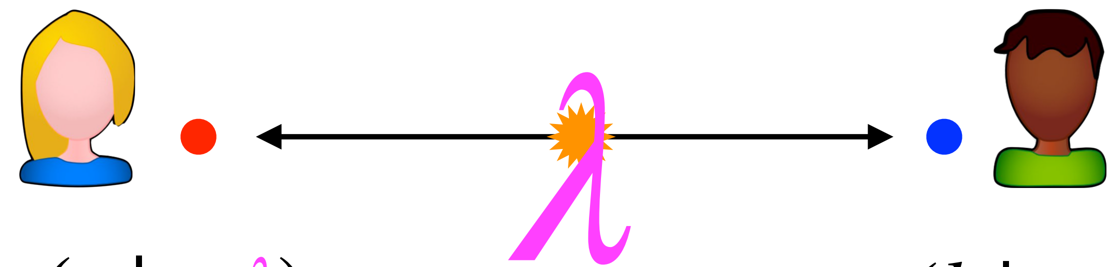

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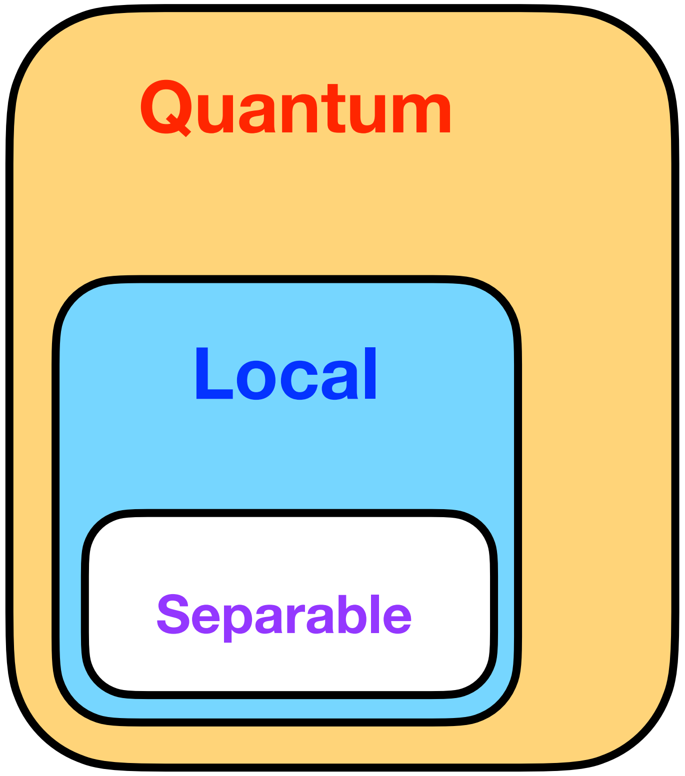
density operator

Quantum Mechanics $\quad p_Q(a, b \mid x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$

For separable quantum states: $\quad$ **Local**

$$\rho_{AB} = \sum_{\lambda} q_{\lambda} \cdot \rho_A^{\lambda} \otimes \rho_B^{\lambda} \Longrightarrow p_{Q_{\mathrm{sep}}}(a, b \mid x, y) = \boxed{\sum_{\lambda} q_{\lambda} \cdot \mathrm{Tr}\left[\rho_A^{\lambda} M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^{\lambda} M_{a|x}\right]}$$



Quantum $\supset$ Local $\supset$ Separable

Local theories

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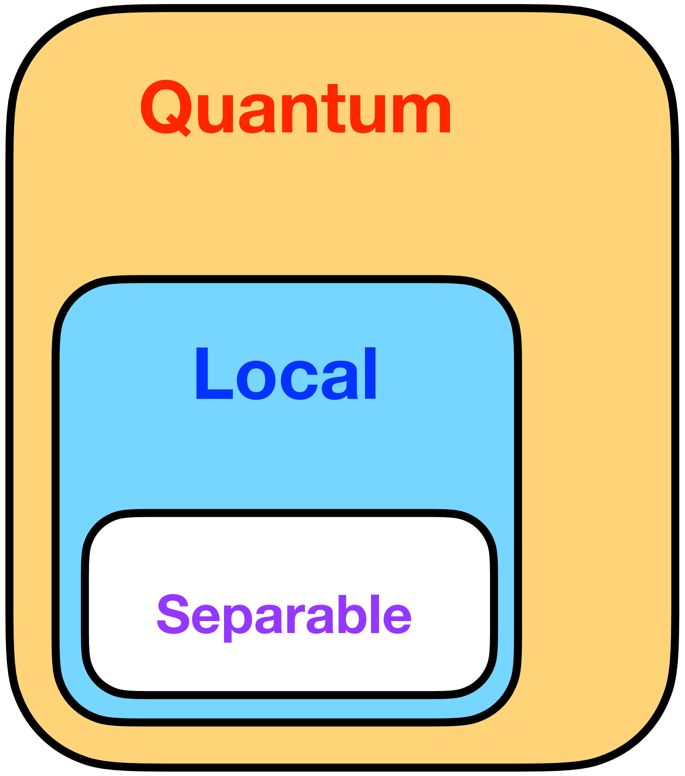

Quantum $\supset$ **Local** $\supset$ **Separable**

$\updownarrow \qquad \updownarrow$

Nonlocal $\subset$ **Entanglement**

Local theories

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- Nonlocal states in QM does not violate causality

$$p(a\,|\,x,y) \equiv \sum_b p(a,b\,|\,x,y)$$

Condition for no causality violation: No-Signalling [Cirel'son(1980), Popescu, Rohrlich(1994)]

$$^\forall a,b,x,x',y,y' \quad \begin{cases} p(a\,|\,x,y) = p(a\,|\,x,y') \\ p(b\,|\,x,y) = p(b\,|\,x',y) \end{cases}$$

Alice's dist. is indep. of Bob's choice for meas. axis

Bob's dist. is indep. of Alice's choice for meas. axis

No-signalling ⊃ Quantum ⊃ Local ⊃ Separable

Bell Inequalities

- Bell-type inequalities (in general) are the inequalities that separate different types of distributions (No-signalling, Quantum, Local).

- Define the correlator $C_{xy} = \langle A_x B_y \rangle \equiv \sum_{a,b} abp(a,b\,|\,x,y)$

- CHSH inequality [Clauser-Horne-Shimony-Holt(1969)]

For $a, b \in \{\pm 1\},\ x \in \{\mathbf{n}_1, \mathbf{n}_2\},\ y \in \{\mathbf{e}_1, \mathbf{e}_2\}$

$$S_{\mathrm{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{e}_1} + C_{\mathbf{n}_1,\mathbf{e}_2} + C_{\mathbf{n}_2,\mathbf{e}_1} - C_{\mathbf{n}_2,\mathbf{e}_2}$$

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$$S_{\text{CHSH}} \leq \begin{cases} 2 & \textbf{Local theories} \quad \text{[CHSH(1969)]} \\ 2\sqrt{2} & \textbf{Quantum Mechanics} \quad \text{[Tsirelson(1987)]} \\ 4 & \textbf{No-signalling} \quad \text{[Popescu, Rohrlich(1994)]} \end{cases}$$

Entanglement detection

- If the state is separable (not entangled),

$$\hat{\rho} = \sum_i q_i \rho_i^A \otimes \rho_i^B \qquad \left(q_i \geq 0, \ \sum_i q_i = 1 \right)$$

- Take a partial transpose:

$$\hat{\rho}^{T_B} = \sum_i q_i \rho_i^A \otimes [\rho_i^B]^T$$

- This matrix is still positive definite.

$\Rightarrow$ if one finds a negative eigenvalue for $\hat{\rho}^{T_B}$, the state has to be entangled.

Peres-Horodecki (1996, 1997)

$$\rho = \frac{1}{4}\left(\mathbf{1}_4 + B_i \cdot \sigma_i \otimes \mathbf{1} + \overline{B}_i \cdot \mathbf{1} \otimes \sigma_i + C_{ij} \cdot \sigma_i \otimes \sigma_j\right)$$

spin-spin correlation

$$\boxed{C_{ij} = \mathrm{Tr}\left[\hat{s}_i^\alpha \hat{s}_j^\beta \hat{\rho}\right] = \langle \hat{s}_i^\alpha \hat{s}_j^\beta \rangle}$$

Entanglement $\Leftarrow$ $\hat{\rho}^{T_B}$ non-positive $\Leftarrow$ $\boxed{D \equiv \frac{\mathrm{Tr}(C)}{3} < -\frac{1}{3}}$

[Afik Nova (2021)]

sufficient condition for entanglement

Entanglement witness

- **Entanglement witness** is a function that distinguishes separable/entangled states

  - **Concurrence** (for bi-qubits) $C[\rho] \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ [Wootters (1998)]

$0 \leq C[\rho] \leq 1$

λ_i are eigenvalues, in descendent order, of

$C[\rho] > 0$ **iff ρ is entangled**

$R = \sqrt{\tilde{\rho}\rho}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\, \rho^* \,(\sigma_y \otimes \sigma_y)$

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$C[\rho] > 0$ **iff ρ is entangled**

No-signalling $\supset$ **Quantum** $\supset$ **Local** $\supset$ **Separable**

$S_{\text{CHSH}} = 2\sqrt{2}$ $\quad$ $S_{\text{CHSH}} = 2$ $\quad$ $C[\rho] = 0_+$ $\quad$ $C[\rho] = 0$

for all separable states

✤ Violation of Bell inequality, $S_{\rm CHSH} > 2$, has been observed at energies $\ll \mathrm{TeV}$

- **Entangled photon pairs** (from decays of Calcium atoms)

  Clauser, Horne, Shimony, Holt (1969), Freedman and Clauser (1972), A. Aspect et. al. (1981, 1982), Y. H. Shih, C. O. Alley (1988), L. K. Shalm et al. (2015) [5σ]

- **Entangled proton pairs** (from decays of ^{2}He)

  M. M. Lamehi-Rachti, W. Mitting (1972), H. Sakai (2006)

- $K^0\overline{K^0}$, $B^0\overline{B^0}$ **flavour oscillation** CPLEAR (1999), Belle (2004, 2007)

- $B^0 \to J/\psi + K^*(892)^0$ **spin correlation,** $S_{\rm CGLMP} > 2$, **[36σ]**

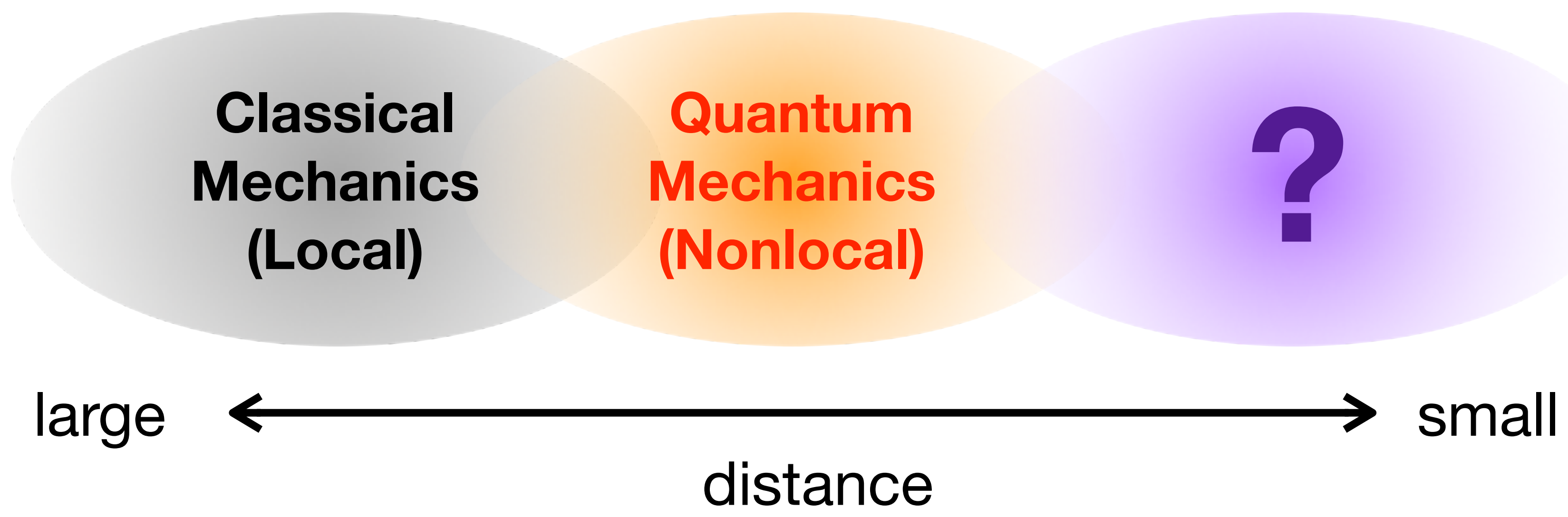
  Fabbrichesi, Floreanini, Gabrielli, Marzola (2023)

Normalised helicity amplitude for $B^0 \to J/\psi + K^*(892)^0$

$$
\begin{aligned}
|A_{||}|^2 &= 0.227 \pm 0.004\ (\mathrm{stat.}) \pm 0.011\ (\mathrm{syst.}),\\
|A_{\perp}|^2 &= 0.201 \pm 0.004\ (\mathrm{stat.}) \pm 0.008\ (\mathrm{syst.}),\\
\delta_{||}\ [\mathrm{rad}] &= -2.94 \pm 0.02\ (\mathrm{stat.}) \pm 0.03\ (\mathrm{syst.}),\\
\delta_{\perp}\ [\mathrm{rad}] &= 2.94 \pm 0.02\ (\mathrm{stat.}) \pm 0.02\ (\mathrm{syst.}).
\end{aligned}
$$

LHCb [1307.2782]

Testing QM at high energy colliders



Motivation

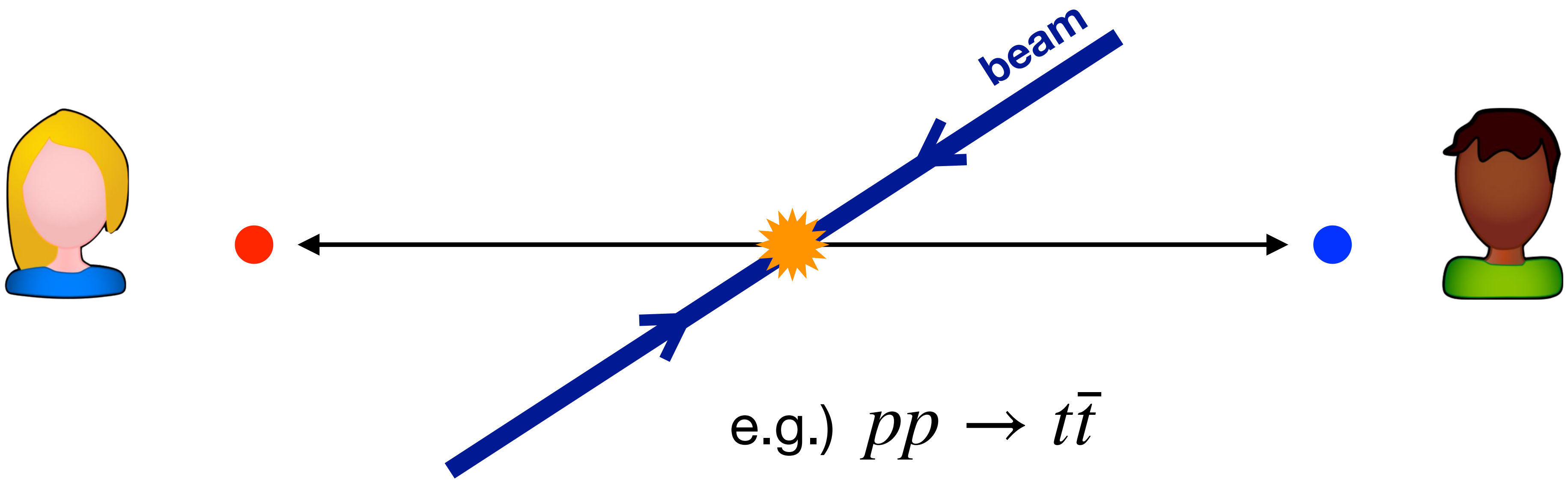
✤ Bell inequalities/Entanglement have not been tested at the TeV energy scale:

➡ LHC (and $FCC_{ee/hh}$) provides the unique opportunity for this test

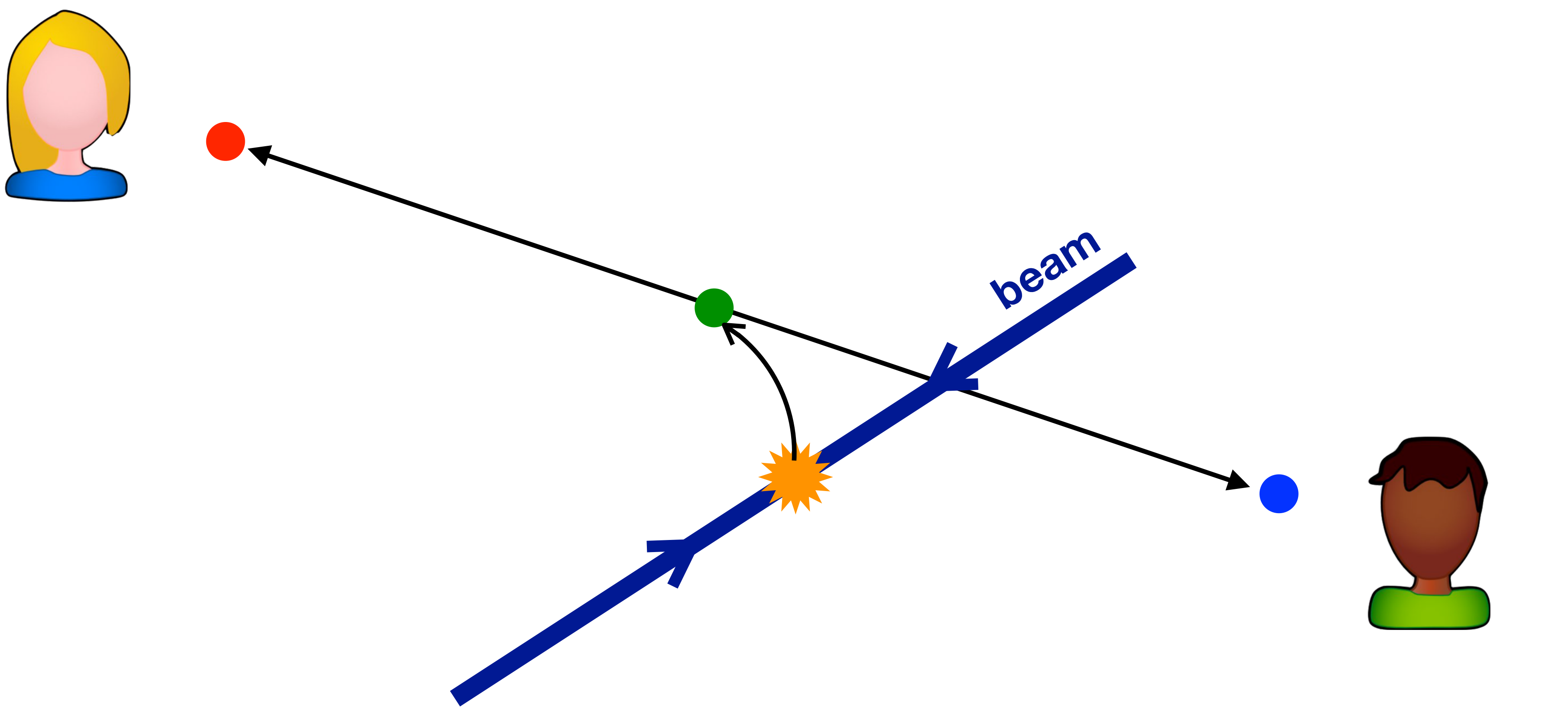
✤ Detection of Entanglement/Bell violation requires a detailed analysis of spin correlation:

➡ provides a very good test for the Standard Model (sensitive to BSM)

Entangled pairs at Colliders

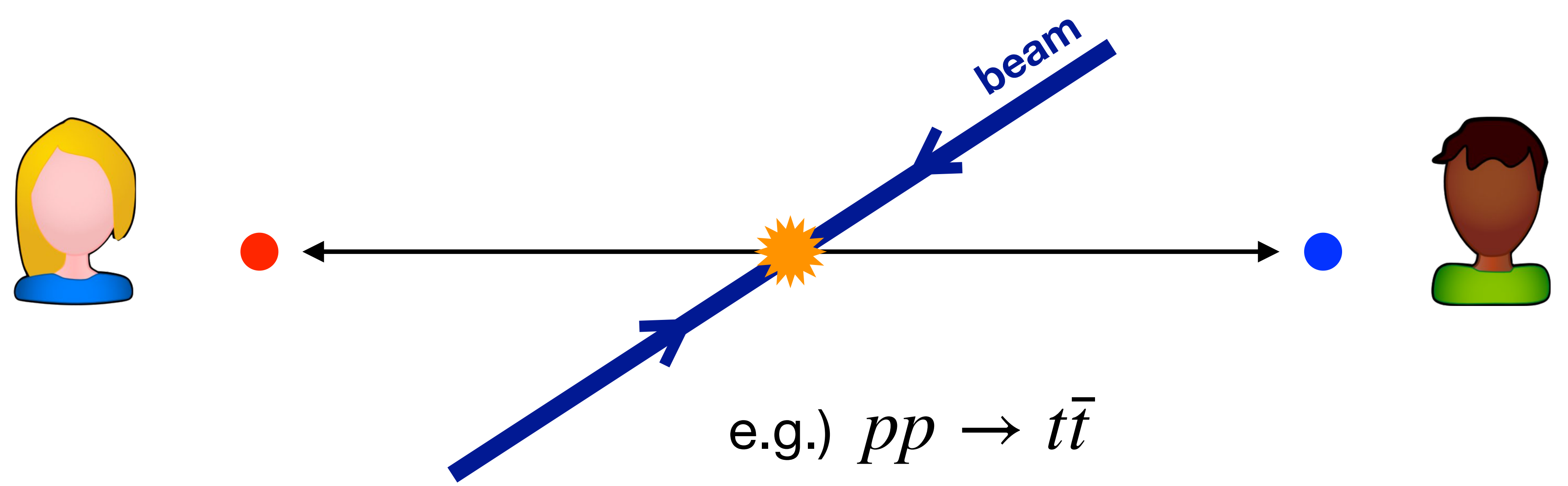


e.g.) $pp \to t\bar{t}$



e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

Entangled pairs at Colliders



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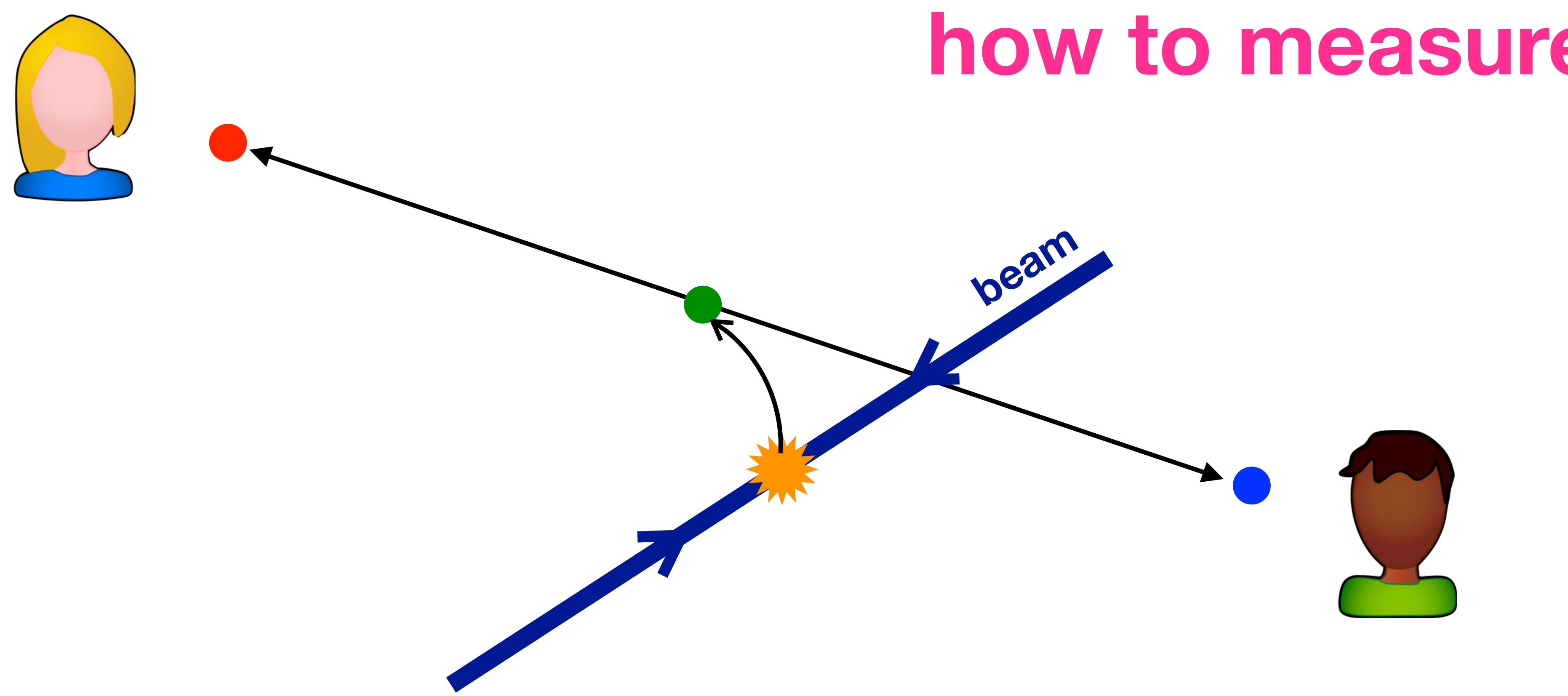
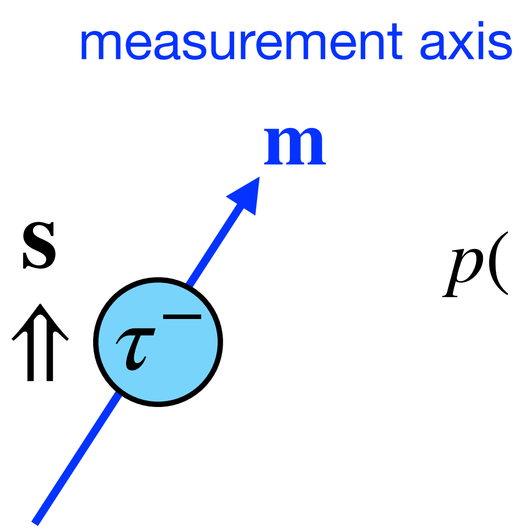
how to measure spin?



e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

Particles with weak decays are their own polarimeters

e.g.) For $\tau^- \to \pi^- + \nu_\tau$ (τ^- rest frame), the spin of τ^- is measured in the direction of π^- $(\vec{\pi})$ and the outcome is +1.



$$p(+\,|\,\mathbf{m}) = |\langle +_{\mathbf{m}}\,|\,+_{\mathbf{s}}\rangle|^2$$

$$= \frac{1 + (\mathbf{m}\cdot\mathbf{s})}{2}$$

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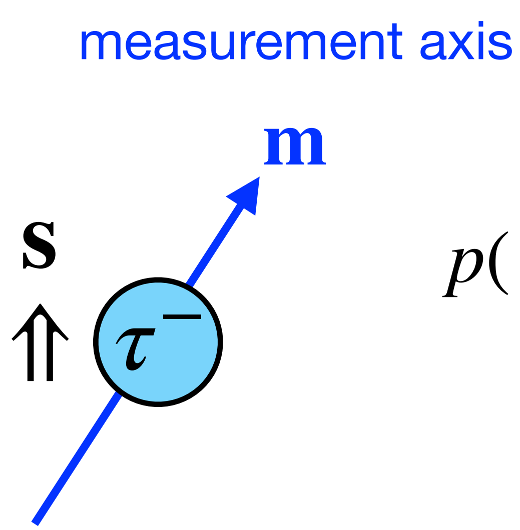

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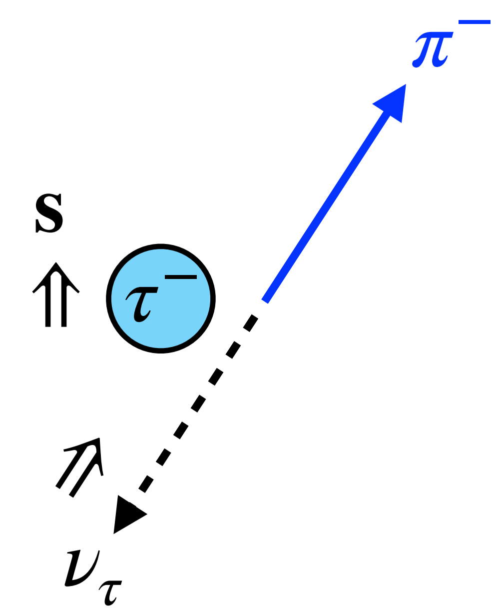

$$\frac{d\Gamma}{d\Omega} = \frac{1 + (\vec{\pi} \cdot \mathbf{s})}{2}$$

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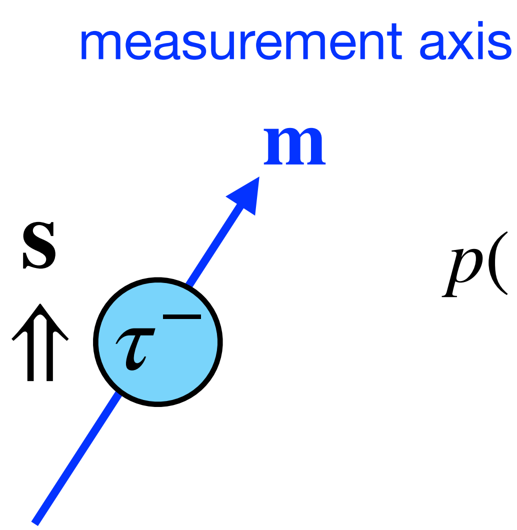

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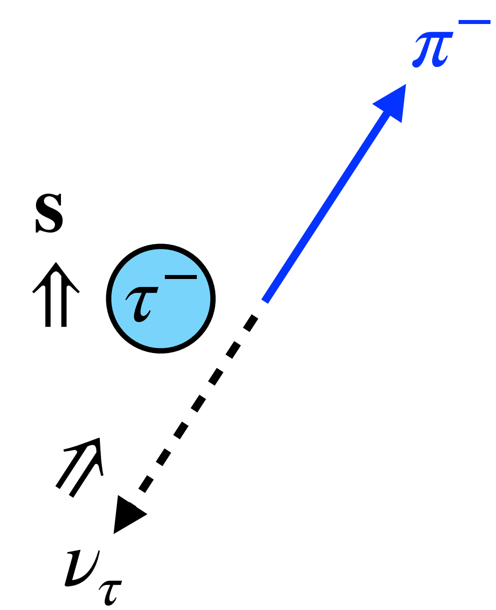


$$\frac{d\Gamma}{d\Omega} = \frac{1 + (\vec{\pi} \cdot \mathbf{s})}{2}$$

For spins to be measurable, one must focus on entangled pairs of weakly decaying particles

$$\tau,\ t,\ W^\pm,\ Z^0$$

Particles with weak decays are their own polarimeters

More generally,

$$\frac{d\Gamma}{d\Omega} = \frac{1 + \alpha_x \cdot (\vec{x} \cdot \mathbf{s})}{2}$$

$\alpha_x \in [-1, +1]$: spin analyzing power

- tau decay

$\alpha_x = 1$ for ($x = \pi^-$ in $\tau^- \to \pi^- \nu$)

- top decay

decay product x	α_x
b	$-0.3925(6)$
W^+	$0.3925(6)$
ℓ^+ (from a W^+)	$0.999(1)$
$\bar{d}$, $\bar{s}$ (from a W^+)	$0.9664(7)$
u, c (from a W^+)	$-0.3167(6)$

Spin correlation:

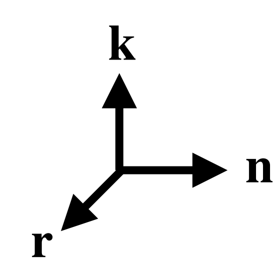
$\mathbf{n}, \mathbf{n}'$: spin measurement axes

$\vec{x}, \vec{y}$: direction of decay products

$$C_{\mathbf{n},\mathbf{n}'} \equiv \langle (\mathbf{s}_A \cdot \mathbf{n})(\mathbf{s}_B \cdot \mathbf{n}') \rangle = \frac{9}{\alpha_x \alpha_y} \langle (\vec{x} \cdot \mathbf{n})(\vec{y} \cdot \mathbf{n}') \rangle$$

$$S_{\mathrm{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{n}'_1} + C_{\mathbf{n}_1,\mathbf{n}'_2} + C_{\mathbf{n}_2,\mathbf{n}'_1} - C_{\mathbf{n}_2,\mathbf{n}'_2} > 2$$ → Bell inequality violation

$$D \equiv \mathrm{Tr}[C]/3 < -\frac{1}{3}$$ → sufficient cond. for entanglement



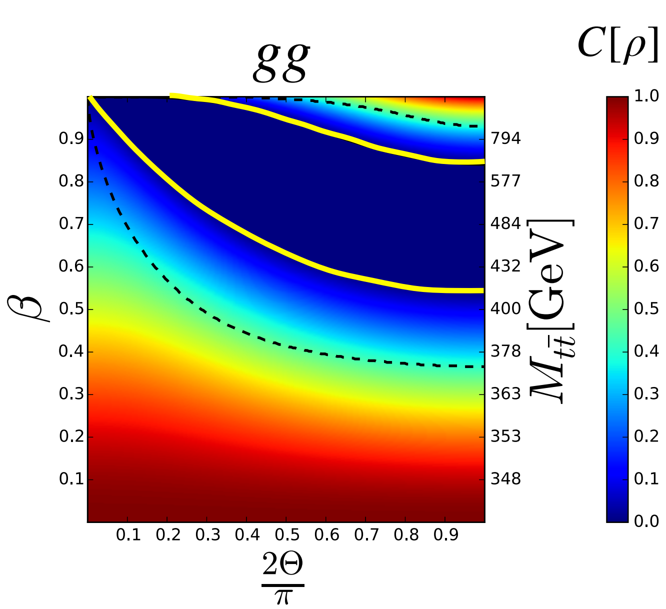
Recent activities to look into entanglements, etc. in HEP

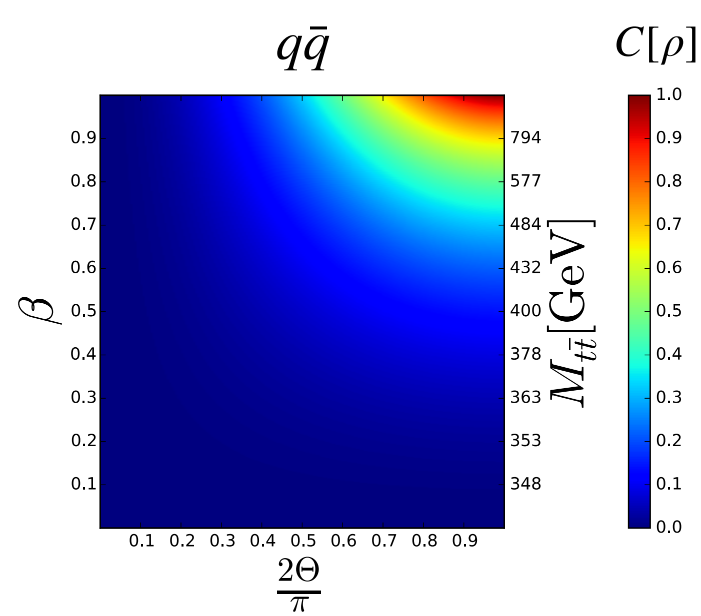
❖ Experimental observation of entanglement and Bell-ineq violation @ LHC

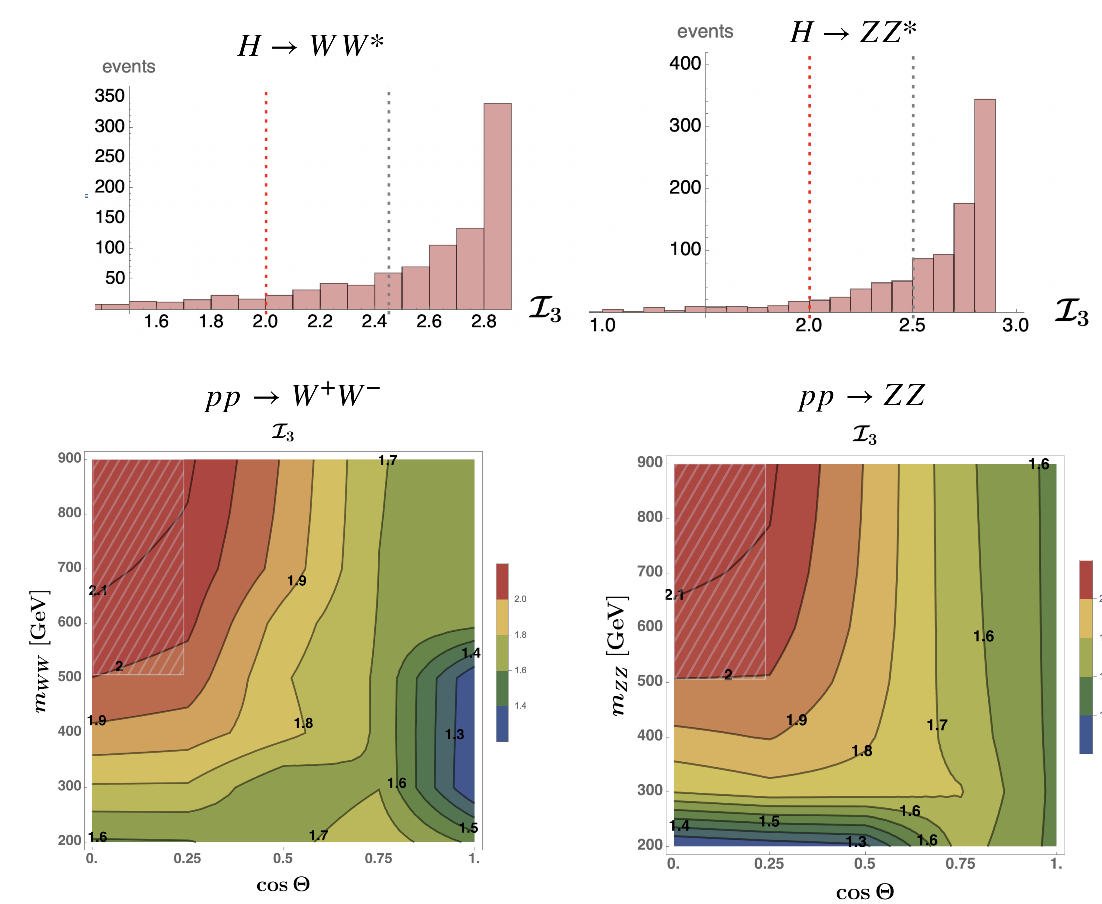
- $pp \to t\bar{t}$ — Y. Afik and J. R. M. de Nova '21, '22, M. Fabbrichesi, R. Floreanini, G. Panizzo '21, Z. Dong, D. Gonçalves, K. Kong, A. Navarro '23

- $H \to WW, ZZ$ — A. J. Barr '21, J.A. Aguilar-Saavedra, A. Bernal, J.A. Casas, J.M. Moreno '22, A. Bernal, P. Caban, J. Rembieliński '23, M. Fabbrichesi, R. Floreanini, E. Gabrielli, Luca Marzola '23

- $H \to \tau^+\tau^-$ (@ e^+e^- colliders) — M. Fabbrichesi, R. Floreanini, E. Gabrielli 22, M. Altakach, P. Lamba, F. Maltoni, K. Mawatari, KS '22, K. Ma, T. Li '23







- $pp \to t\bar{t}$

$$D = \frac{\mathrm{tr}[\mathbf{C}]}{3} < -\frac{1}{3}$$

(sufficient cond. for entanglement)

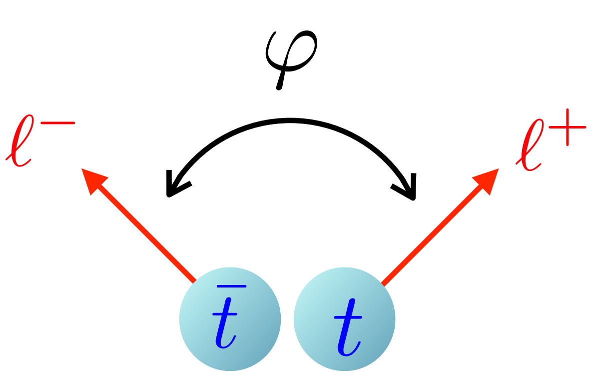
Observation of quantum entanglement in top-quark pairs using the ATLAS detector

The ATLAS Collaboration

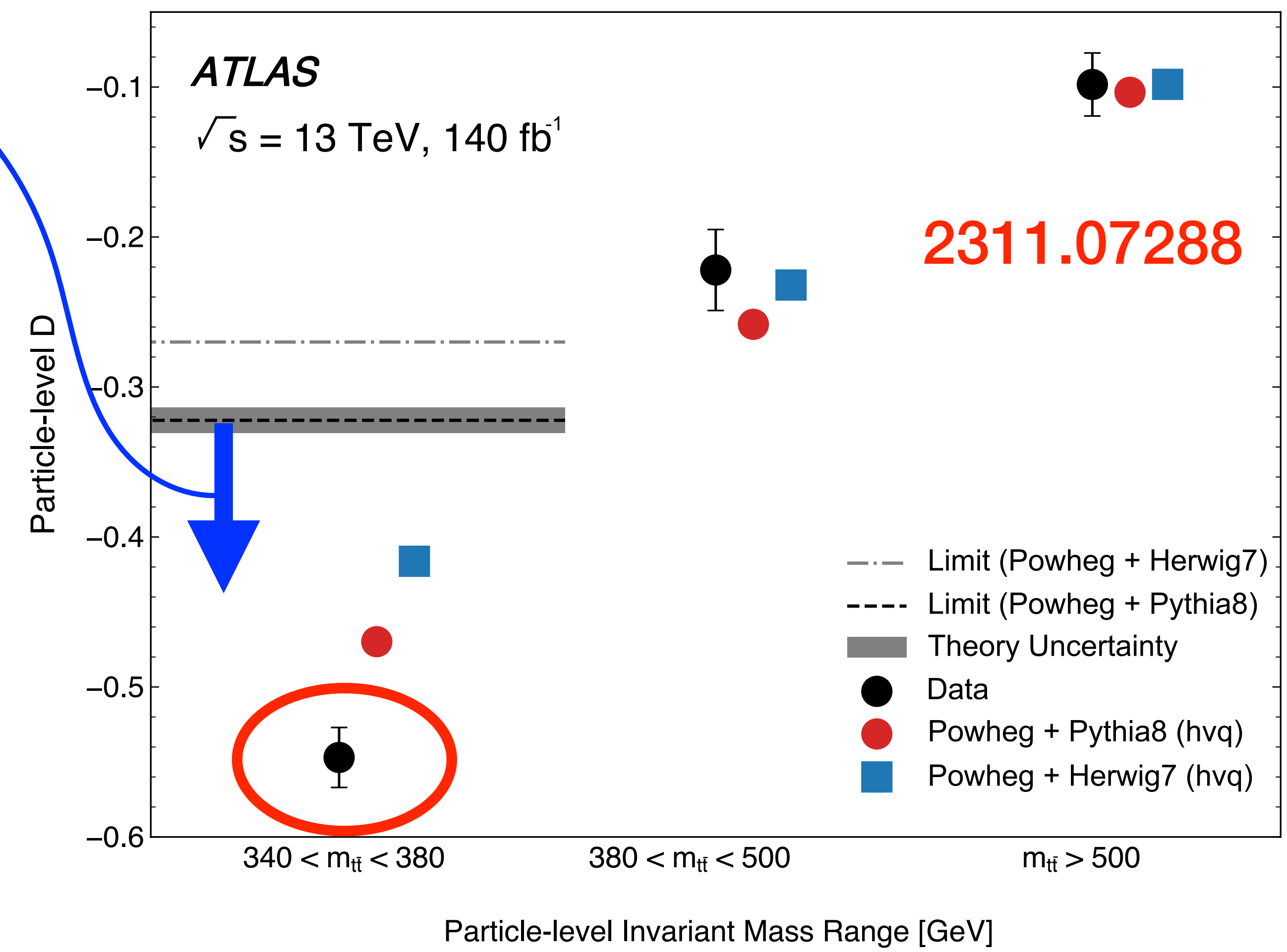
We report the highest-energy observation of entanglement, in top–antitop quark events produced at the Large Hadron Collider, using a proton–proton collision data set with a center-of-mass energy of $\sqrt{s}$ = 13 TeV and an integrated luminosity of 140 fb^{-1} recorded

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1}{2}(1 - D\cos\varphi)$$

$$\Rightarrow \quad D = -3 \cdot \langle \cos\varphi \rangle$$

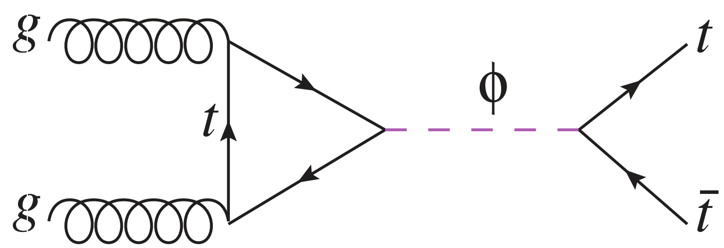


entangled

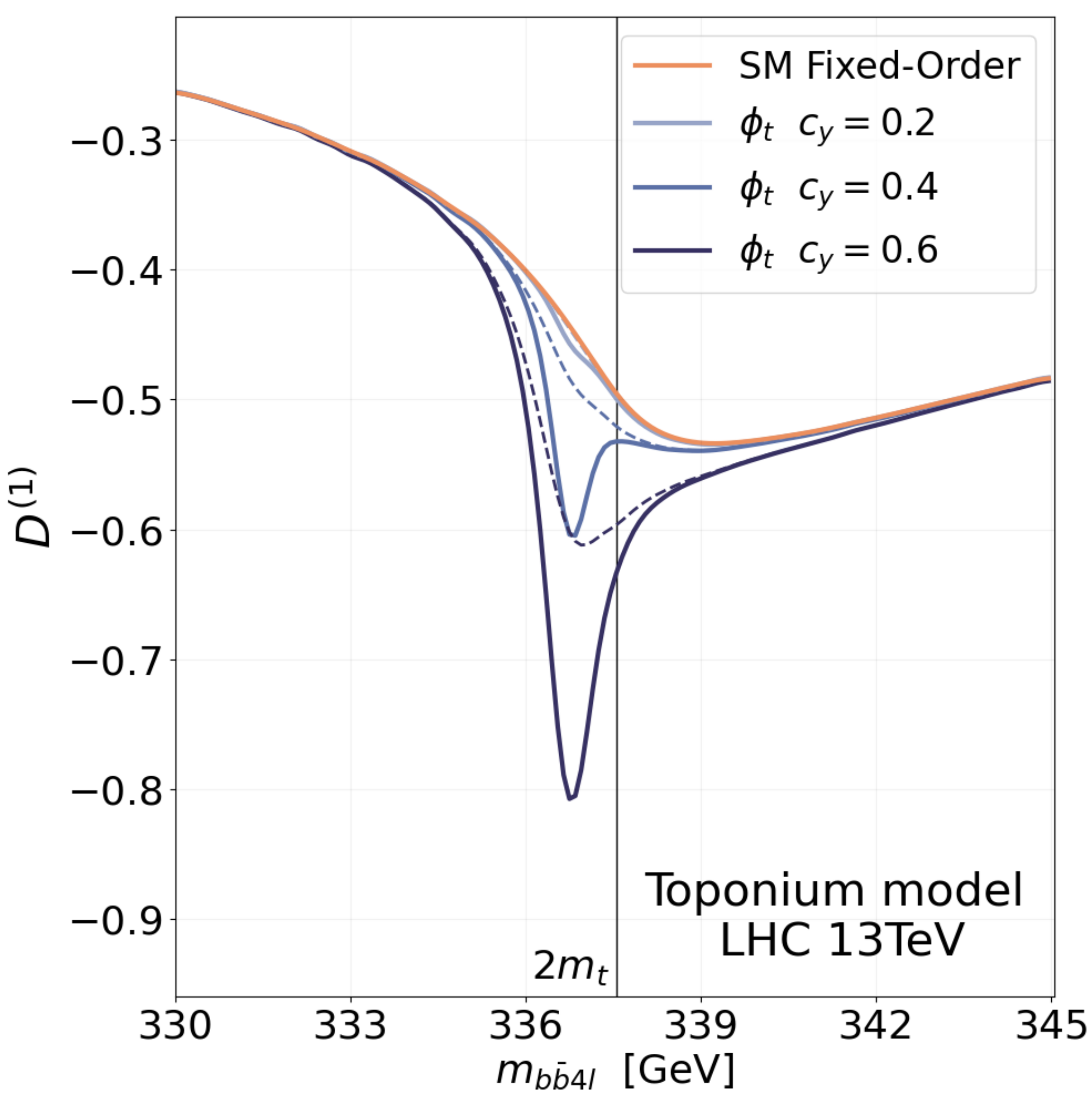


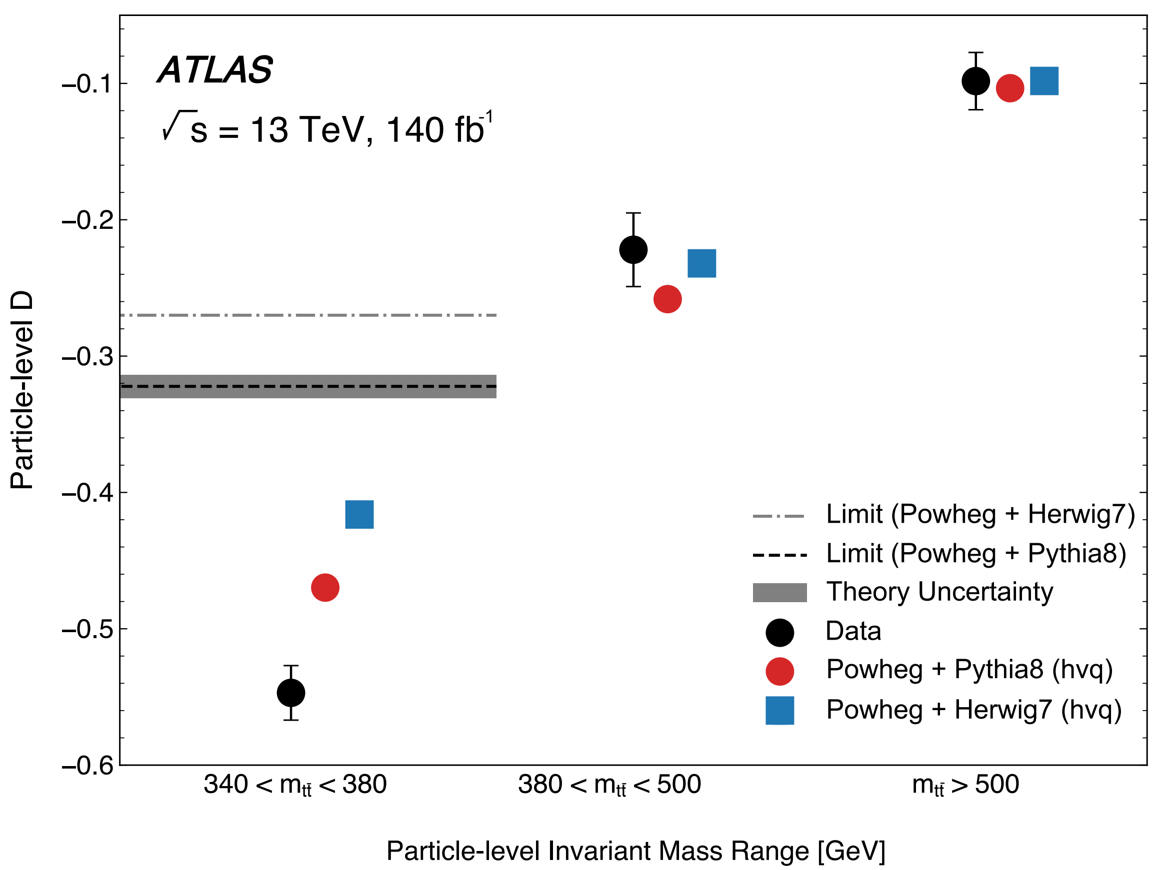
Maltoni, Severi, Tentori, Vryonidou [2401.08751]

$$\mathcal{L} = \mathcal{L}_{\rm SM} - \frac{1}{2}\phi(\partial^2 + M_\phi^2)\phi + c_y\,\frac{y_t}{\sqrt{2}}\,\phi\,\bar{t}\,(\cos\alpha + i\gamma^5\sin\alpha)\,t$$



$M_\phi = 343.5\,\mathrm{GeV}, \quad \alpha = \pi/2$

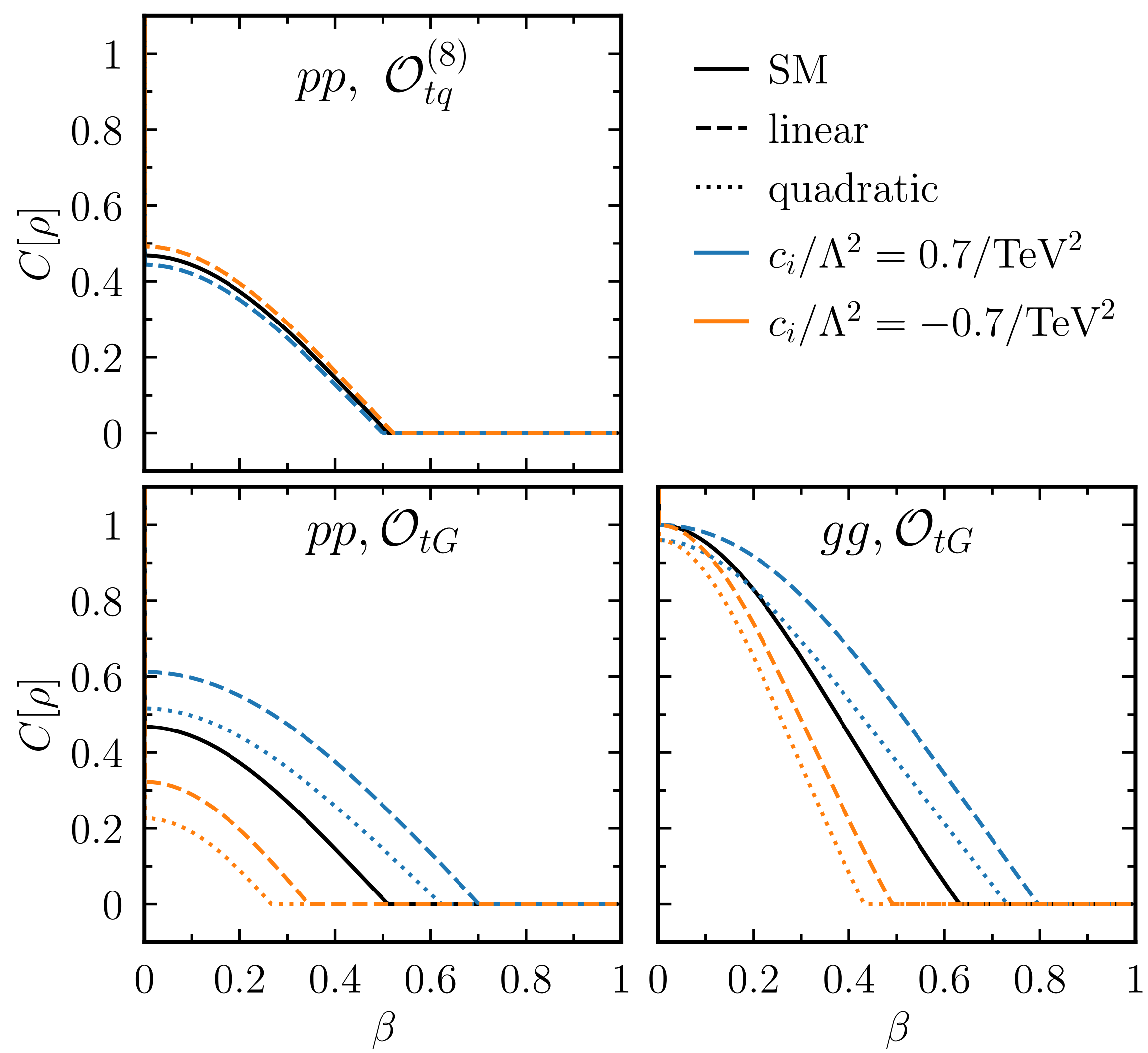




Effect of BSM $pp \to t\bar{t}$

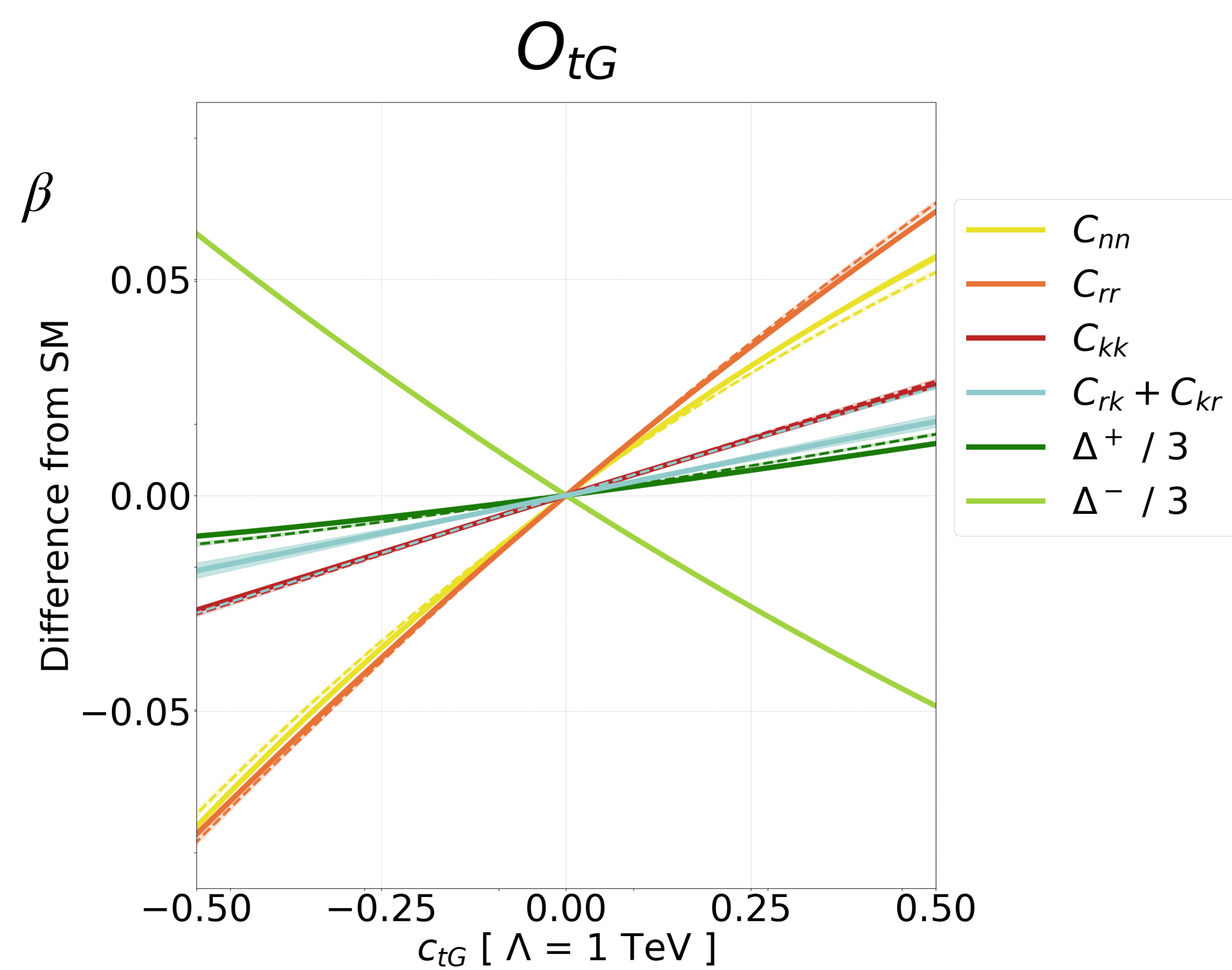
$$\mathcal{O}_{tG} = g_S\, \overline{Q} T_A \tilde{\varphi} \sigma^{\mu\nu} t\, G^A_{\mu\nu} \qquad \mathcal{O}^8_{tq} = \sum_{\mathrm{f}=1}^{2} (\bar{q}_{\mathrm{f}} \gamma_\mu T_A q_{\mathrm{f}})(\bar{t}\gamma^\mu T^A t)$$

[Aoude Madge Maltoni Mantani (2022)]



β: top velocity in the $t\bar{t}$ rest frame

[Severi Vryonidou (2023)]



$$\Delta^\pm = \pm(C_{kk} + C_{rr}) - C_{nn} - 1$$

$H \to \tau^+\tau^-$ @ e^+e^- colliders

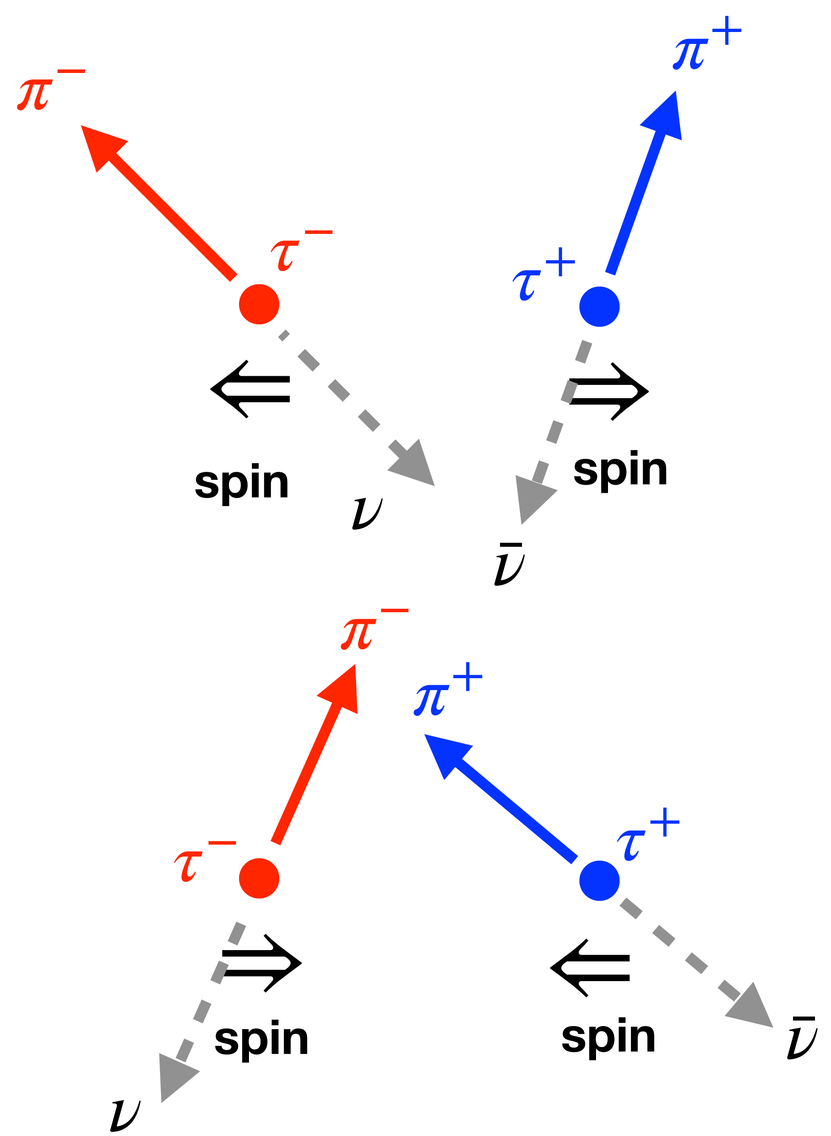
M. Altakach, F. Maltoni, K. Mawatari, P. Lamba, KS, *Phys.Rev.D* 107 (2023) 9, 093002 [2211.10513]

Experimental Challenge in $H \to \tau^+\tau^-$

Among weakly decaying particles, $\tau,\ t,\ W^{\pm},\ Z^0$, the tau-lepton is special because $m_\tau \ll m_H$

One has to measure the direction of pions at the rest frame of each tau.

→ Reconstruction of the tau rest frames (i.e. neutrino reconstruction) is necessary



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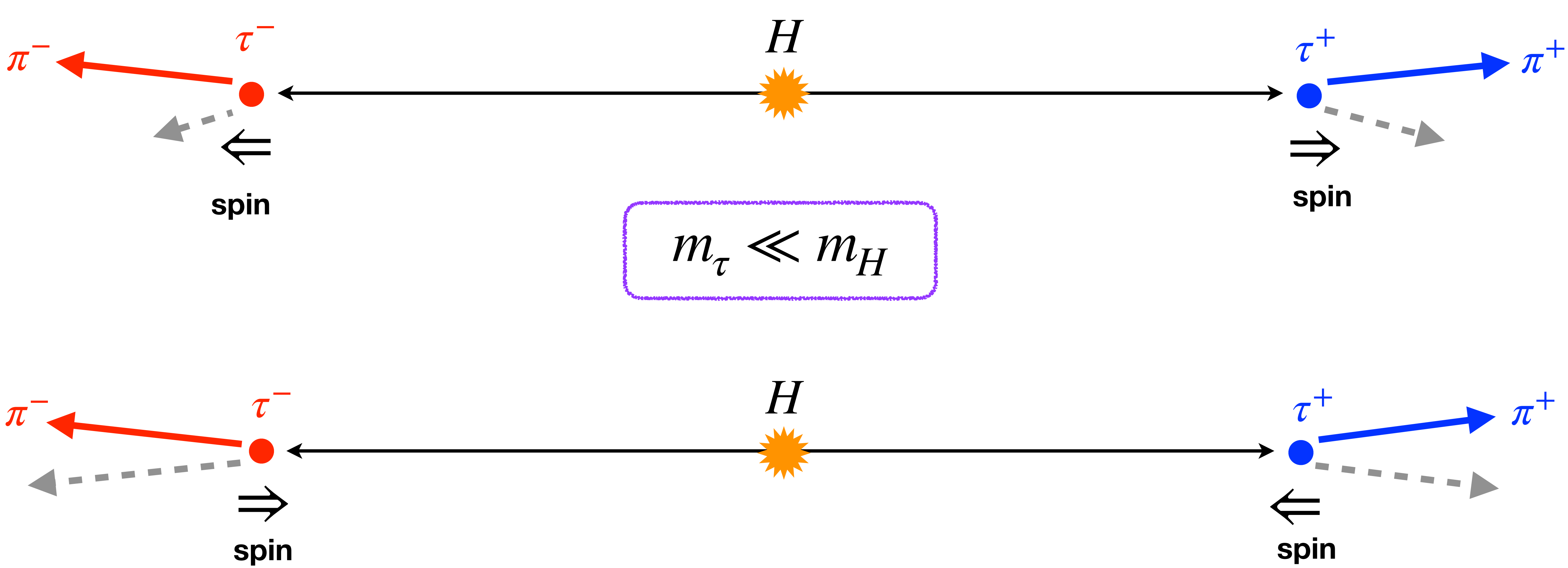
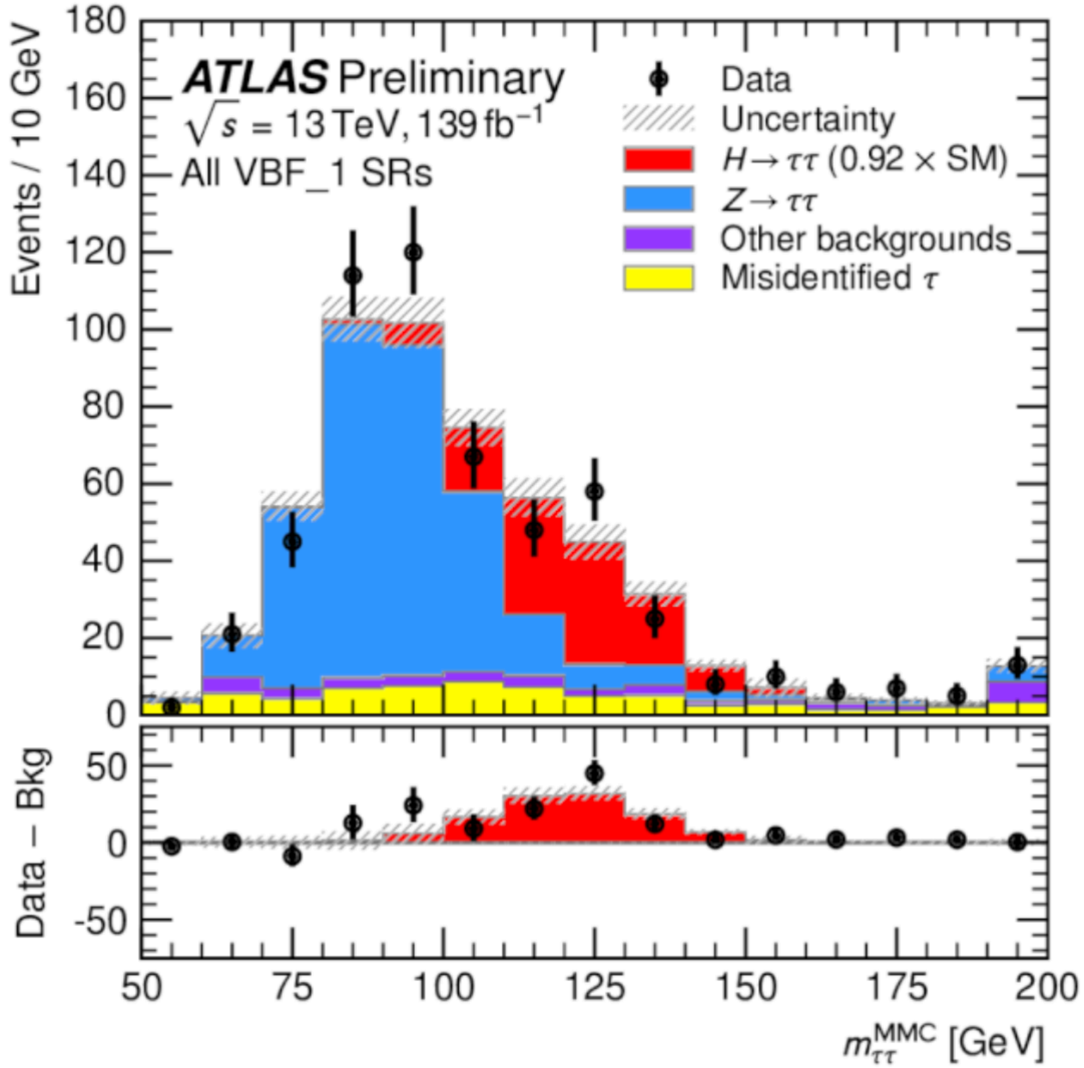


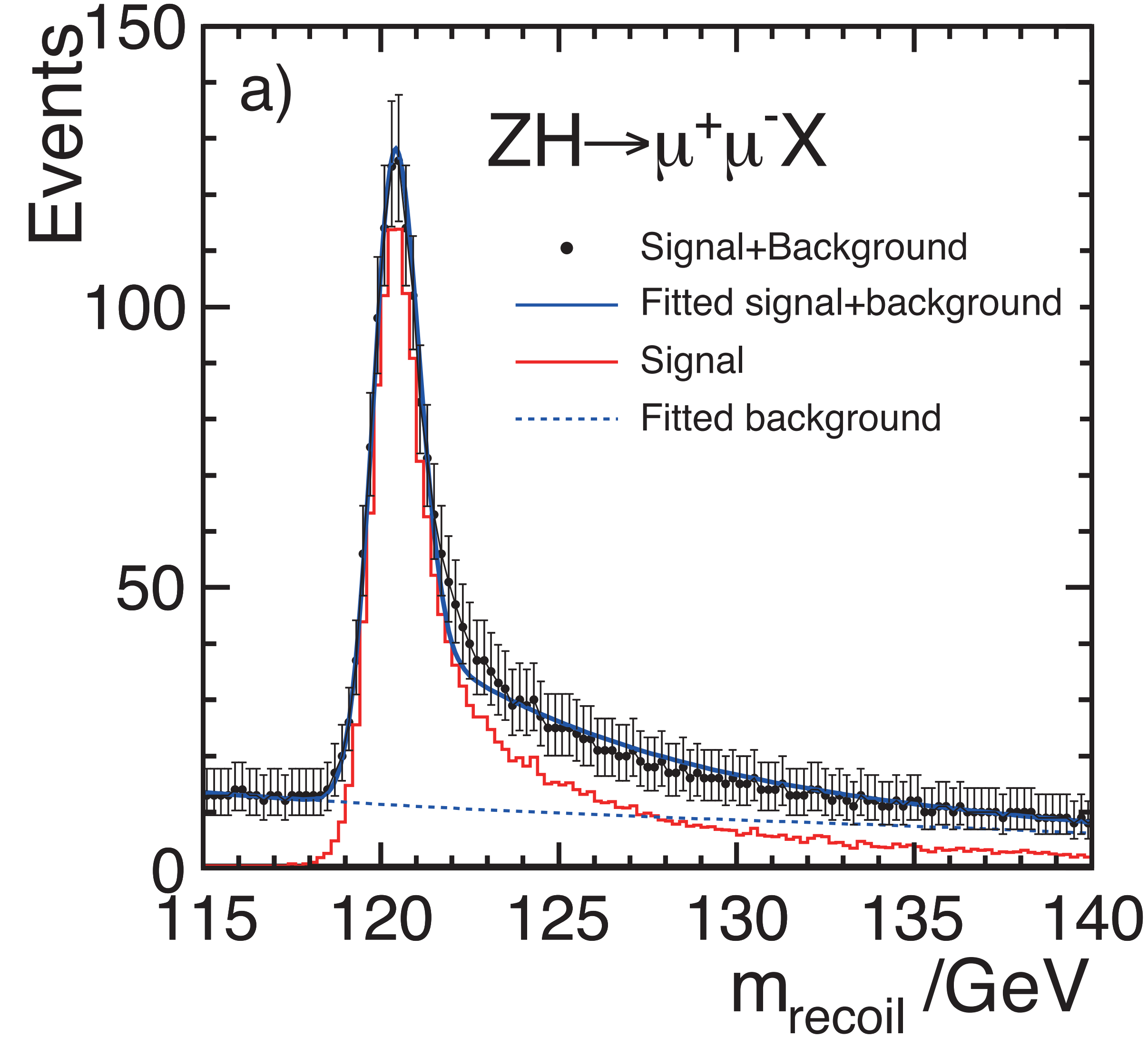
$H \to \tau^+\tau^-$ @ lepton colliders

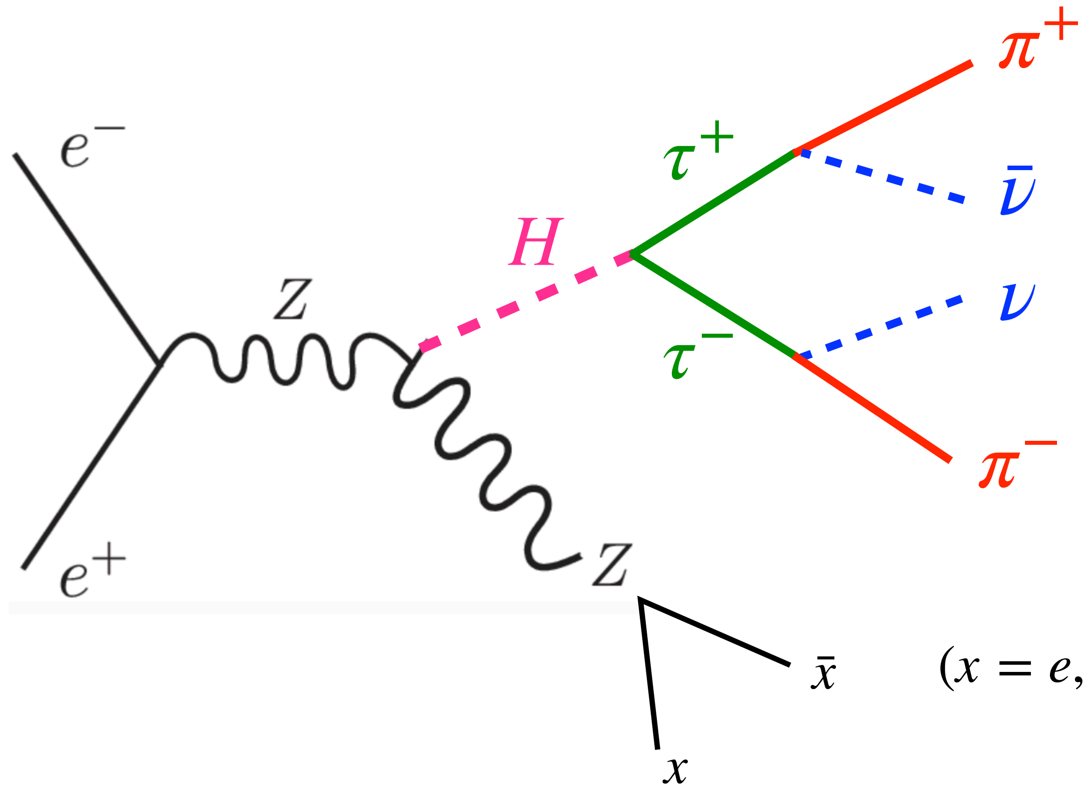
- For precise event reconstruction and for much smaller background, we consider lepton colliders.

LHC



ILC





$(x = e, \mu, j)$

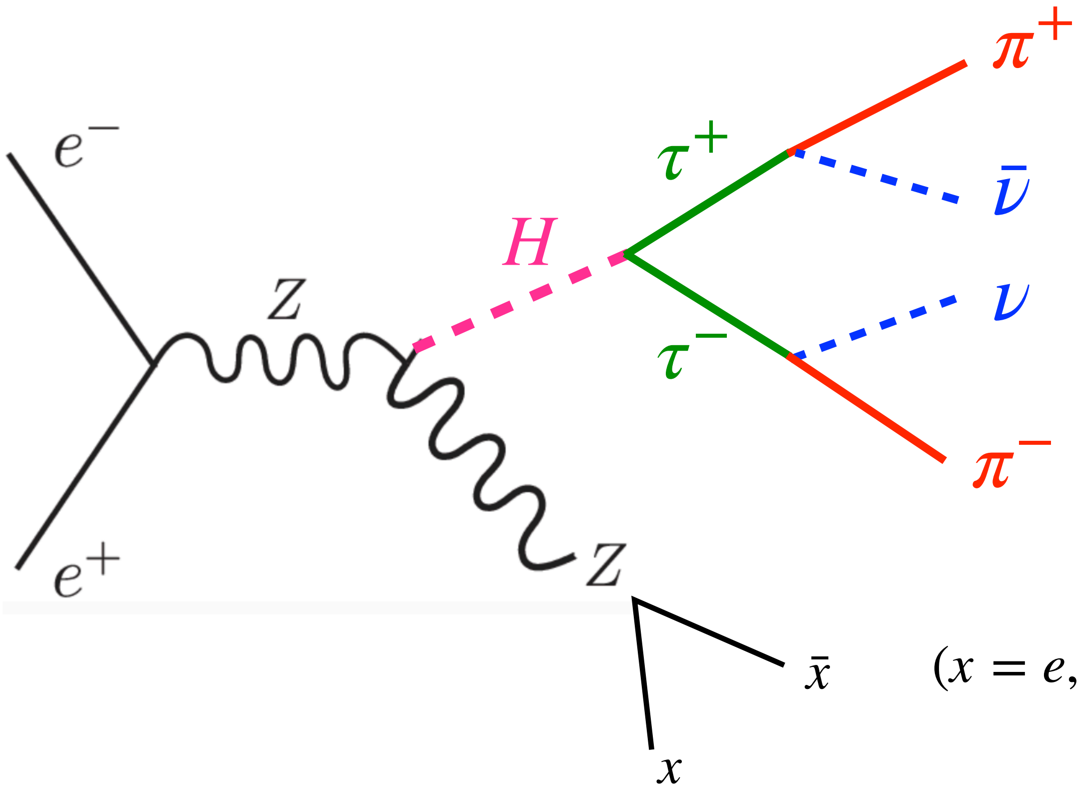
$$(P_H^{\text{reco}})^\mu \equiv P^\mu_{e^+e^-} - P^\mu_{Z\to x\bar{x}} \qquad M^2_{\text{recoil}} \equiv (P_H^{\text{reco}})^2$$

Event selection: $|M_{\text{recoil}} - 125\,\text{GeV}\,| < 5\,\text{GeV}$

$e^+e^- \to Z + (Z^*/\gamma^*) \to f\bar{f} + \tau^+\tau^-$

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	0.83×10^{-4}
beam resolution e^- (%)	0.27	0.83×10^{-4}
$\sigma(e^+e^- \to HZ)$ (fb)	240.1	240.3
# of signal ($\sigma \cdot \text{BR} \cdot L \cdot \epsilon$)	385	663
# of background ($\sigma \cdot \text{BR} \cdot L \cdot \epsilon$)	20	36

- Generate the SM events $(\kappa, \delta) = (1{,}0)$ with **MadGraph5.**
- **100 pseudo-experiments** to estimate the statistical uncertainties



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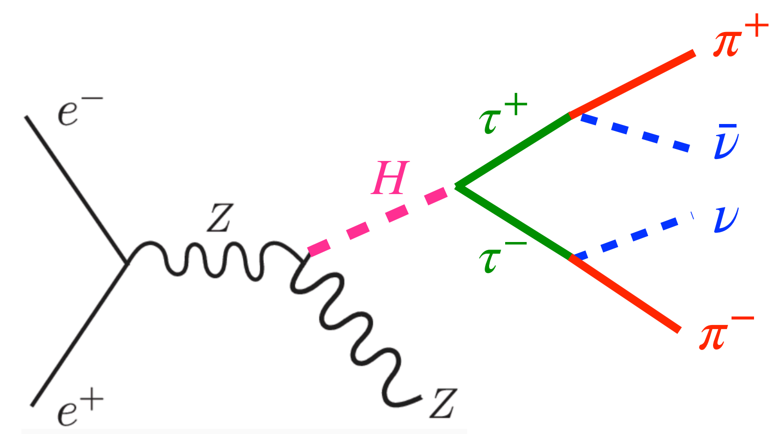
- Generate the SM events $(\kappa, \delta) = (1,0)$ with **MadGraph5**.
- **100 pseudo-experiments** to estimate the statistical uncertainties

- To determine the tau momenta, we have to reconstruct the unobserved neutrino momenta $(p_x^{\nu}, p_y^{\nu}, p_z^{\nu})$, $(p_x^{\bar{\nu}}, p_y^{\bar{\nu}}, p_z^{\bar{\nu}})$.



- **6** unknowns can be constrained by **2** mass-shell conditions and **4** energy-momentum conservation.

$$m_\tau^2 = (p_{\tau^+})^2 = (p_{\pi^+} + p_{\bar{\nu}})^2$$

$$m_\tau^2 = (p_{\tau^-})^2 = (p_{\pi^-} + p_{\nu})^2$$

$$(p_{ee} - p_Z)^\mu = p_H^\mu = \left[(p_{\pi^-} + p_\nu) + (p_{\pi^+} + p_{\bar{\nu}})\right]^\mu$$

=> 2-fold solutions.

$$C_{ij}^{\mathrm{SM}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$C_{\mathrm{SM}}[\rho] = 1$$

$$S_{\mathrm{CHSH}} = 2\sqrt{2}$$

reproduced very accurately in the simulation

→ we found that false solutions also give the correct correlations! (?)

Effect of momentum mismeasurement

$$E_i^{\rm true} \to E_i^{\rm obs} = (1 + \sigma_E \cdot \omega) \cdot E_i^{\rm true} \qquad \sigma_E = 0.03 \qquad (i = \pi^\pm, e^\pm, \mu^\pm, j)$$

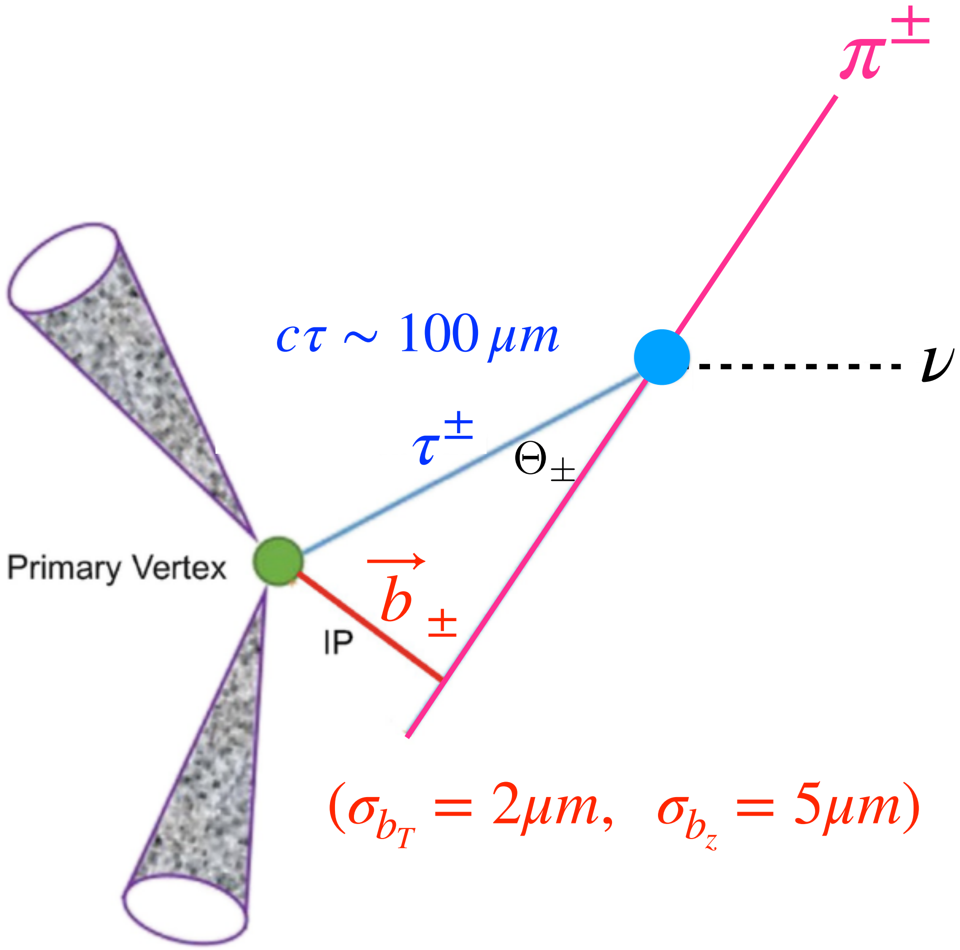
random number drawn from the normal distribution

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} -0.600 \pm 0.210 & 0.003 \pm 0.125 & 0.020 \pm 0.149 \\ 0.003 \pm 0.125 & -0.494 \pm 0.190 & 0.007 \pm 0.128 \\ 0.048 \pm 0.174 & 0.0007 \pm 0.156 & 0.487 \pm 0.193 \end{pmatrix}$	$\begin{pmatrix} -0.559 \pm 0.143 & -0.010 \pm 0.095 & -0.014 \pm 0.122 \\ -0.010 \pm 0.095 & -0.494 \pm 0.139 & -0.002 \pm 0.111 \\ 0.012 \pm 0.124 & 0.020 \pm 0.105 & 0.434 \pm 0.134 \end{pmatrix}$
$\mathcal{C}[\rho]$	0.030 ± 0.071	0.005 ± 0.023
$S_{\rm CHSH}/2$	0.769 ± 0.189	0.703 ± 0.134

$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\rm SM}[\rho] = 1 \qquad S_{\rm CHSH}^{\rm SM}/2 = \sqrt{2}$$

Momentum smearing spoils the previous good result…

Use impact parameter information



Goal:

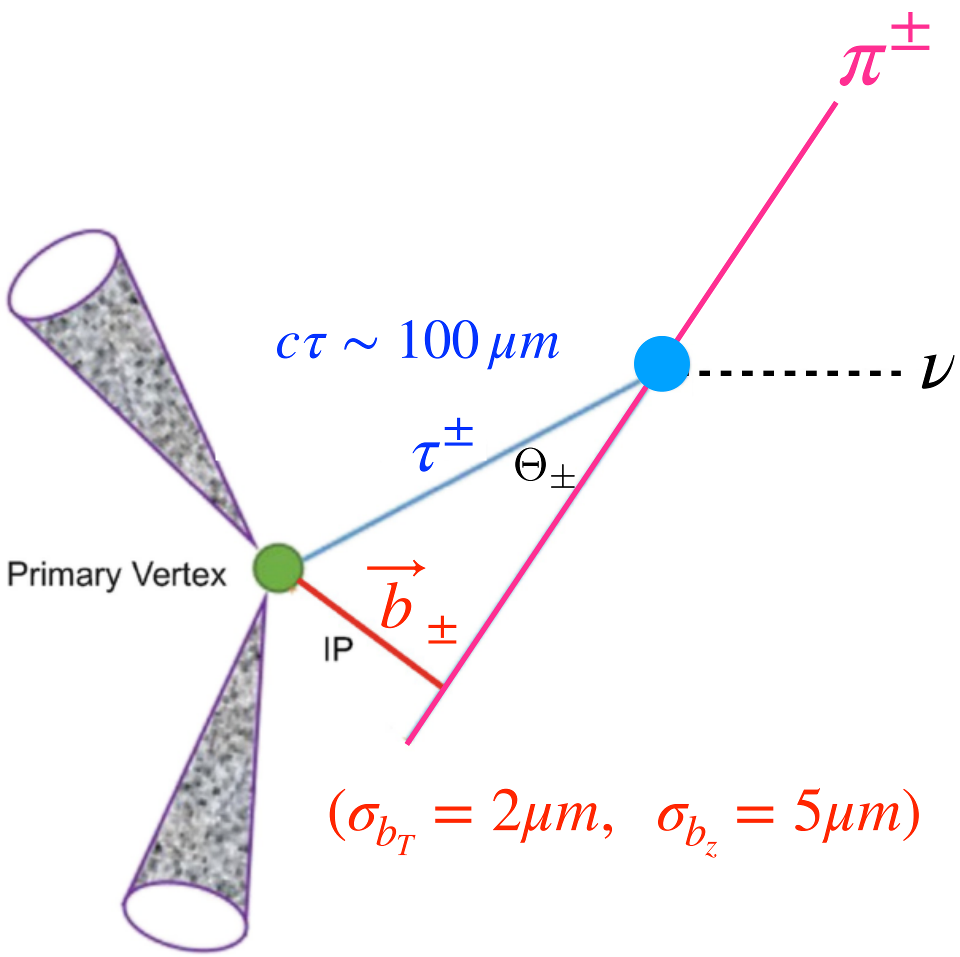
$$E_i^{\text{true}} \to E_i^{\text{obs}} \to E_i^{\text{true}} \quad (i = \pi^{\pm}, e^{\pm}, \mu^{\pm}, j)$$

What we do:

- modify E_i^{obs} for some amount by δ

$$E_i^{\text{obs}} \to E_i(\delta_i) = (1 + \delta_i\,\sigma_E) \cdot E_i^{\text{obs}}$$

Use impact parameter information



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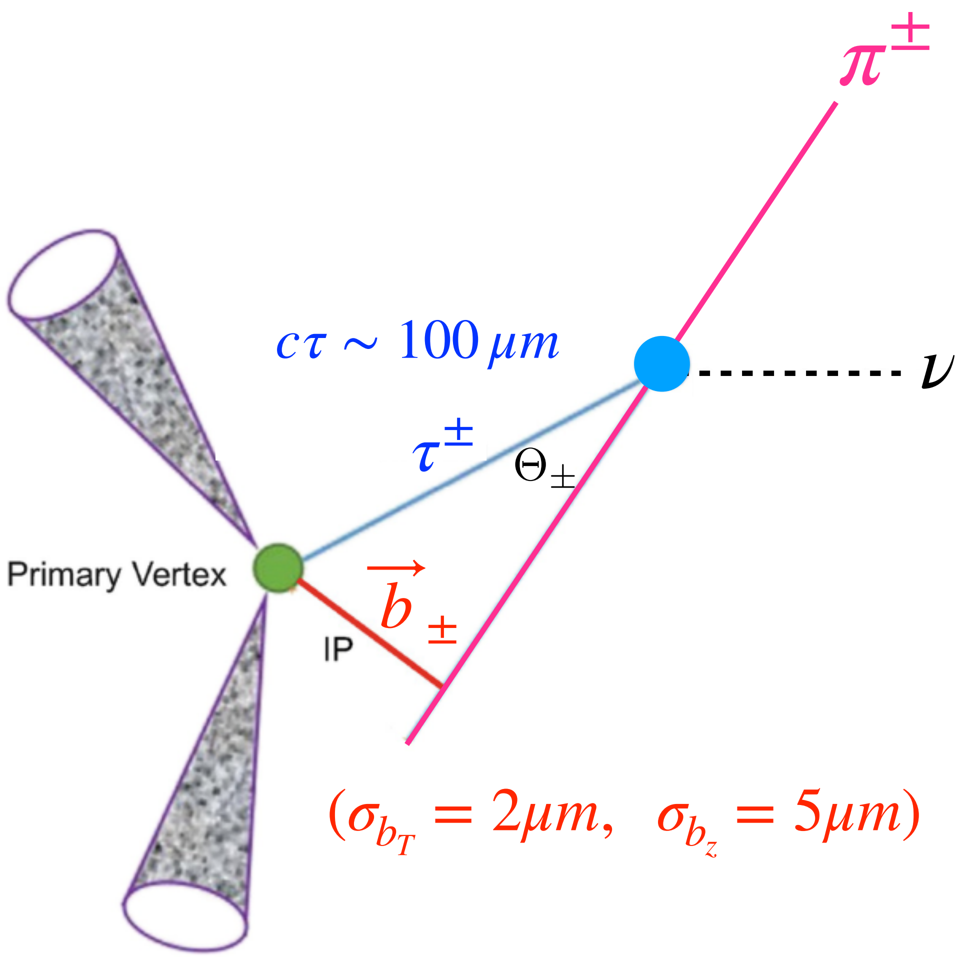
- solve tau direction $\mathbf{e}_{\tau^{\pm}}(\boldsymbol{\delta})$

→ lets us calculate $\overrightarrow{b}_{\pm}$ as functions of δ

$$\vec{b}_{\pm}^{\text{reco}}(\mathbf{e}_{\tau^{\pm}}) = |\vec{b}_{\pm}| \cdot \left[\mathbf{e}_{\tau^{\pm}} \cdot \sin^{-1}\Theta_{\pm} - \mathbf{e}_{\pi^{\pm}} \cdot \tan^{-1}\Theta_{\pm}\right]$$

Use impact parameter information



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- solve tau direction $\mathbf{e}_{\tau^{\pm}}(\boldsymbol{\delta})$

→ lets us calculate $\overrightarrow{b}_{\pm}$ as functions of δ

$$\vec{b}_{\pm}^{\rm reco}\,(\mathbf{e}_{\tau^{\pm}}) \ = \ |\vec{b}_{\pm}| \cdot \left[\mathbf{e}_{\tau^{\pm}} \cdot \sin^{-1}\Theta_{\pm} \ - \ \mathbf{e}_{\pi^{\pm}} \cdot \tan^{-1}\Theta_{\pm}\right]$$

- compare the calculated $\overrightarrow{b}_{\pm}^{\rm reco}(\boldsymbol{\delta})$ and measured $\overrightarrow{b}_{\pm}^{\rm obs}$ and construct the likelihood function

2 fold solutions: $i_s = 1{,}2$

$$\vec{\Delta}_{b_{\pm}}^{i_s}(\boldsymbol{\delta}) \ \equiv \ \vec{b}_{\pm} - \vec{b}_{\pm}^{\rm reco}\left(\mathbf{e}_{\tau^{+}}^{i_s}(\boldsymbol{\delta})\right)$$

$$L_{\pm}^{i_s}(\boldsymbol{\delta}) \ = \ \frac{[\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_x^2 + [\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_y^2}{\sigma_{b_T}^2} \ + \ \frac{[\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_z^2}{\sigma_{b_z}^2} \ + \ \delta_{\pi^+}^2 \ + \ \delta_{\pi^-}^2 \ + \ \delta_x^2 \ + \ \delta_{\bar{x}}^2 \ .$$

minimizing $L^{i_s}(\boldsymbol{\delta})$ would give us the correct set of $\boldsymbol{\delta}s$ and solution i_s

Result

2211.10513

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} 0.830 \pm 0.176 & 0.020 \pm 0.146 & -0.019 \pm 0.159 \\ -0.034 \pm 0.160 & 0.981 \pm 0.1527 & -0.029 \pm 0.156 \\ -0.001 \pm 0.158 & -0.021 \pm 0.155 & -0.729 \pm 0.140 \end{pmatrix}$	$\begin{pmatrix} 0.925 \pm 0.109 & -0.011 \pm 0.110 & 0.038 \pm 0.095 \\ -0.009 \pm 0.110 & 0.929 \pm 0.113 & 0.001 \pm 0.115 \\ -0.026 \pm 0.122 & -0.019 \pm 0.110 & -0.879 \pm 0.098 \end{pmatrix}$
$\mathcal{C}[\rho]$	0.778 ± 0.126	0.871 ± 0.084
$S_{\rm CHSH}/2$	1.103 ± 0.163	1.276 ± 0.094

$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\rm SM}[\rho] = 1 \qquad S_{\rm CHSH}^{\rm SM}/2 = \sqrt{2}$$

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$\mathcal{C}[\rho]$	0.778 ± 0.126 $\sim 5\sigma$	0.871 ± 0.084 $\gg 5\sigma$
$S_{\text{CHSH}}/2$	1.103 ± 0.163	1.276 ± 0.094 $\sim 3\sigma$

$$C_{ij}^{\text{SM}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\text{SM}}[\rho] = 1 \qquad S_{\text{CHSH}}^{\text{SM}}/2 = \sqrt{2}$$

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2211.10513

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$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\rm SM}[\rho] = 1 \qquad S_{\rm CHSH}^{\rm SM}/2 = \sqrt{2}$$

Superiority of FCC-ee over ILC is due to a better beam resolution

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	$0.83 \cdot 10^{-4}$
beam resolution e^- (%)	0.27	$0.83 \cdot 10^{-4}$

CP measurement

- Under CP, the spin correlation matrix transforms: $C \xrightarrow{CP} C^T$
- This can be used for a *model-independent* test of CP violation. We define:

$$A \equiv (C_{rn} - C_{nr})^2 + (C_{nk} - C_{kn})^2 + (C_{kr} - C_{rk})^2 \geq 0$$

- Observation of $A \neq 0$ immediately confirms CP violation.

- From our simulation, we observe

$$A = \begin{cases} 0.204 \pm 0.173 & \text{(ILC)} \\ 0.112 \pm 0.085 & \text{(FCC-ee)} \end{cases} \longleftarrow$$

consistent with absence of CPV

- This model independent bounds can be translated to the constraint on the CP-phase δ

$$\mathscr{L}_{\text{int}} \propto H \bar{\psi}_\tau (\cos\delta + i\gamma_5 \sin\delta)\,\psi_\tau \;\Rightarrow\; C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix} \;\Rightarrow\; A(\delta) = 4\sin^2 2\delta$$

CP measurement

- Focusing on the region near $|\delta| = 0$, we find the 1-σ bounds:

$$|\delta| < \begin{cases} 8.9^o & \text{(ILC)} \\ 6.4^o & \text{(FCC-ee)} \end{cases}$$

- Other studies:

$$\Delta\delta \sim 11.5^o \quad \text{(HL-LHC)}$$ [Hagiwara, Ma, Mori 2016]

$$\Delta\delta \sim 4.3^o \quad \text{(ILC)}$$ [Jeans and G. W. Wilson 2018]

So far, the literature focuses on **two**-particle entanglement

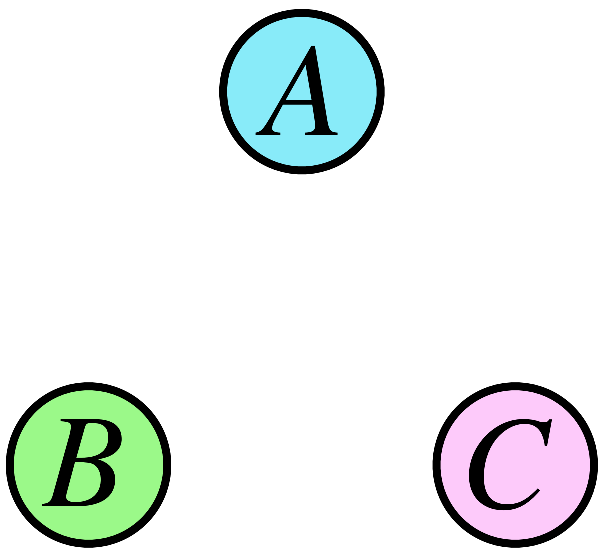
↓

what about **three**-particle entanglement?

✦ KS, M. Spannowsky [2310.01477]

3-Particle Entanglement

3-particle entanglement has a much richer structure that 2-PE !



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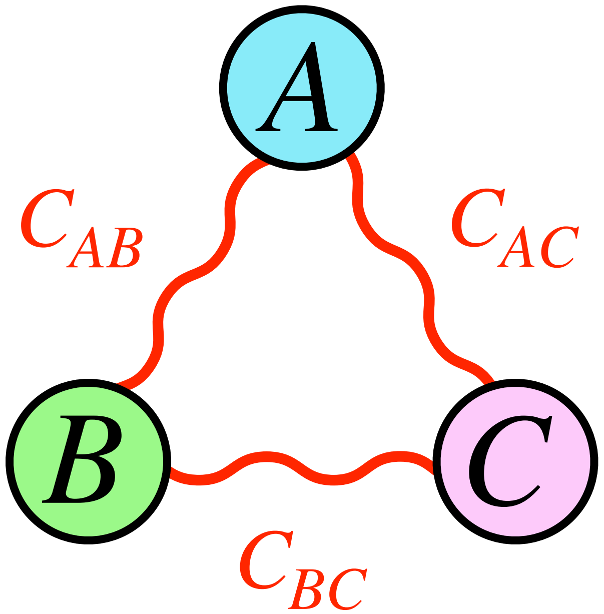
- Ent. btw 2-individual particles



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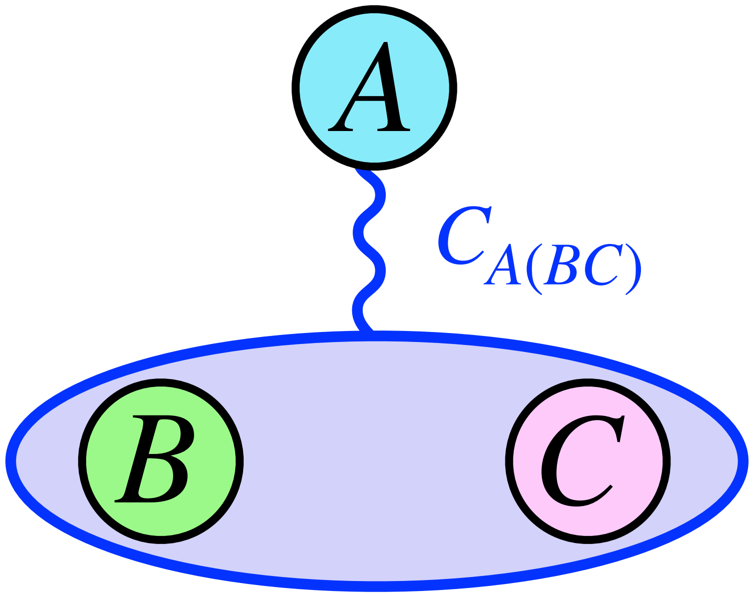

- Ent. btw 2-individual particles
- Ent. btw one-to-other

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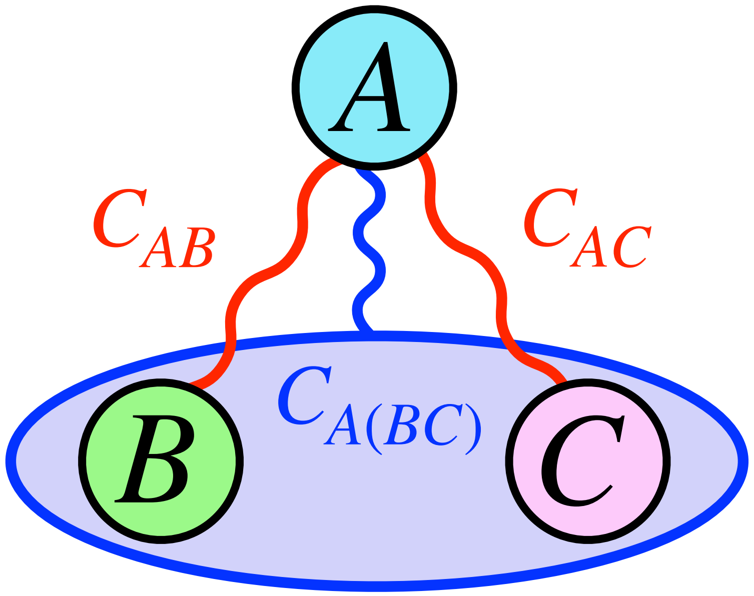
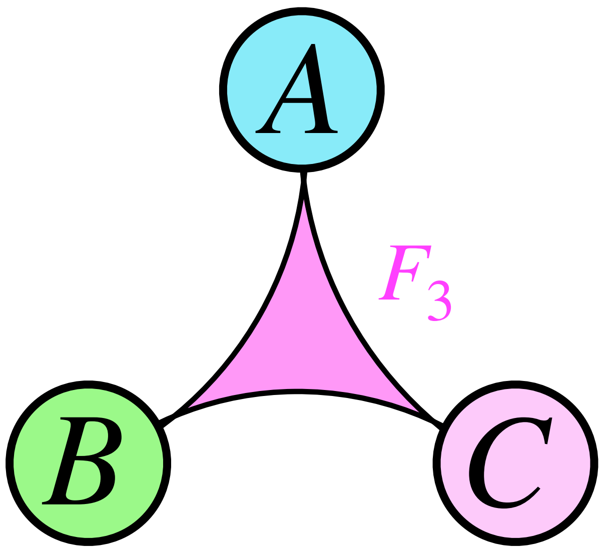


- Ent. btw 2-individual particles
- Ent. btw one-to-other
- "Monogamy" $C^2_{A(BC)} \geq C^2_{AB} + C^2_{AC}$

3-Particle Entanglement

3-particle entanglement has a much richer structure that 2-PE !



- Ent. btw 2-individual particles
- Ent. btw one-to-other
- "Monogamy" $C^2_{A(BC)} \geq C^2_{AB} + C^2_{AC}$
- "Genuine" 3-particle entanglement F_3

  (non-separable even partially)

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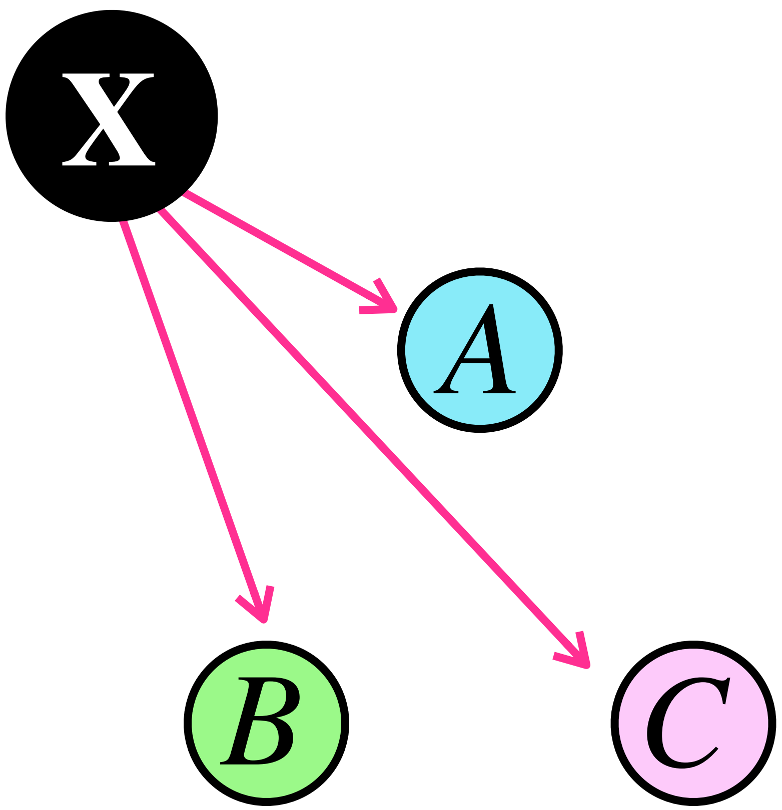

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3-body decay: $X \to ABC$

explore all possible Lorentz invariant interactions

How to quantify entanglement?

Ex.) **Concurrence** [for 2 qubit system]

$$\mathcal{C}[\rho] = \max(0, \eta_1 - \eta_2 - \eta_3 - \eta_4) \in [0,1]$$

$\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4$ are eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

density matrix $\longrightarrow$ $\rho \equiv \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$

$(p_i \geq 0, \sum_i p_i = 1)$

$$\mathcal{C}[\rho] \begin{cases} = 0 & \longleftarrow \textbf{not}\text{-entangled} \\ > 0 & \longleftarrow \textbf{entangled} \end{cases}$$

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$$|\Psi\rangle = \sum_{a,b,c} c_{abc} \cdot |a\rangle_A \otimes |b\rangle_B \otimes |c\rangle_C$$

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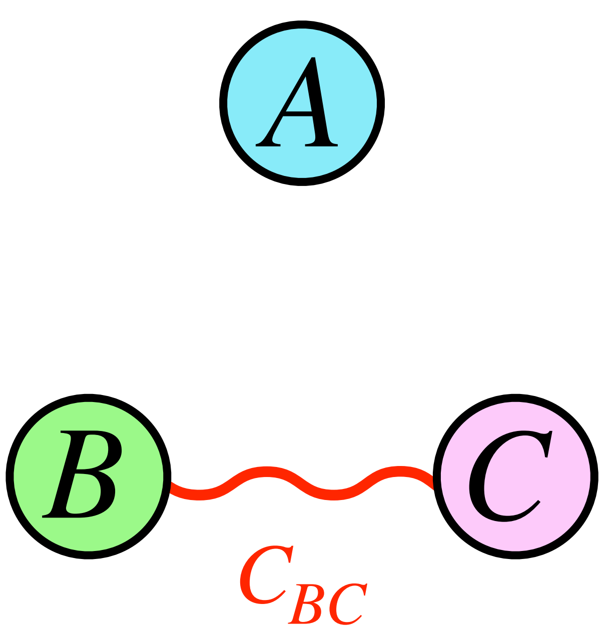

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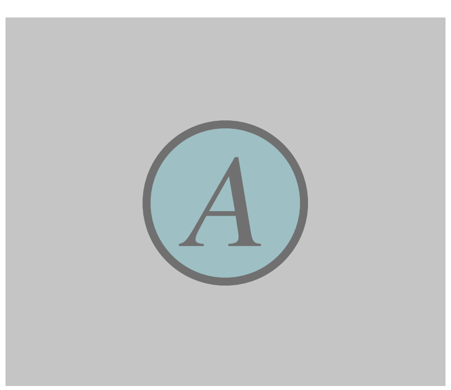

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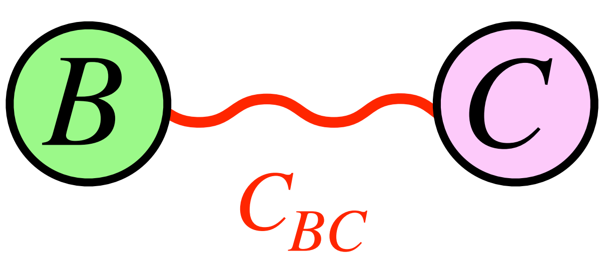
trace out A

$$\Longrightarrow \quad \rho_{BC} = \mathrm{Tr}_A |\Psi\rangle\langle\Psi|$$



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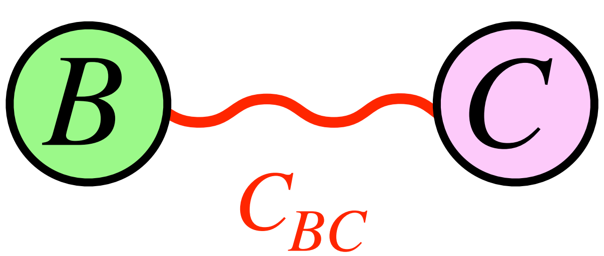

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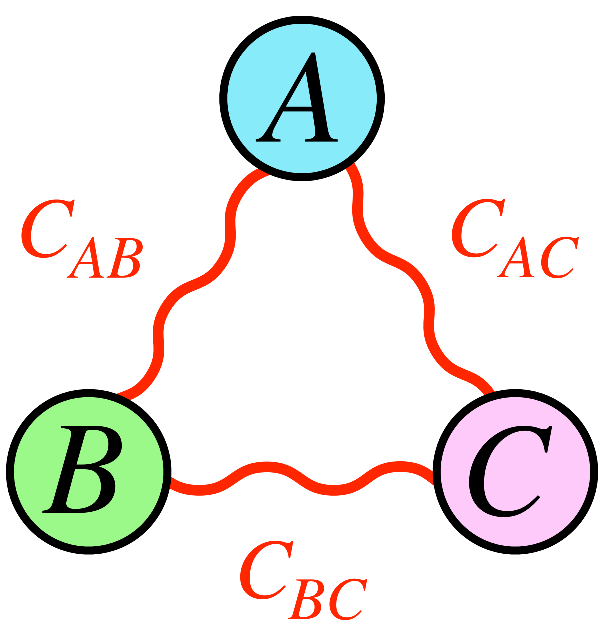
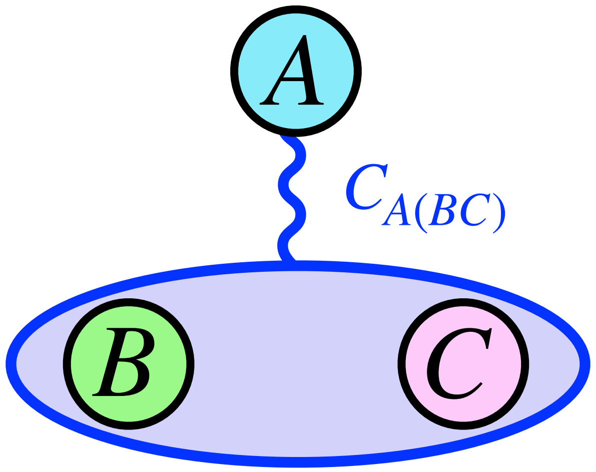


C_{AB}, C_{BC}, C_{AC}

How to calculate entanglement btw A and (BC) ?



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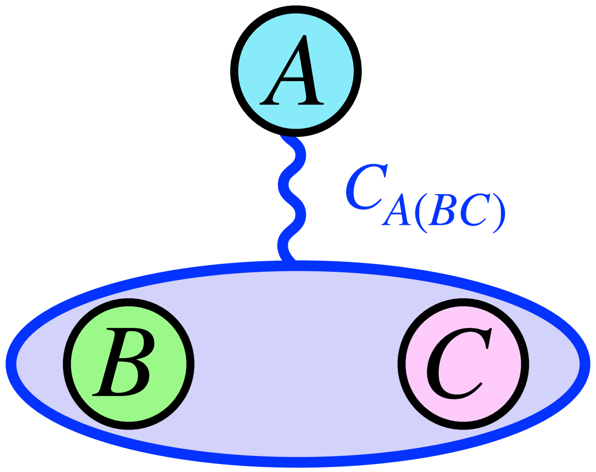

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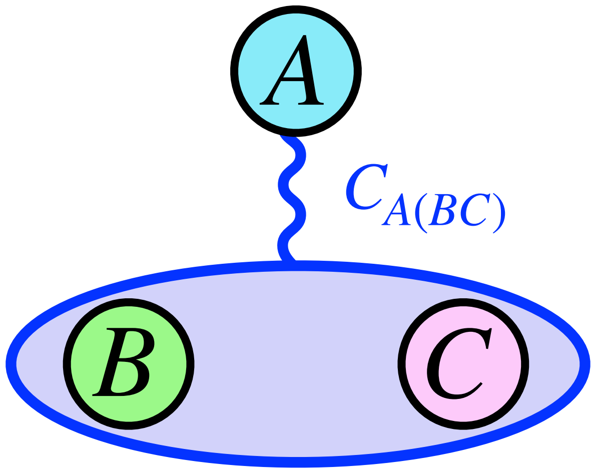

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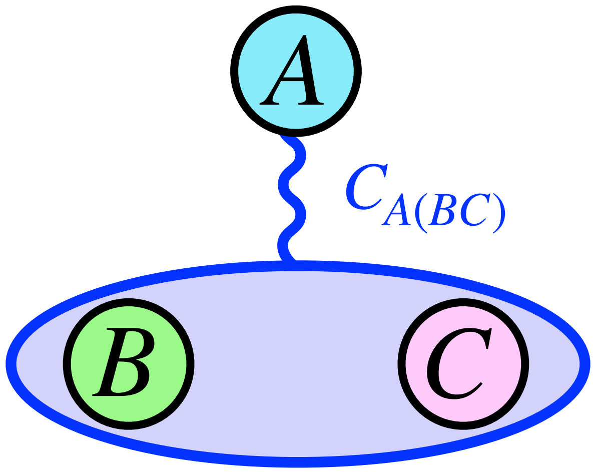

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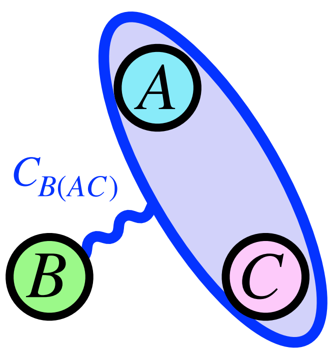
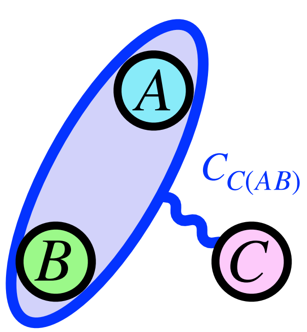


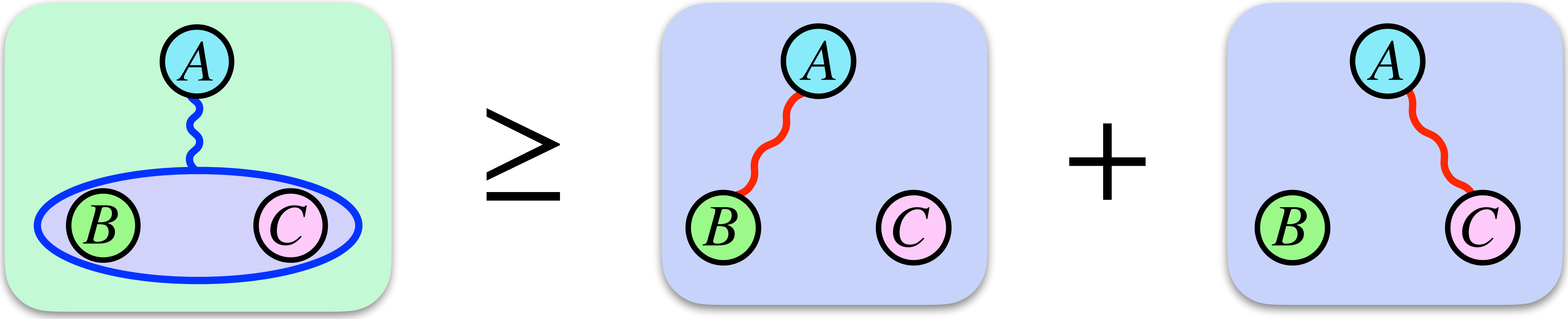
$$\mathcal{C}_{B(AC)} \equiv \mathcal{C}[|\Psi\rangle] = \sqrt{2(1 - \mathrm{Tr}\rho^2_{AC})} \qquad \rho_{AC} \equiv \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$



$$\mathcal{C}_{C(AB)} \equiv \mathcal{C}[|\Psi\rangle] = \sqrt{2(1 - \mathrm{Tr}\rho^2_{AB})} \qquad \rho_{AB} \equiv \mathrm{Tr}_C |\Psi\rangle\langle\Psi|$$

Monogamy

- A-(BC) entanglement limits A-B and A-C entanglements



[Coffman, Kundu, Wootters '99]

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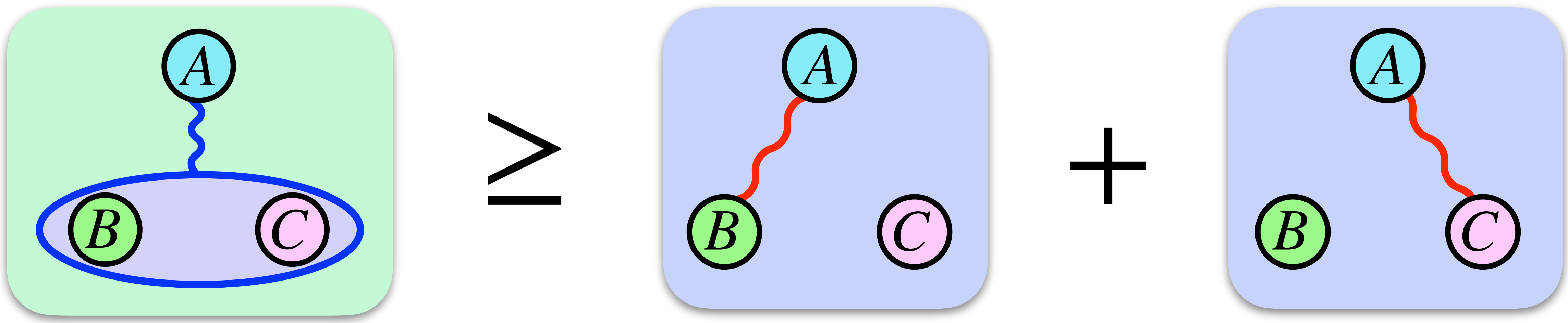

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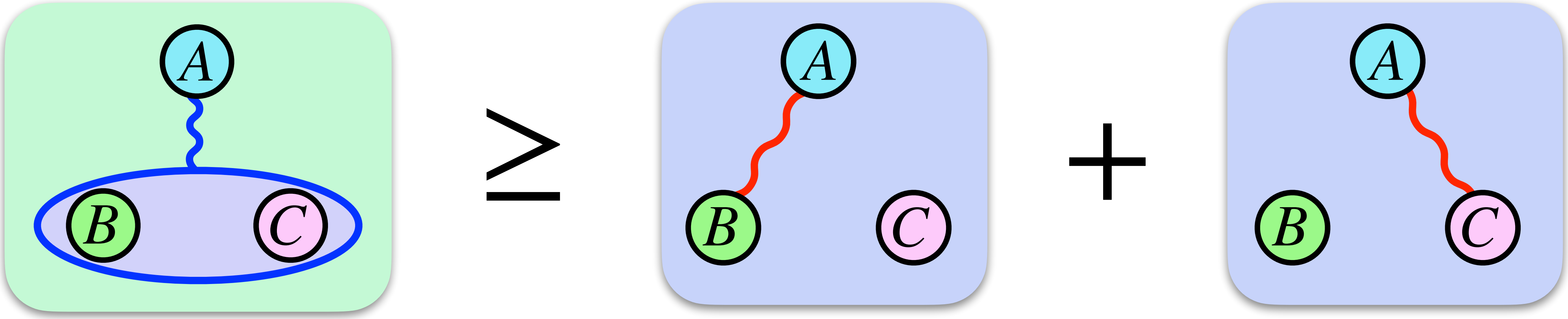

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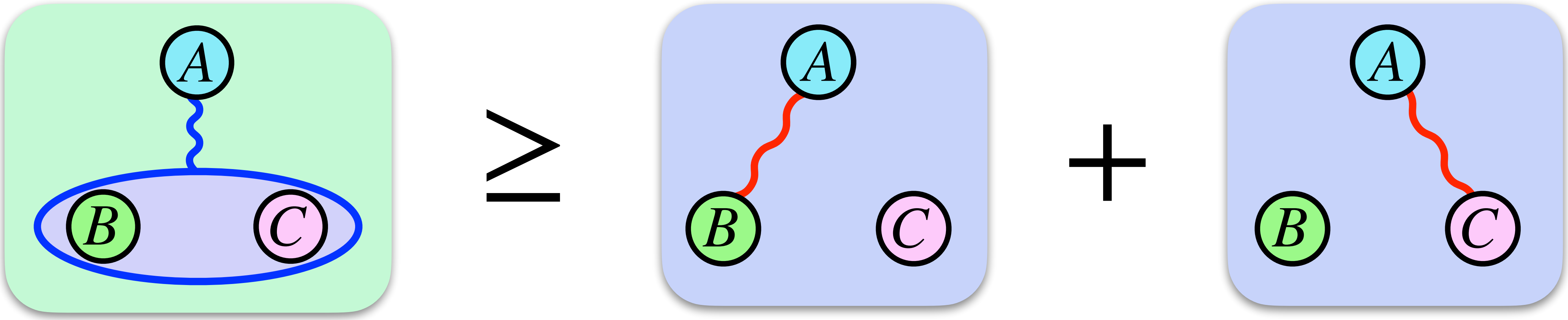

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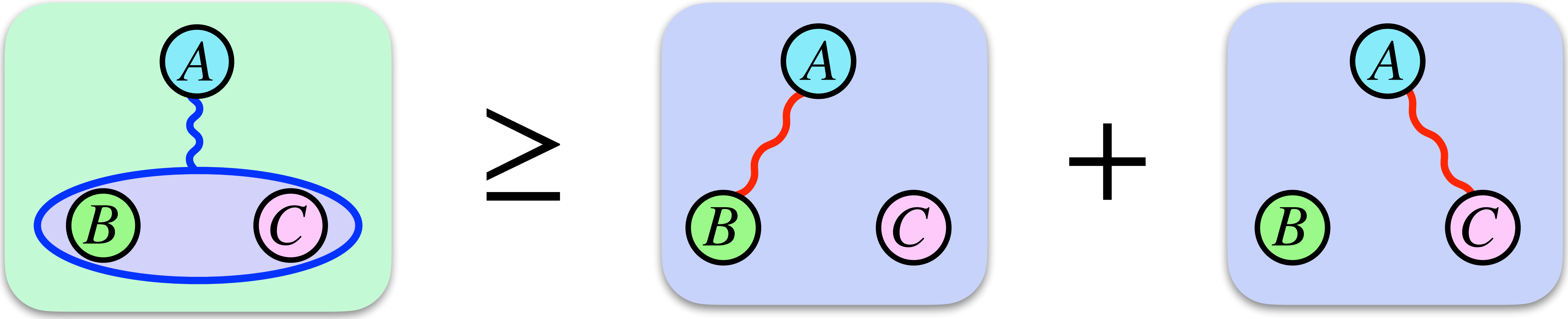

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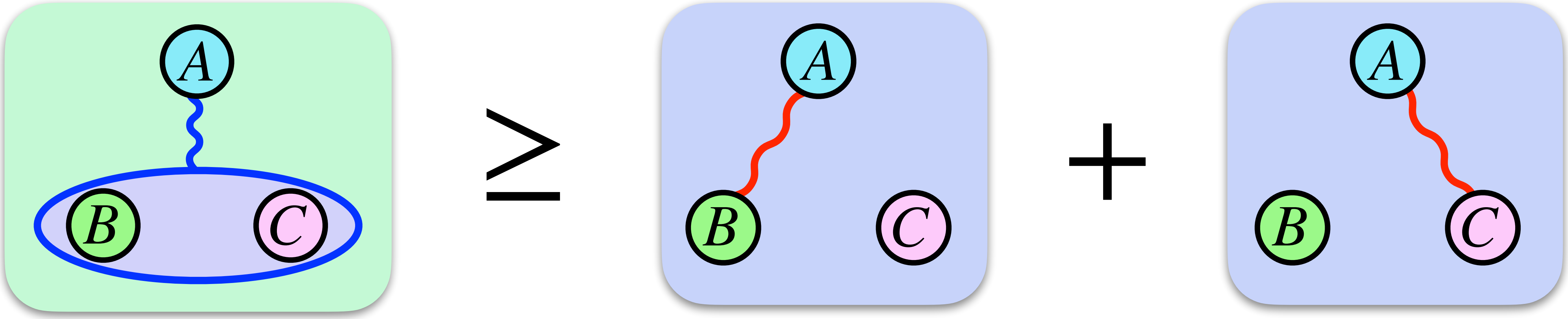

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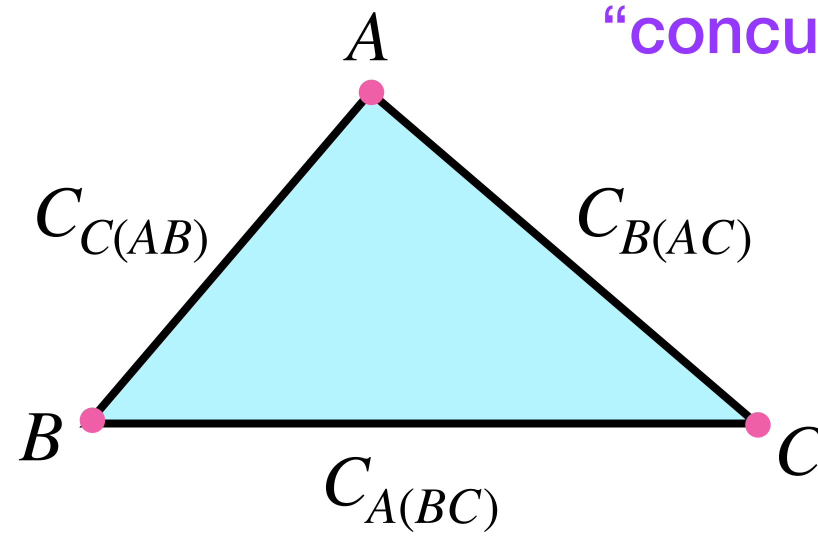
$$C^2_{A(BC)} + C^2_{B(AC)} \geq C^2_{C(AB)}$$

$$\Downarrow$$

$$\boxed{C_{A(BC)} + C_{B(AC)} \geq C_{C(AB)}}$$

“concurrence triangle”

$$C_{\mathbf{A}(\mathbf{BC})} + C_{\mathbf{B}(\mathbf{AC})} \geq C_{\mathbf{C}(\mathbf{AB})} \quad \Rightarrow$$



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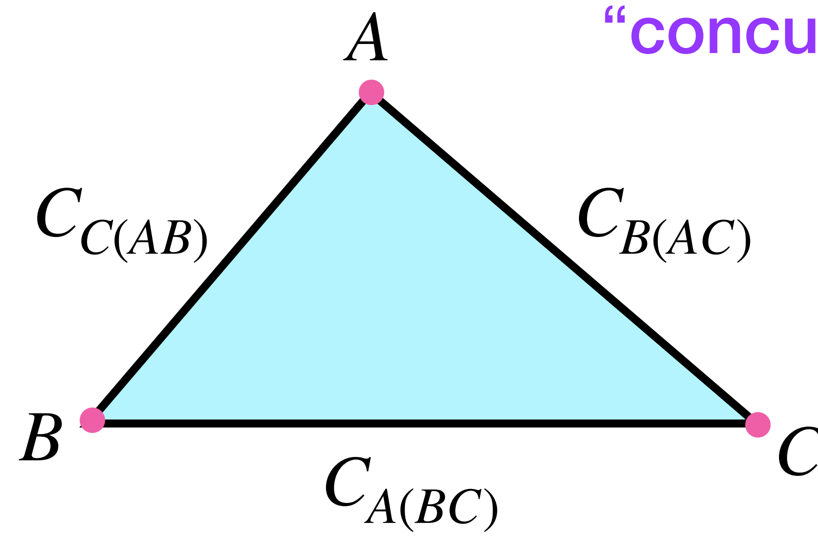

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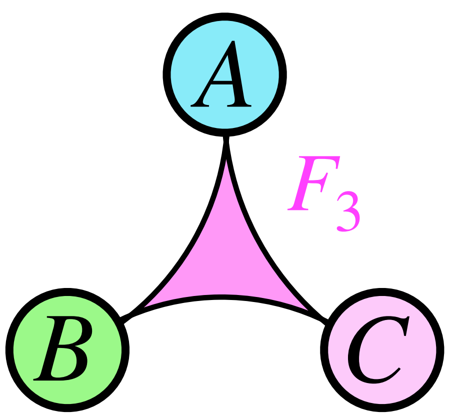
Genuine Multi-particle Entanglement (GME) measure: [Dur, Vidal, Cirac '00, Ma, Chen, Chen, Spengler, Gabriel, Huber '11, Xie, Eberly '21]



GME should satisfy the following properties:

(1) vanish for all product and biseparable states $\Rightarrow$ unseparable even partially

(2) positive for all non-biseparable states

(3) not increase under LOCC

$$|\psi\rangle_A \otimes (|00\rangle_{BC} + |11\rangle_{BC})$$

$$\Rightarrow F_3 = 0$$

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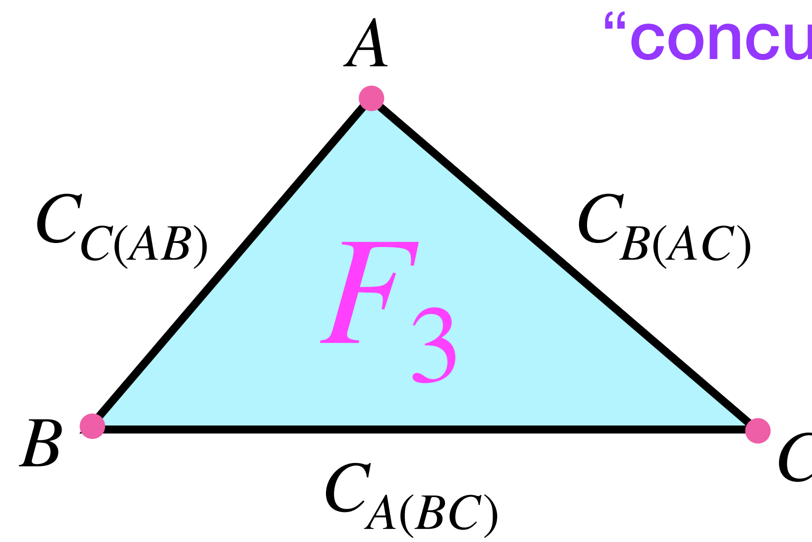

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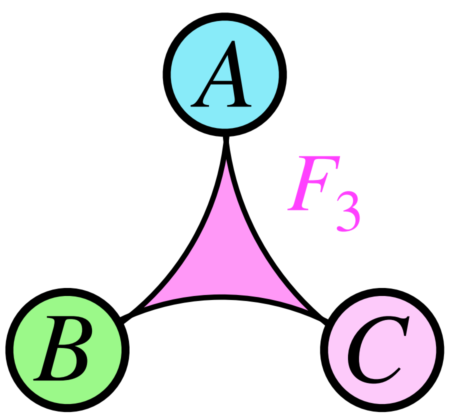

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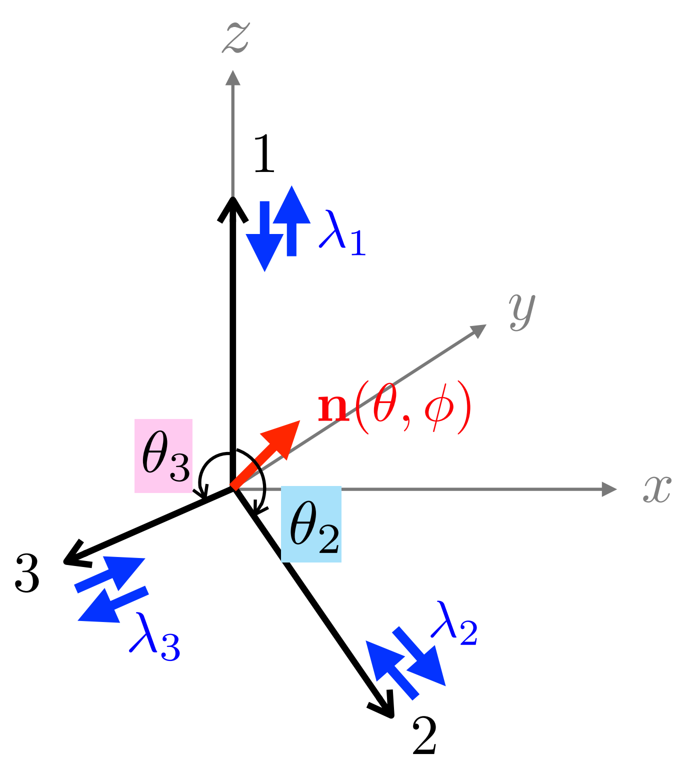
The **area** of the “concurrence triangle” satisfies (1), (2), (3) ! [Jin, Tao, Gui, Fei, Li-Jost, Qiao (2023)]

$$F_3 \equiv \left[\tfrac{16}{3} Q\left(Q - \mathcal{C}_{A(BC)}\right)\left(Q - \mathcal{C}_{B(AC)}\right)\left(Q - \mathcal{C}_{C(AB)}\right)\right]^{\frac{1}{2}} \in [0, 1]$$

$$Q \equiv \frac{1}{2}\left[\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} + \mathcal{C}_{C(AB)}\right]$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$



[KS, M.Spannowsky 2310.01477]

Assumptions:

- all particles have spin 1/2
- all final particles 1,2,3 are massless

Kinematics:

- rest frame of the initial particle 0
- p_1 is in the z-axis
- decay is in the x-z plane

$$p_1^\mu = p_1(1, 0, 0, 1)$$
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$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

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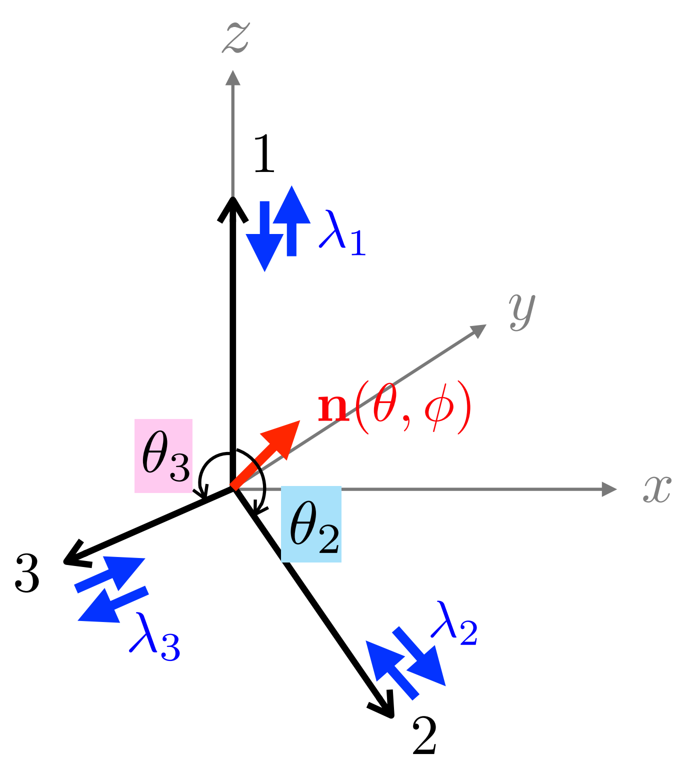

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initial state

$$|\mathbf{n}(\theta,\phi)\rangle$$

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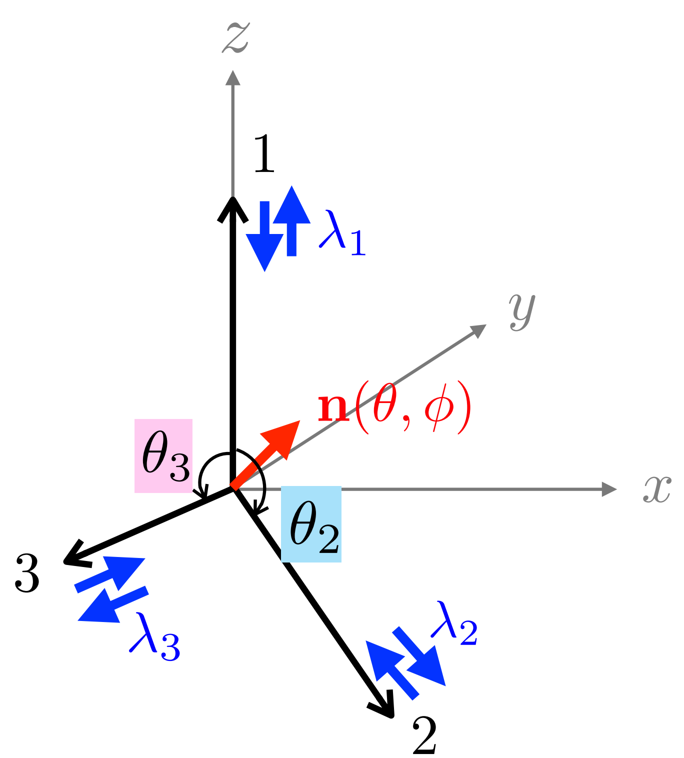

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amplitude

$$\hat{\mathbf{1}} = \sum_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle\langle\lambda_1, \lambda_2, \lambda_3| \qquad \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} = \langle\lambda_1, \lambda_2, \lambda_3|\mathbf{n}(\theta, \phi)\rangle$$

initial state

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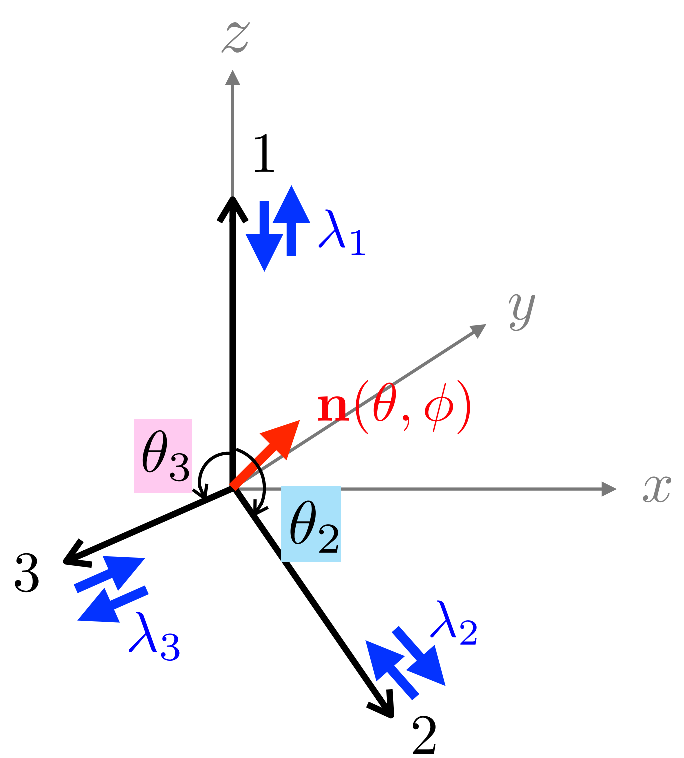
final state

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



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pure (entangled)
3-spin state

Interaction

- Consider **most general** Lorentz invariant 4-fermion interactions

$$\boxed{\begin{gathered}\mathcal{L}_{\rm int} = (\bar{\psi}_1 \Gamma_A \psi_0)(\bar{\psi}_3 \Gamma_B \psi_2) \\ \Gamma_{A/B} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}\end{gathered}}$$

$$\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$$

❖ Scalar-type

$$[\bar{\psi}_1 (c_S + i c_A \gamma_5)\psi_0][\bar{\psi}_3 (d_S + i d_A \gamma_5)\psi_2]$$

$$c \equiv c_S + i c_A = e^{i\delta_1}$$
$$d \equiv d_S + i d_A = e^{i\delta_2}$$

❖ Vector-type

$$[\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R)\psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R)\psi_2]$$

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$
$$c_L, c_R, d_L, d_R \in \mathbb{R}$$

❖ Tensor-type

$$[\bar{\psi}_1 (c_M + i c_E \gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3 (d_M + i d_E \gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + i c_E = e^{i\omega_1}$$
$$d \equiv d_M + i d_E = e^{i\omega_2}$$

[KS, M.Spannowsky 2310.01477]

Scalar

$$\mathcal{L}_{\rm int} = \;[\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

$$c \equiv c_S + ic_A = e^{i\delta_1}$$
$$d \equiv d_S + id_A = e^{i\delta_2}$$

$$\Rightarrow \; |\Psi\rangle \;=\; \frac{cd}{\sqrt{2}}\cdot e^{i\phi}s_{\frac{\theta}{2}}|---\rangle - \frac{cd^*}{\sqrt{2}}\cdot e^{i\phi}s_{\frac{\theta}{2}}|-++\rangle + \frac{c^*d}{\sqrt{2}}\cdot c_{\frac{\theta}{2}}|+--\rangle - \frac{c^*d^*}{\sqrt{2}}\cdot c_{\frac{\theta}{2}}|+++\rangle$$

[KS, M.Spannowsky 2310.01477]

Scalar

$$\mathcal{L}_{\rm int} = \ [\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

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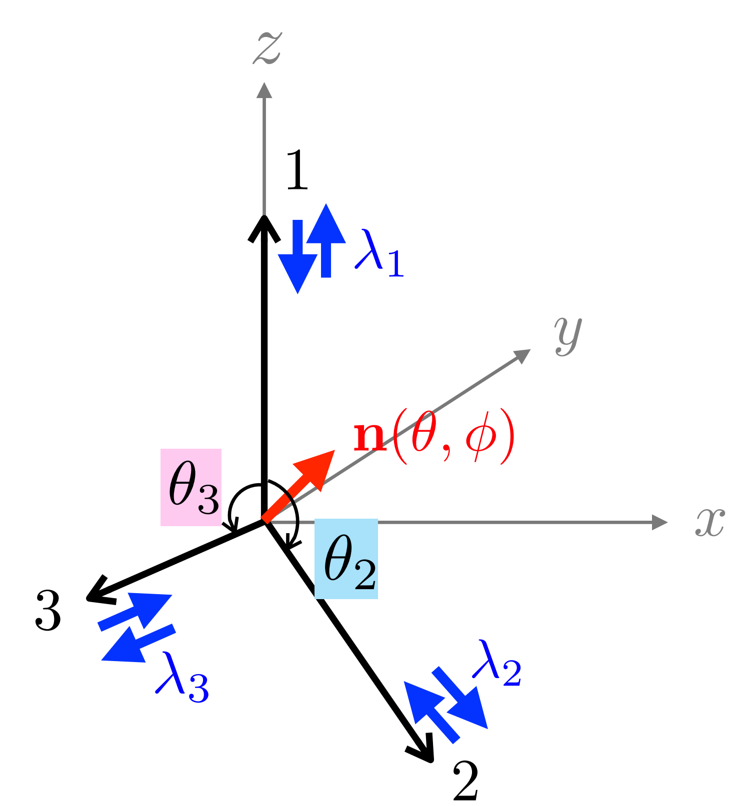
independent of final state momenta θ_2, θ_3



[KS, M.Spannowsky 2310.01477]

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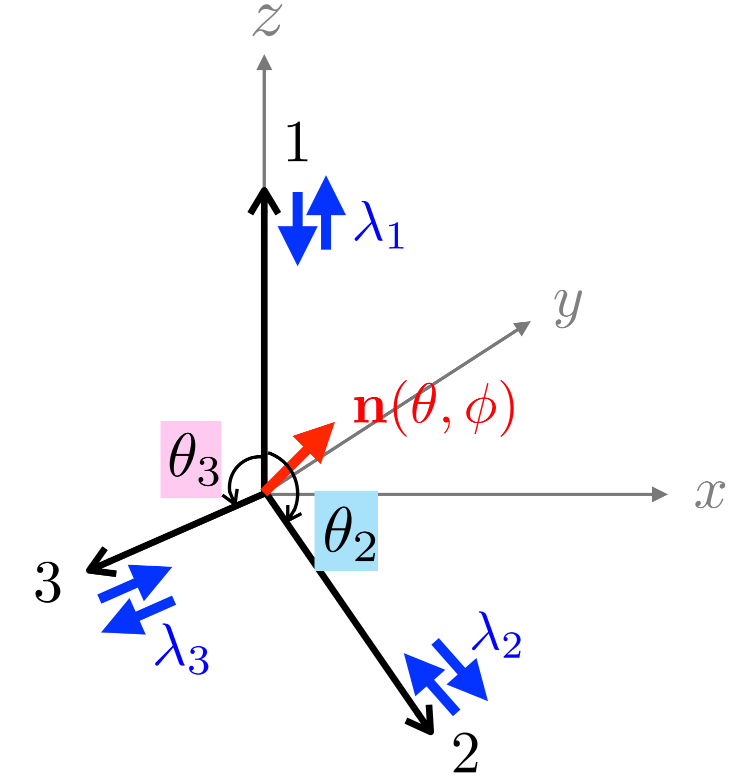
independent of final state momenta θ_2, θ_3

$$= \left[c e^{i\phi}s_{\frac{\theta}{2}}|-\rangle_1 + c^* c_{\frac{\theta}{2}}|+\rangle_1\right]\otimes\frac{1}{\sqrt{2}}\left[d|--\rangle_{23} - d^*|++\rangle_{23}\right]$$

bi-separable



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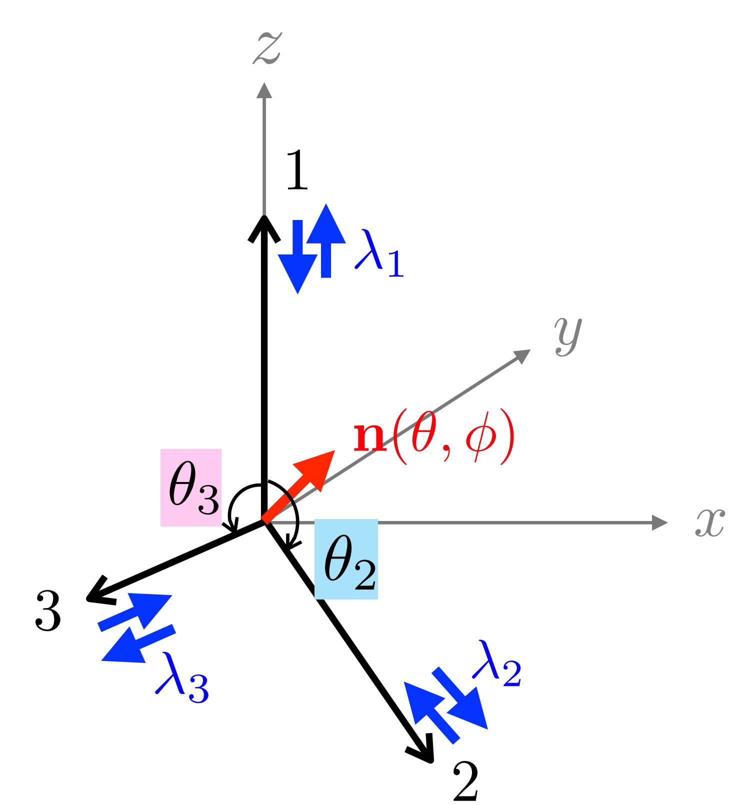
independent of final state momenta θ_2, θ_3

$$= \left[ce^{i\phi}s_{\frac{\theta}{2}}|-\rangle_1 + c^*c_{\frac{\theta}{2}}|+\rangle_1\right] \otimes \frac{1}{\sqrt{2}}\left[d|--\rangle_{23} - d^*|++\rangle_{23}\right]$$

bi-separable

$$\Rightarrow F_3 = 0$$



[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

$$= \left[c e^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1\right] \otimes \frac{1}{\sqrt{2}}\left[d|-\,-\rangle_{23} - d^*|+\,+\rangle_{23}\right]$$

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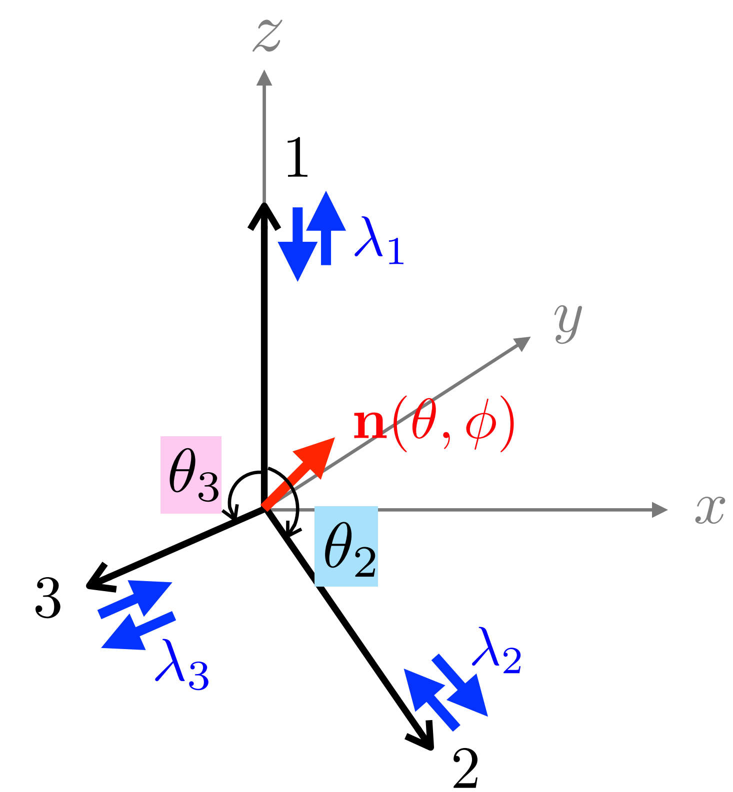
✤ **1** is **not entangled** with **2** and **3** in any way:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$$



[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

$$= \left[c e^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1 \right] \otimes \frac{1}{\sqrt{2}} \left[d|-\,-\rangle_{23} - d^* |+\,+\rangle_{23} \right]$$

bi-separable

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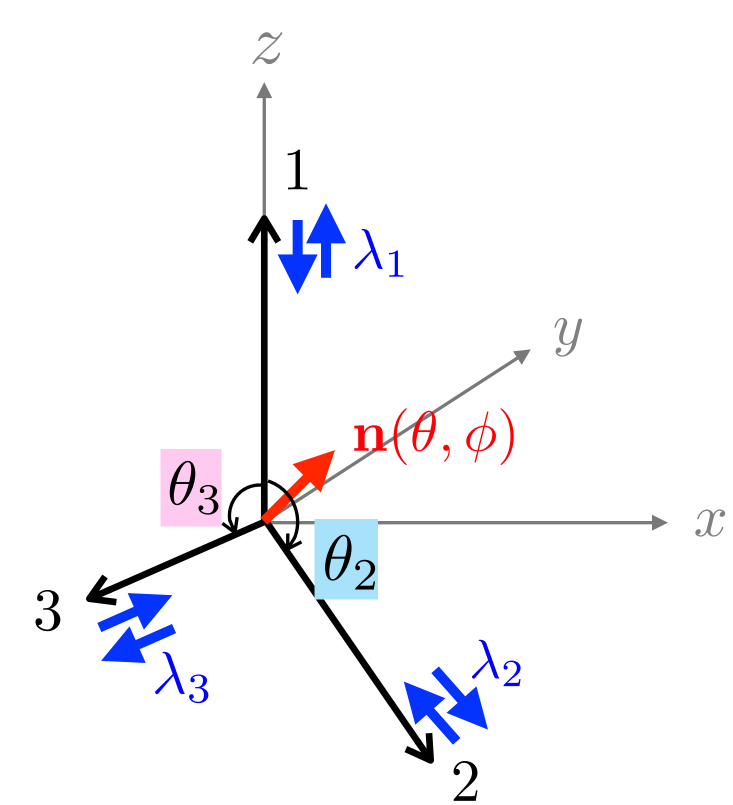
✤ **1** is **not entangled** with **2** and **3** in any way:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$$

✤ **2** and **3** are **maximally entangled**

$$\mathcal{C}_{23} = 1$$



[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

$$= \left[c e^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1 \right] \otimes \frac{1}{\sqrt{2}} \left[d | - - \rangle_{23} - d^* | + + \rangle_{23} \right]$$

bi-separable

$$\Rightarrow F_3 = 0$$

✤ **1** is **not entangled** with **2** and **3** in any way:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$$

✤ **2** and **3** are **maximally entangled**

$$\mathcal{C}_{23} = 1$$

✤ Due to **monogamy**, **2** and **3** are **maximally entangled** with the rest

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 1$$

Monogamy

$$\mathcal{C}^2_{2(13)} \geq \underbrace{\mathcal{C}^2_{12}}_{0} + \underbrace{\mathcal{C}^2_{23}}_{1}$$

$$\mathcal{C}^2_{3(12)} \geq \underbrace{\mathcal{C}^2_{13}}_{0} + \underbrace{\mathcal{C}^2_{23}}_{1}$$

[KS, M.Spannowsky 2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

[KS, M.Spannowsky
2310.01477]

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$$\Rightarrow \quad |\Psi\rangle = M_{LL}|-+-\rangle + M_{LR}|--+\rangle + M_{RL}|++-\rangle + M_{RR}|+-+\rangle$$

[KS, M.Spannowsky
2310.01477]

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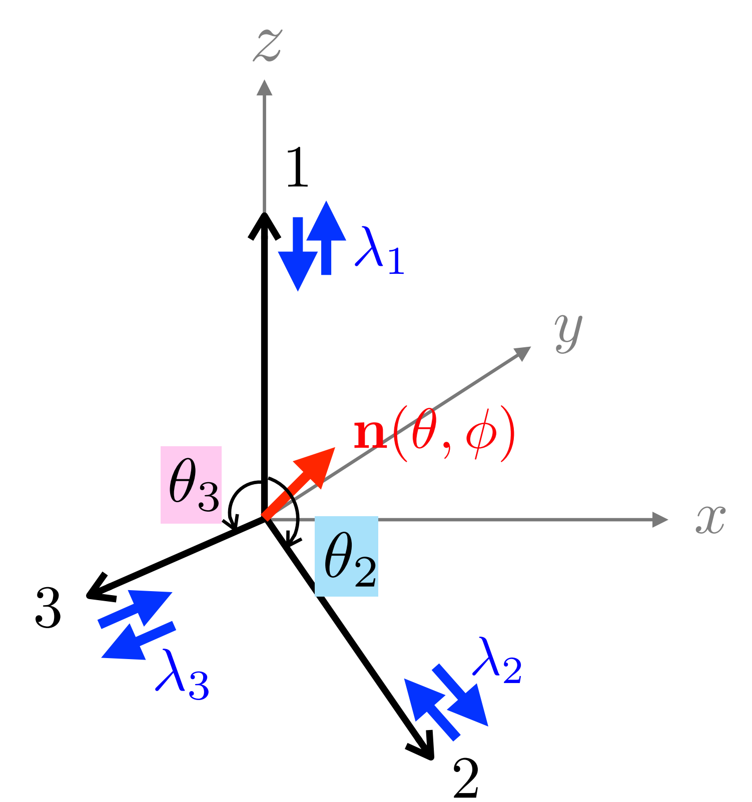
$$\propto \; c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}}\right] |-+-\rangle \; + \; c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}}\right] |--+\rangle$$

$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$



[KS, M.Spannowsky 2310.01477]

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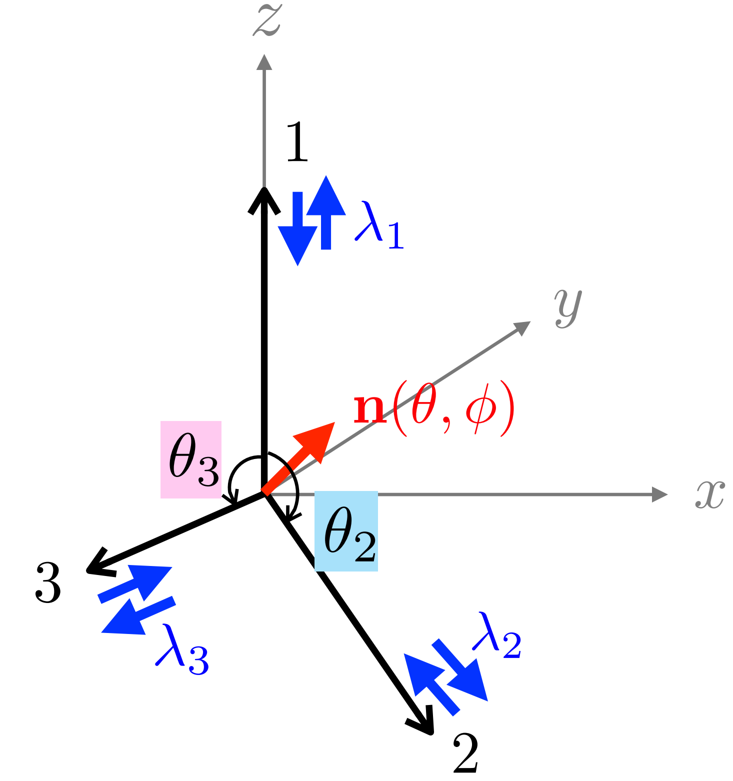
✤ Individual 2-party entanglement:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0 \,, \quad \mathcal{C}_{23} = 2|M_{LL}M^*_{LR} + M_{RL}M^*_{RR}|$$



[KS, M.Spannowsky 2310.01477]

Vector

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$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle + c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$

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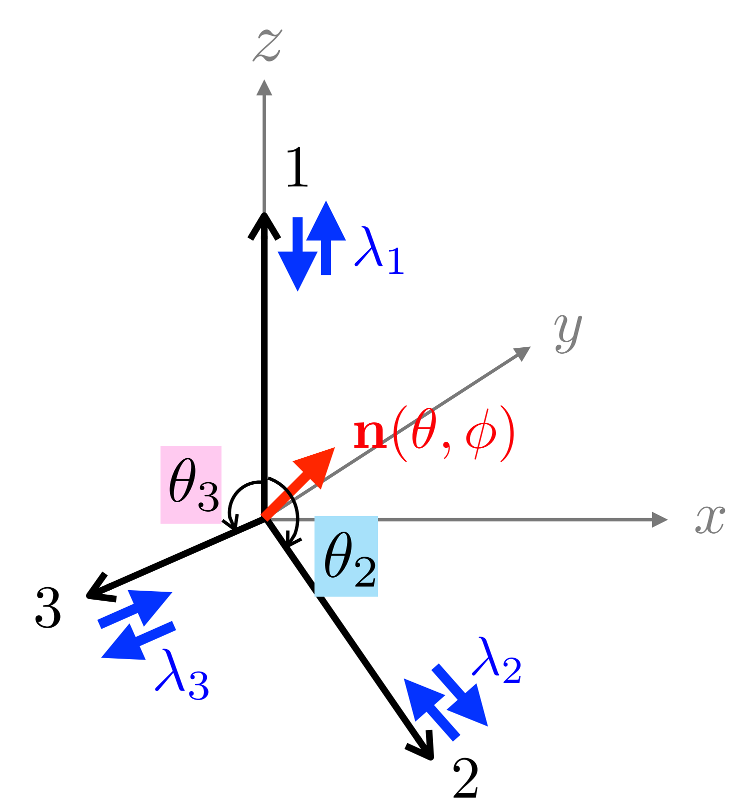
$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M^*_{LR} + M_{RL}M^*_{RR}|$$

✤ one-to-other entanglement:

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{\left(|M_{LL}|^2 + |M_{RL}|^2\right)\left(|M_{LR}|^2 + |M_{RR}|^2\right)}$$

$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big|$$



Vector

[KS, M.Spannowsky 2310.01477]

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$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle + c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$

✤ **Individual 2-party entanglement:**

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M^*_{LR} + M_{RL}M^*_{RR}|$$

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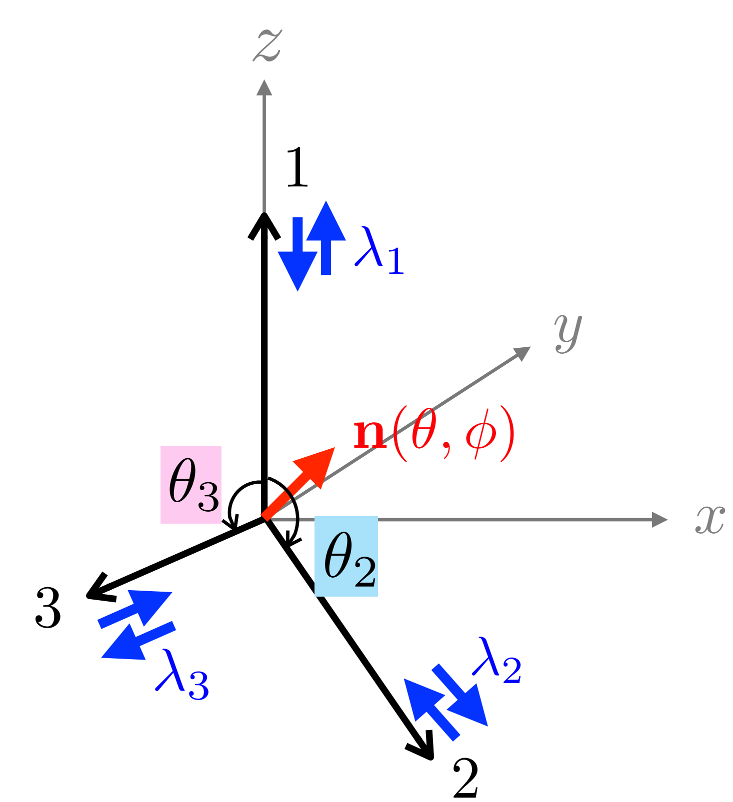
$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big|$$

✤ **Monogamy**

$$M_i \equiv \mathcal{C}^2_{i(jk)} - [\mathcal{C}^2_{ij} + \mathcal{C}^2_{ik}] \quad \Rightarrow \quad M_1 = M_2 = M_3 = \mathcal{C}^2_{1(23)} \geq 0$$



[KS, M.Spannowsky 2310.01477]

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$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} \right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} \right] |+-+\rangle$$

- **Individual 2-party entanglement:**

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M_{LR}^* + M_{RL}M_{RR}^*| \quad \longleftarrow \text{vanish if } d_L d_R = 0$$

- **one-to-other entanglement:**

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{(|M_{LL}|^2 + |M_{RL}|^2)(|M_{LR}|^2 + |M_{RR}|^2)} \quad \longleftarrow \text{vanish if } c_L c_R = d_L d_R = 0$$

$$\mathcal{C}_{1(23)} = 2\left|M_{RR}M_{LL} - M_{LR}M_{RL}\right| \quad \longleftarrow \text{vanish if } c_L c_R d_L d_R = 0$$

- **Monogamy**

$$M_i \equiv \mathcal{C}_{i(jk)}^2 - [\mathcal{C}_{ij}^2 + \mathcal{C}_{ik}^2] \;\Rightarrow\; M_1 = M_2 = M_3 = \mathcal{C}_{1(23)}^2 \geq 0$$

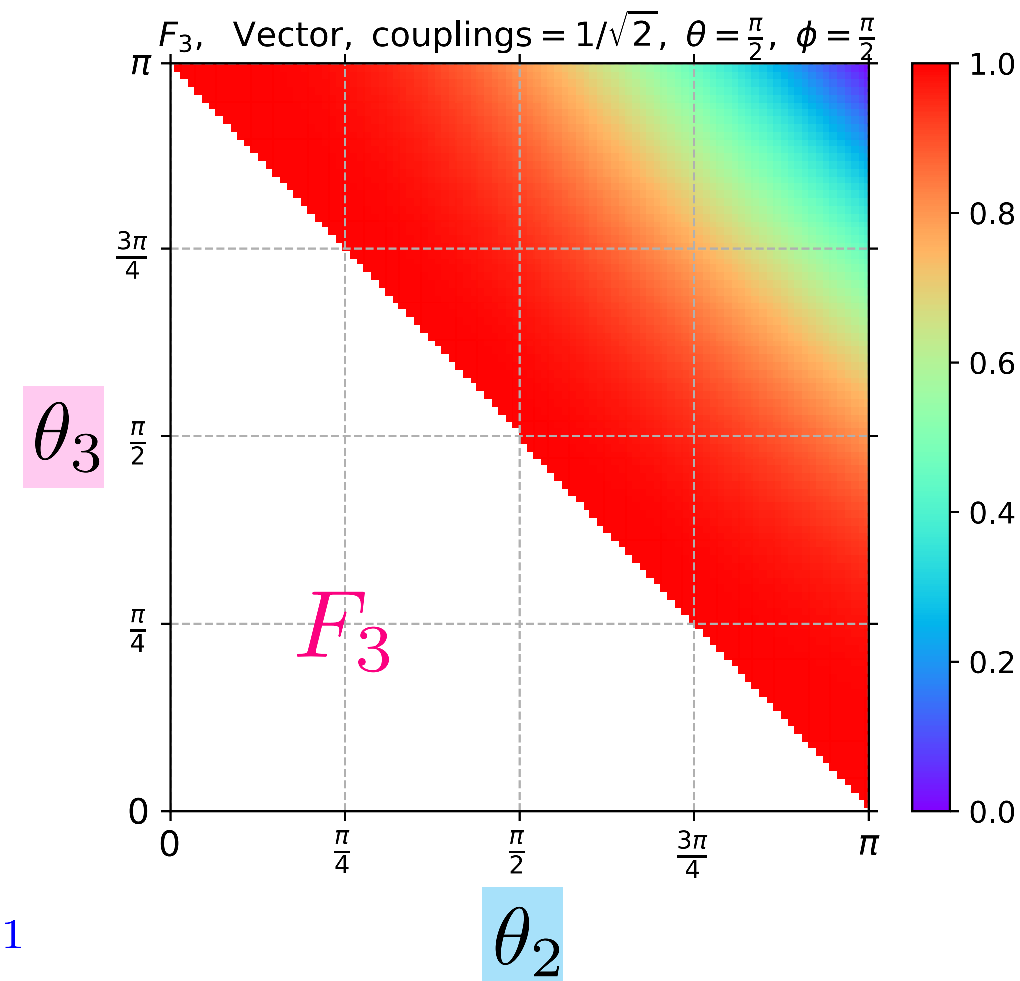
$\Rightarrow$ **All** entanglements **vanish** for **weak decays**

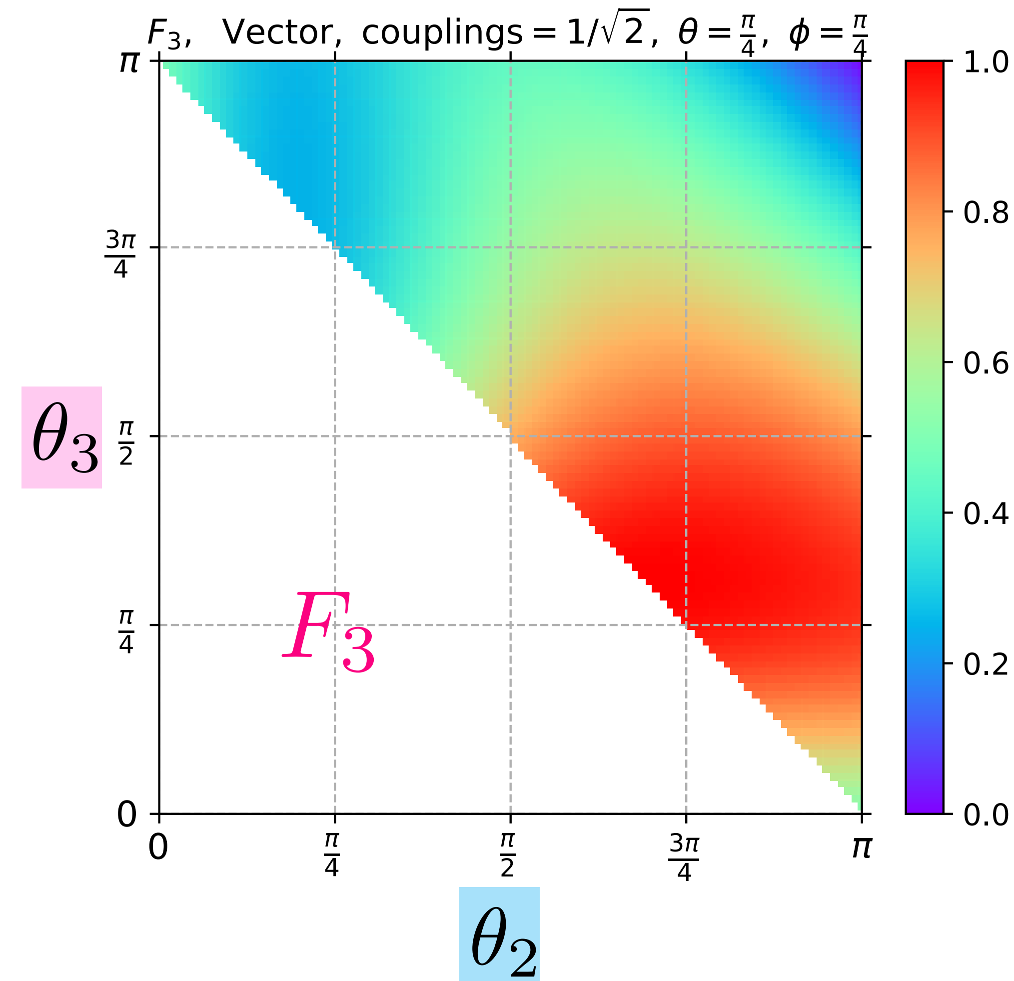
$$c_R = d_R = 0$$

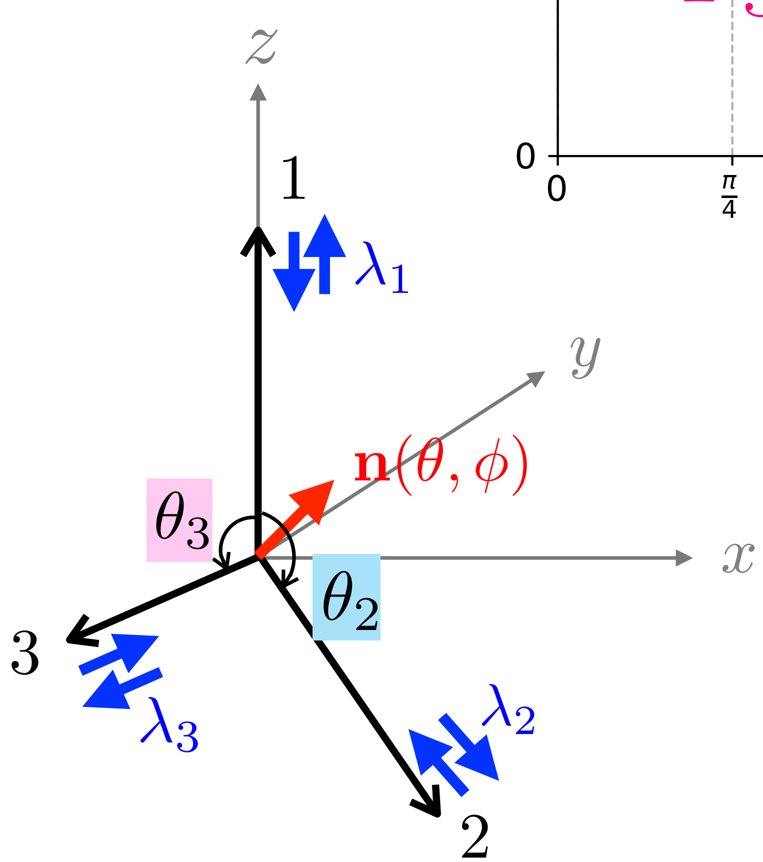
F_3 for Vector

$\mathbf{n} = \mathbf{e}_y$

$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$







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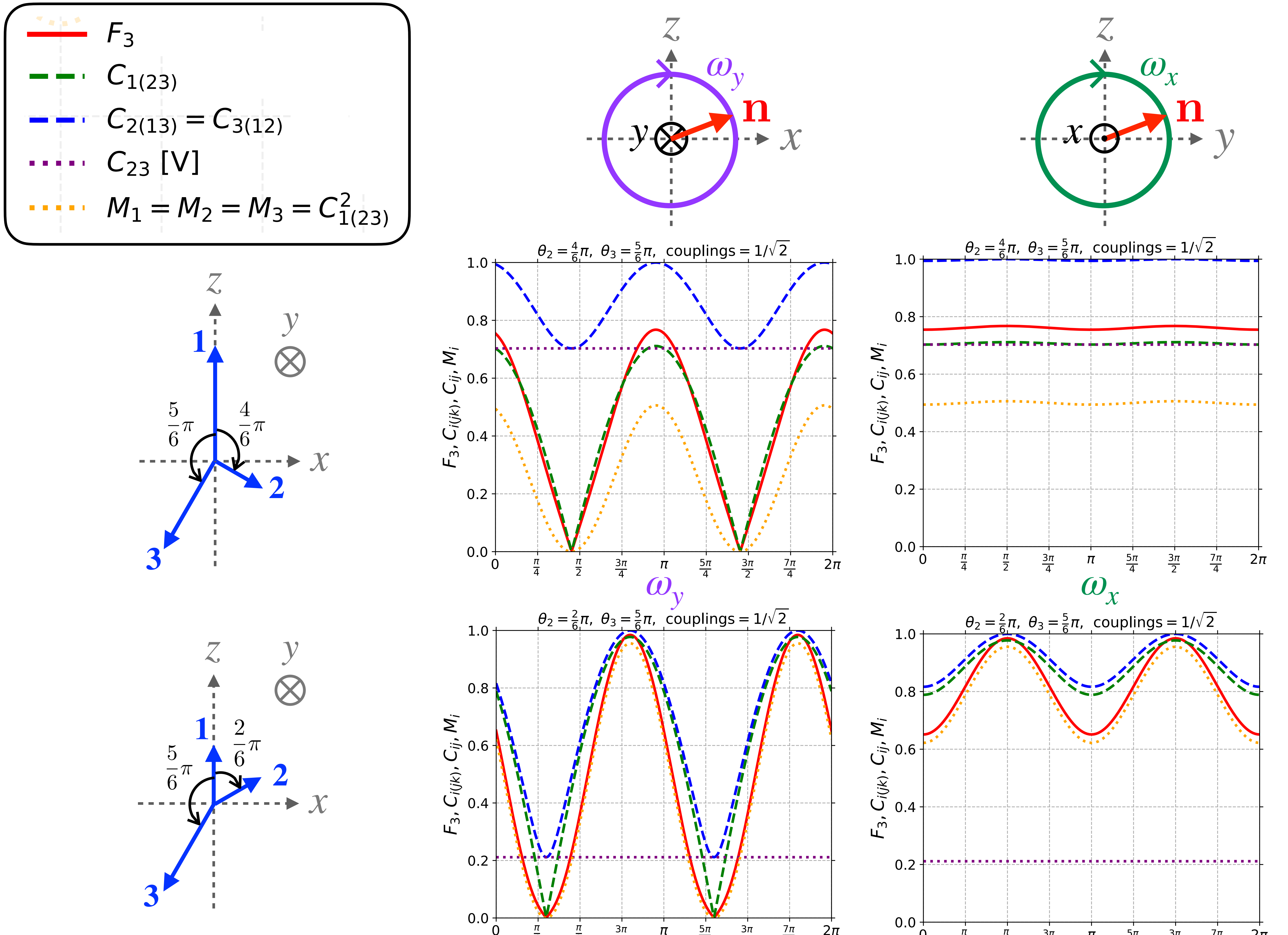

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[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
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[KS, M.Spannowsky 2310.01477]

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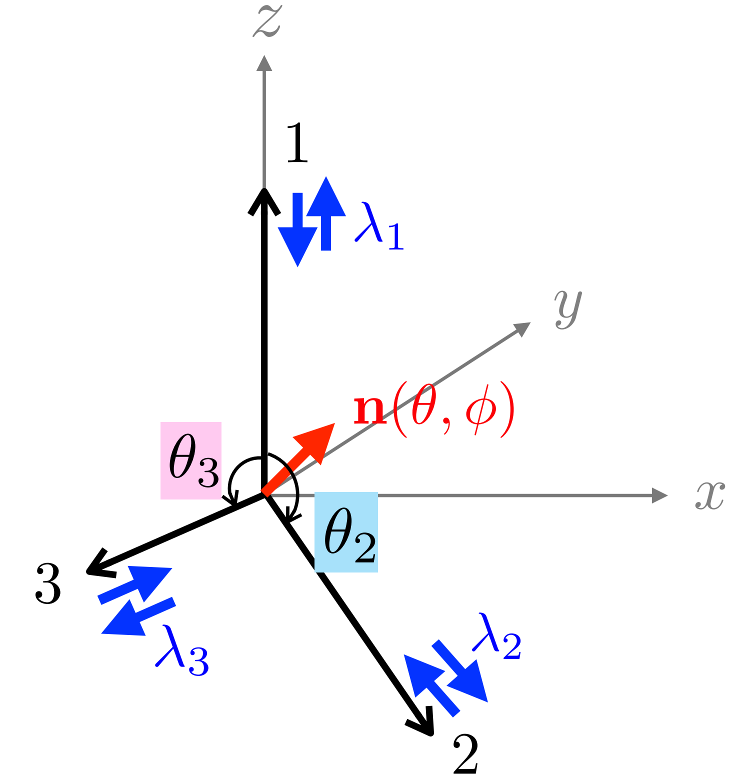
$$|\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto c^* d^* [2e^{i\phi}s_{\frac{\theta}{2}}s_{\frac{\theta_2}{2}}s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}}s_{\frac{\theta_3-\theta_2}{2}}]|+++\rangle + cd[-e^{i\phi}s_{\frac{\theta}{2}}s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}}s_{\frac{\theta_2}{2}}s_{\frac{\theta_3}{2}}]|---\rangle$$



[KS, M.Spannowsky 2310.01477]

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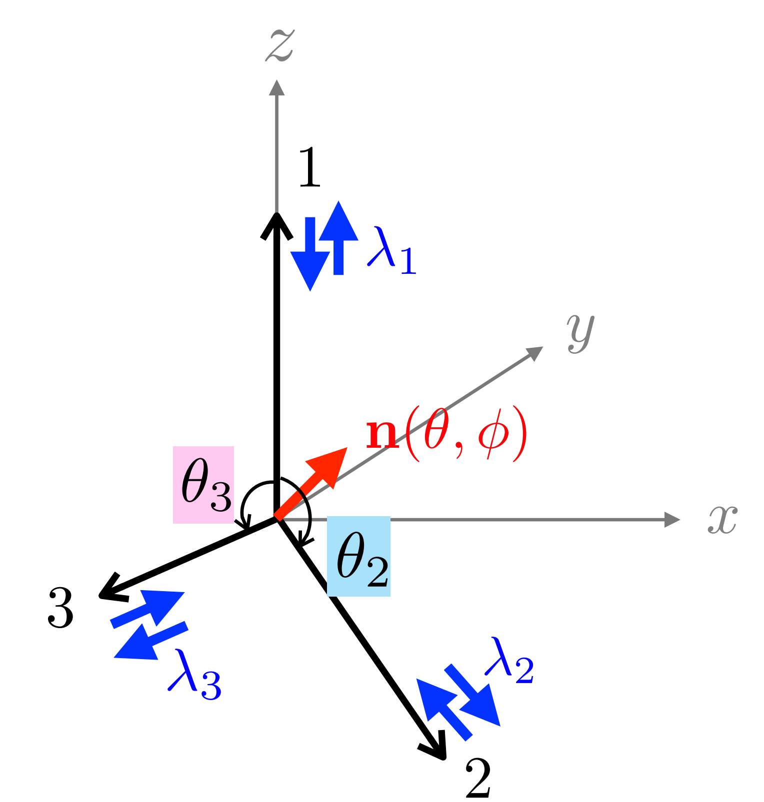
✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$



[KS, M.Spannowsky 2310.01477]

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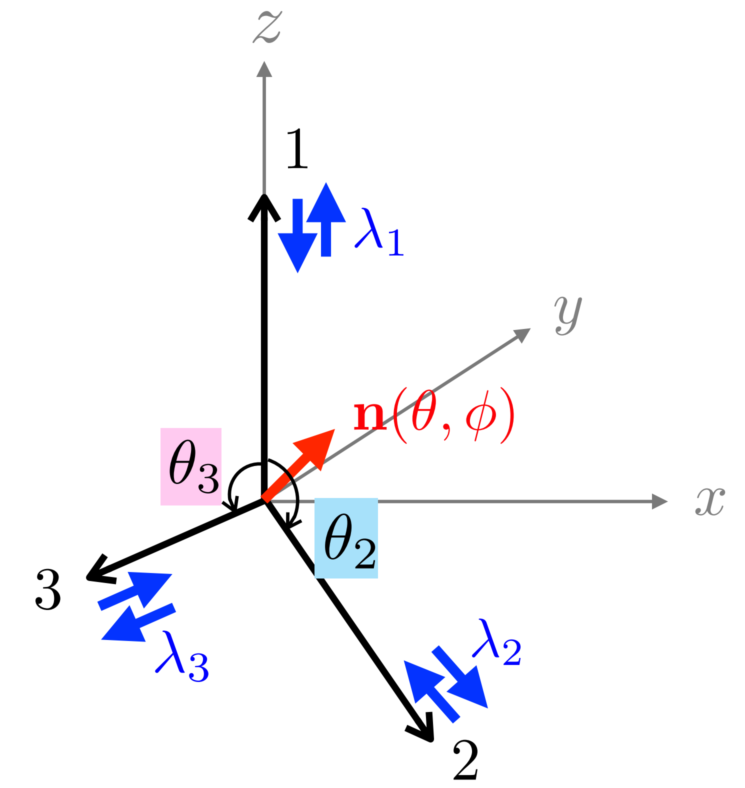
✤ **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0$$



[KS, M.Spannowsky 2310.01477]

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[KS, M.Spannowsky 2310.01477]

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✤ one-to-other entanglements are **universal**:

$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L|$$

$$F_3 = 4|M_R M_L|^2$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\text{int}} = [\bar\psi_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar\psi_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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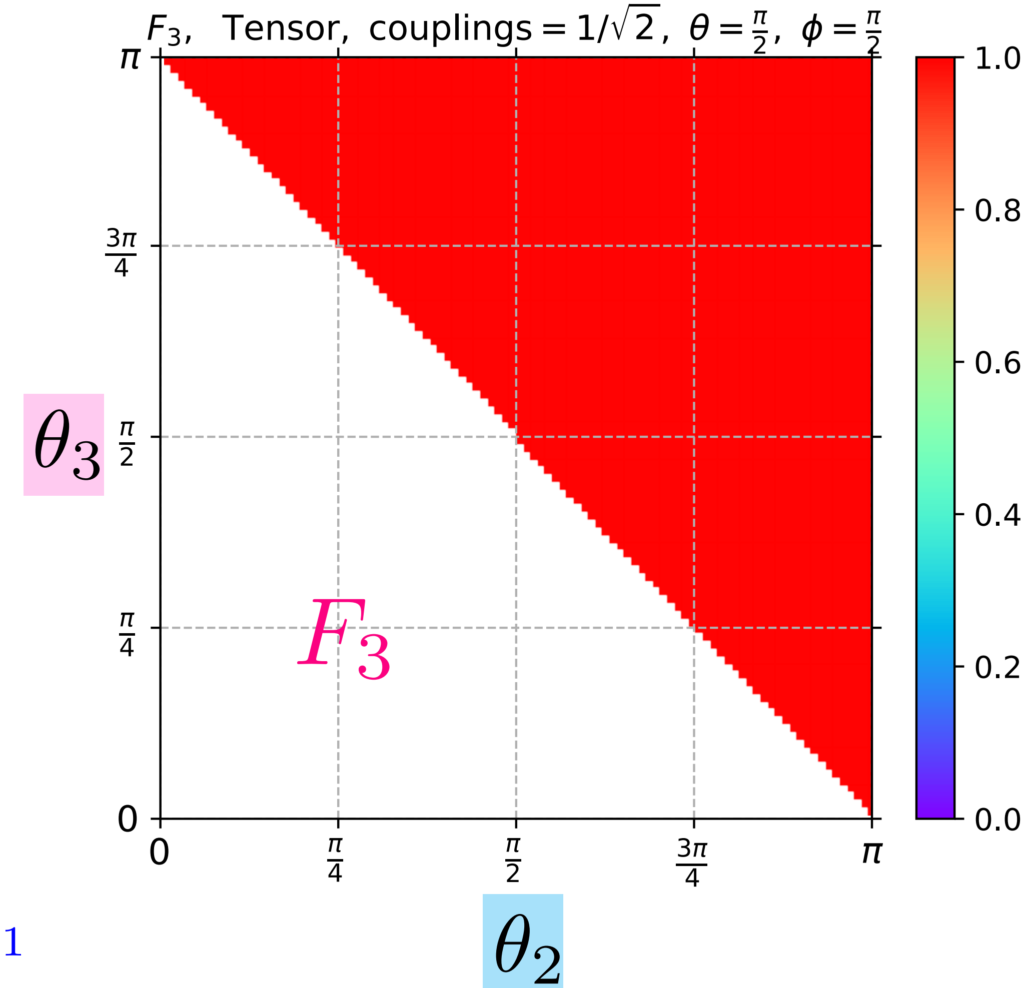
independent of the coupling structure (CP phases)

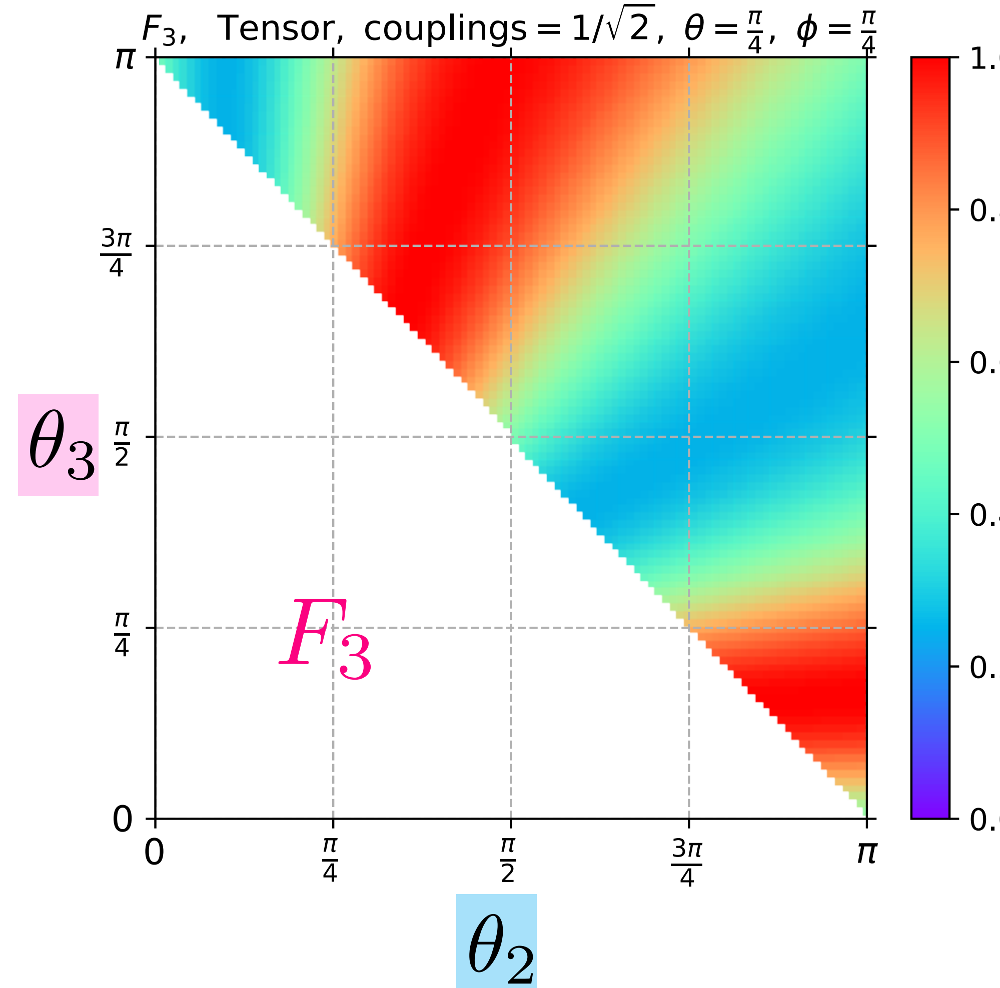
$$\omega_1, \omega_2$$

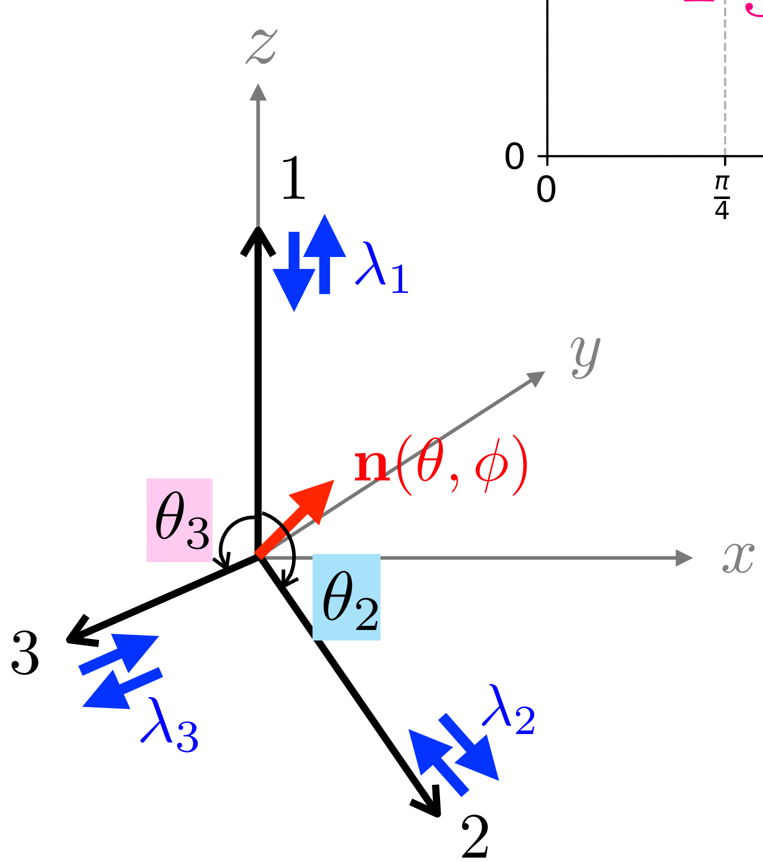
F_3 for Tensor

$$\mathbf{n} = \mathbf{e}_y$$



$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$





[KS, M.Spannowsky 2310.01477]

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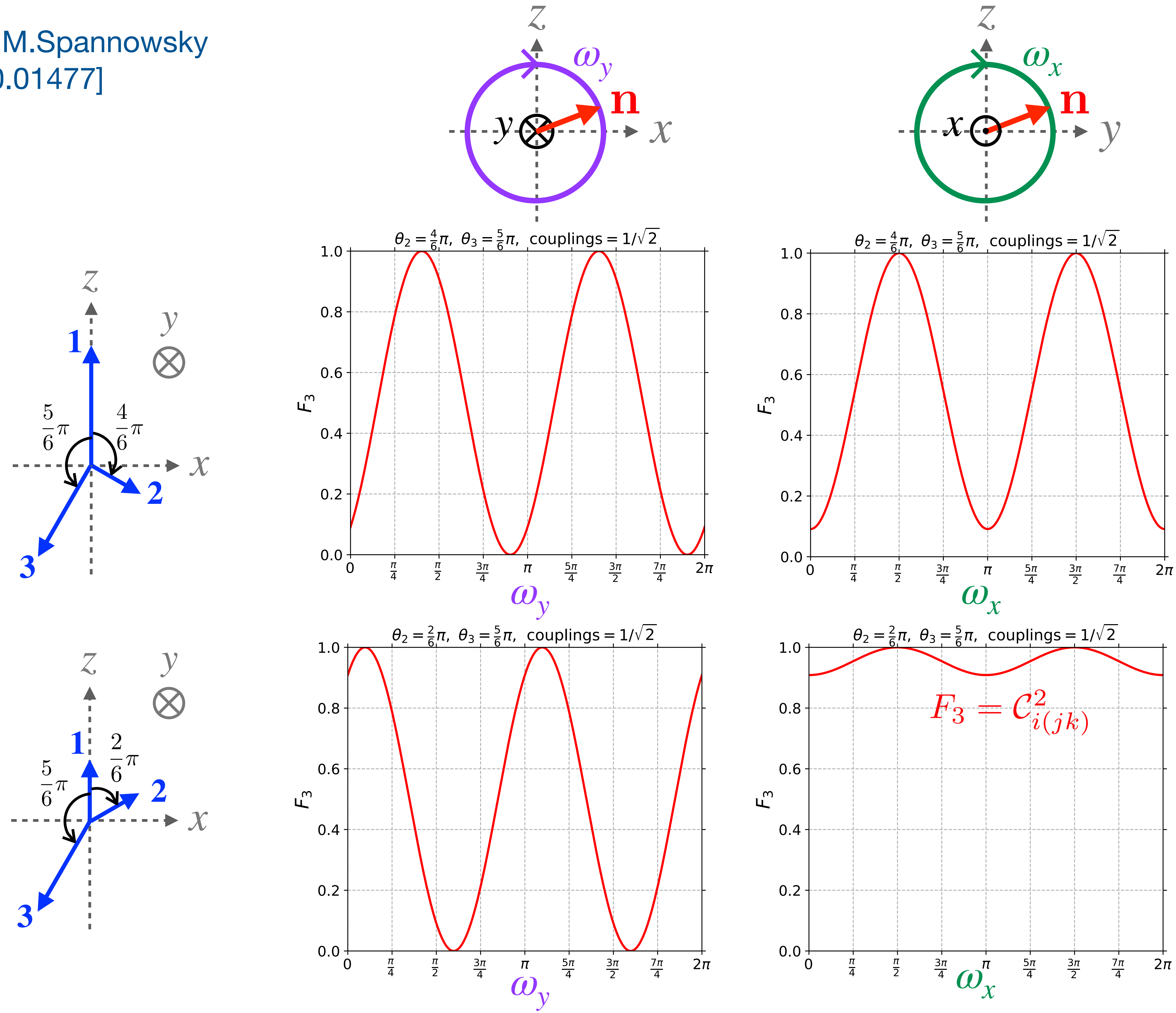

Discussion

What to do with it?

✤ **measure**/study 3-body entanglements **experimentally** *e.g.* in **hadron decays**

✤ look for **theories** to **maximise/minimise** the **entanglement**

Carena, Low, Wagner, Xiao [2307.08112]

$$\boxed{\text{general 2HD}} \xrightarrow[\text{in } \Phi_a\Phi_b \to \Phi_c\Phi_d]{\text{minimising entanglement}} \boxed{\text{maximum SO(8) sym.}} \longrightarrow \text{SM-like Higgs}$$

Future directions:

✤ Effect of **masses** in the final particles

✤ More **spin structures**: $SFFV,\ VVFF,\ SFVF_{3/2},\ SVVT \cdots$

✤ 3-body **non-locality** [Mermin '90, Svetlichny '87]

Mermin ineq: $\langle \mathcal{B}_{\rm M} \rangle_{\rm LR} \le 2 \quad \langle \mathcal{B}_{\rm M} \rangle_{\rm QM} \le 4 \qquad \mathcal{B}_{\rm M} = abc' + ab'c + a'bc - a'b'c'$

Horodecki, KS, Spannowsky, *in progress*

Thank you for listening!

Norway grants

NATIONAL SCIENCE CENTRE
POLAND

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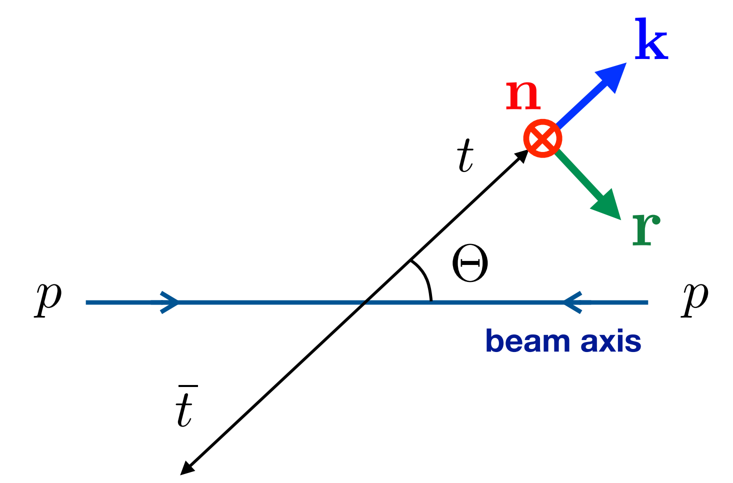
Understanding the Early Universe:
interplay of theory and collider experiments

Joint research project between the University of Warsaw & University of Bergen

[Afik, Nova (2021, 2022)]

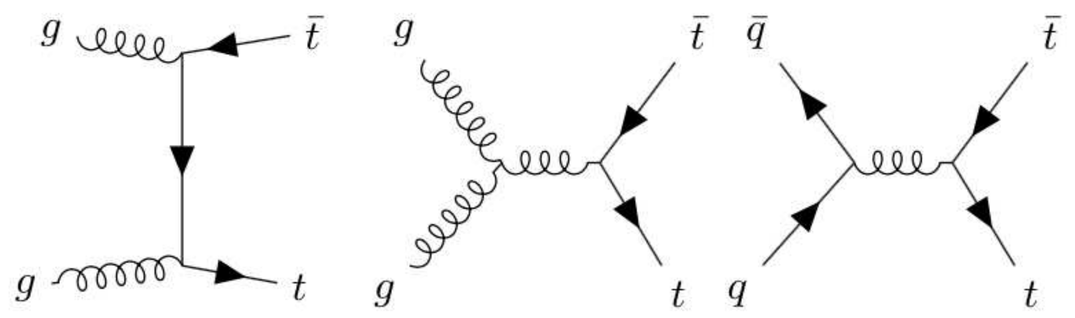
$pp \to t\bar{t}$ @ LHC



- At the rest frame of $t\bar{t}$, the kinematics is determined by:

  Θ : the angle between t and the beam line ($0 \le \Theta \le \pi/2$)

  $M_{t\bar{t}}$: the inv. mass of $t\bar{t}$



- gg and $q\bar{q}$ initial states contribute stochastically
  $\Rightarrow$ the $t\bar{t}$ spin state is necessarily **mixed**

$$\rho(M_{t\bar{t}}, \Theta) = \sum_{I=gg,q\bar{q}} w_I(M_{t\bar{t}}, \Theta) \cdot \rho^I(M_{t\bar{t}}, \Theta)$$

$$w_I(M_{t\bar{t}}, \Theta) = \frac{L_I(M_{t\bar{t}})\tilde{A}^I(M_{t\bar{t}}, \Theta)}{\sum_J L_J(M_{t\bar{t}})\tilde{A}^J(M_{t\bar{t}}, \Theta)}$$

$\tilde{A}^I(M_{t\bar{t}}, \Theta)$: partonic differential x-section

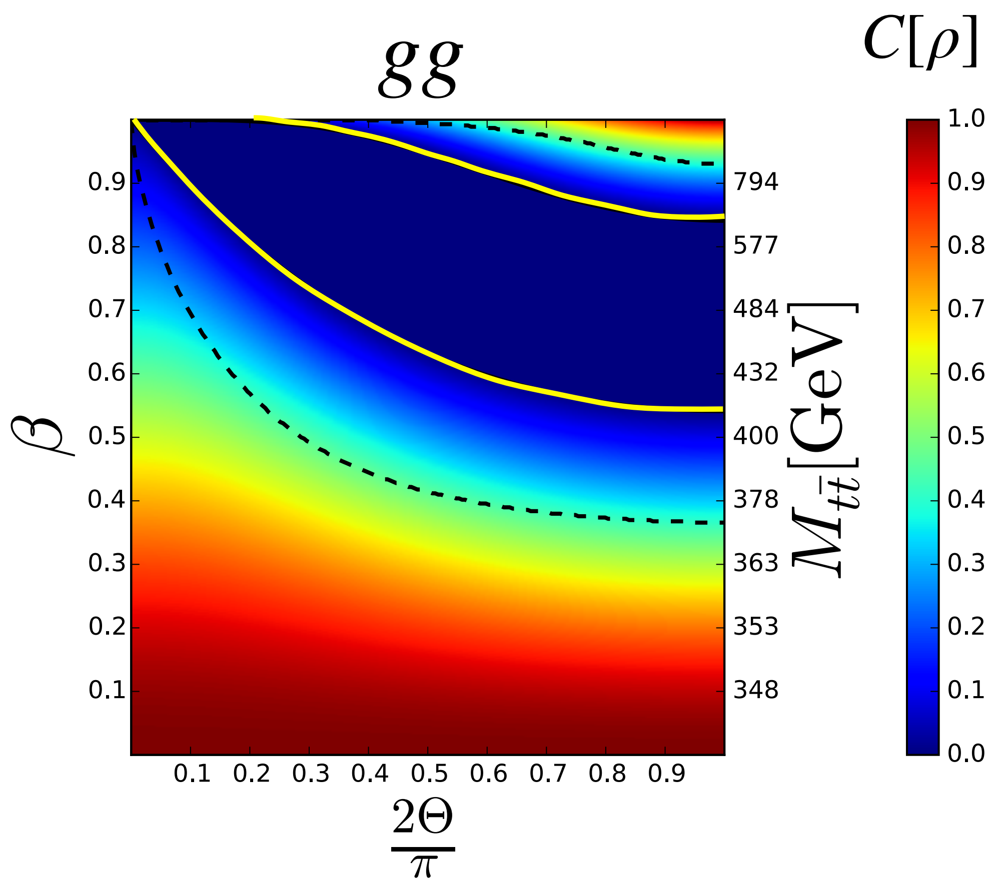
$L_I(M_{t\bar{t}})$: luminosity function

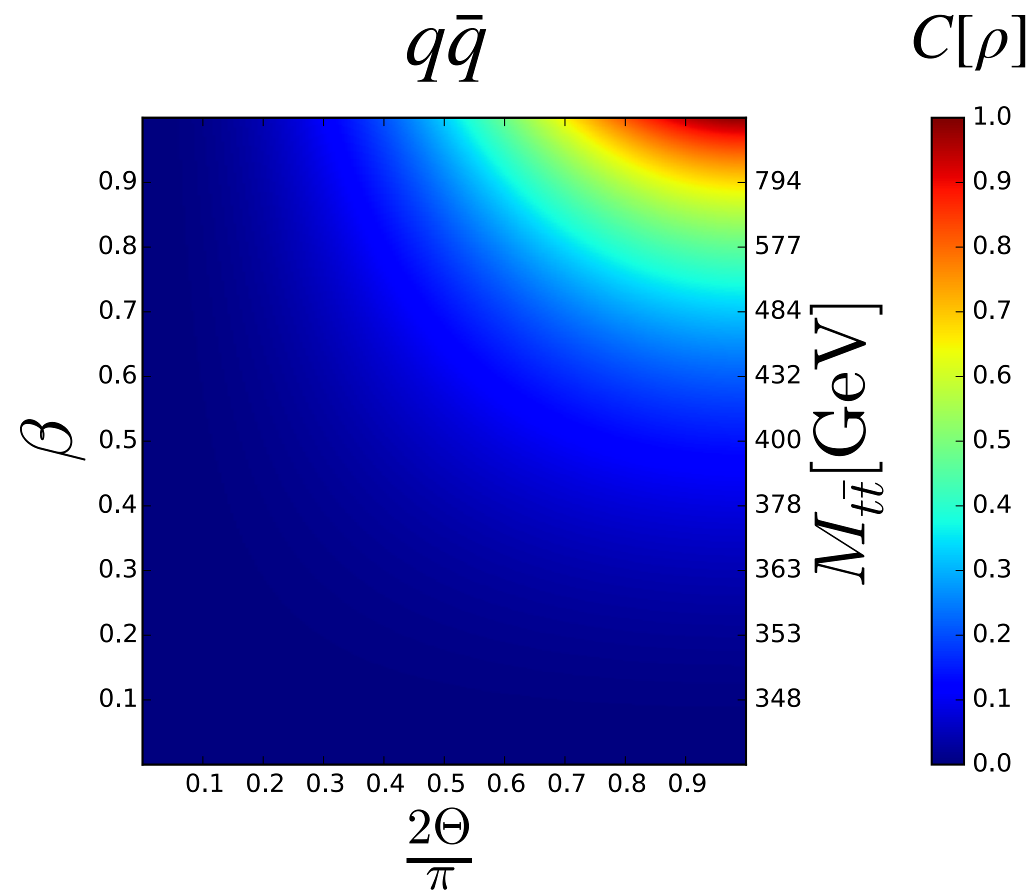
Yellow solid: $C[\rho] \geq 0$

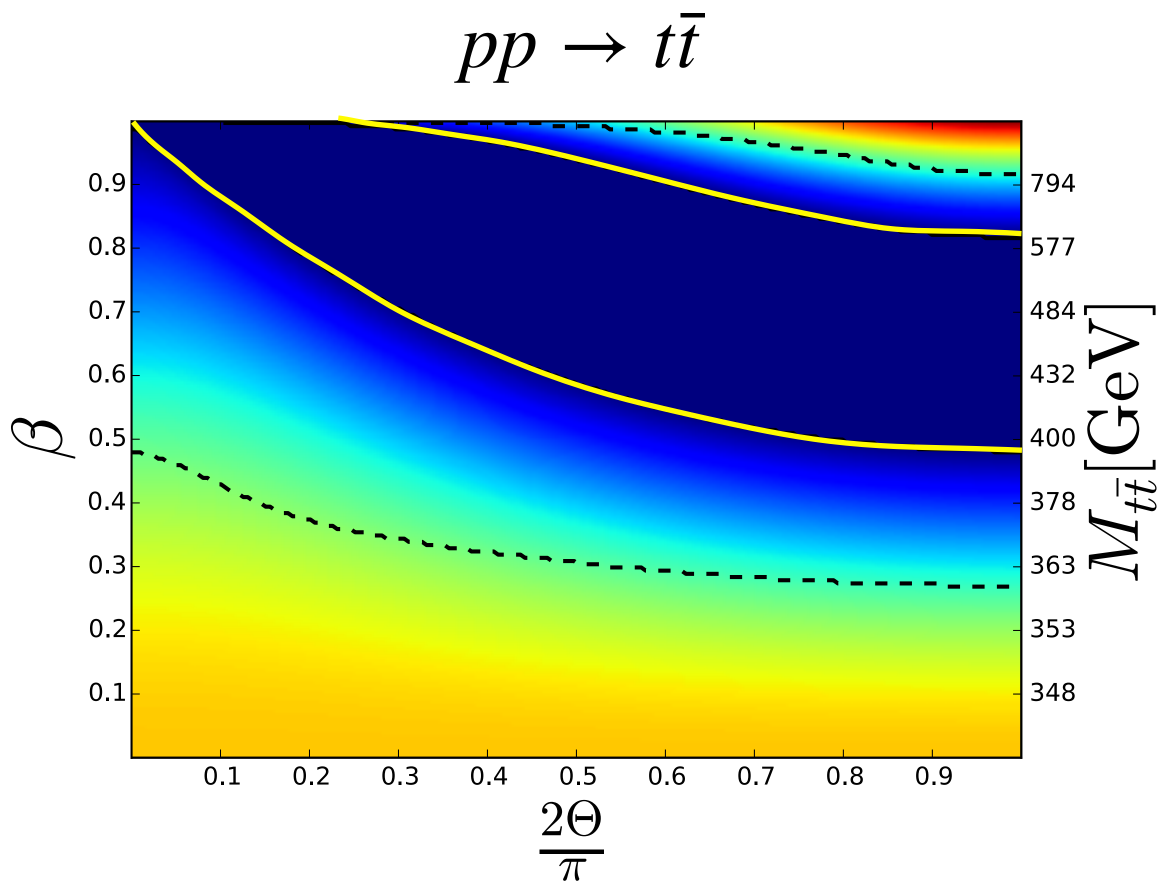
Black dashed: $S_{\text{CHSH}} \geq 2$

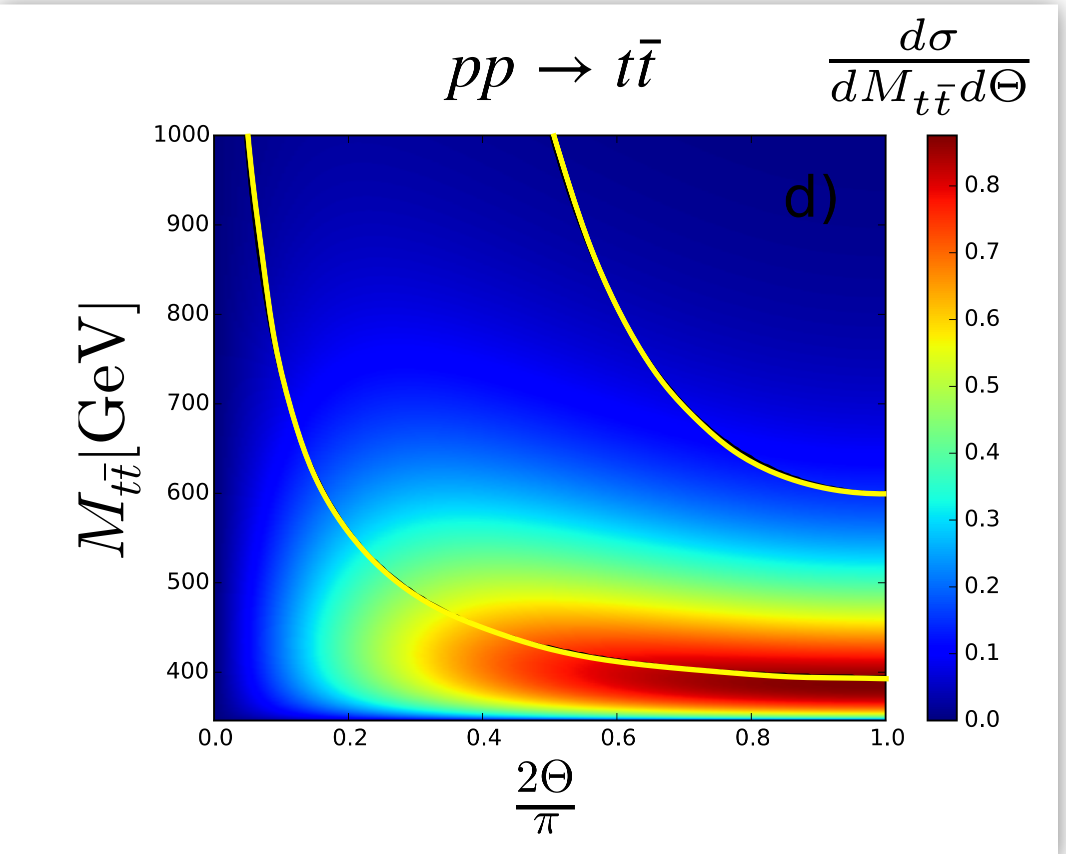
Concurrence

[Afik, Nova (2021, 2022)]

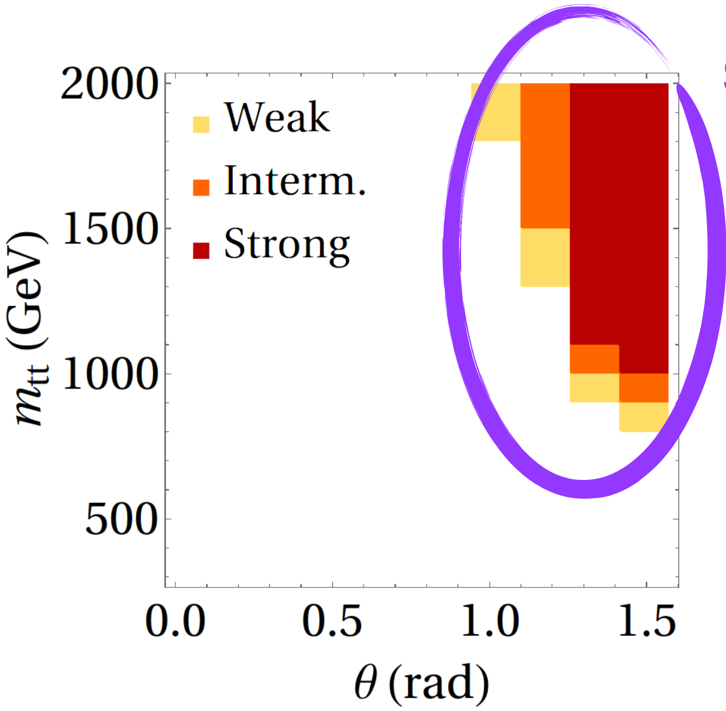








MC-sim: di-leptonic decay, $pp \to t\bar{t} \to (b\ell^+\nu)(\bar{b}\ell^-\bar{\nu})$



selecting events here

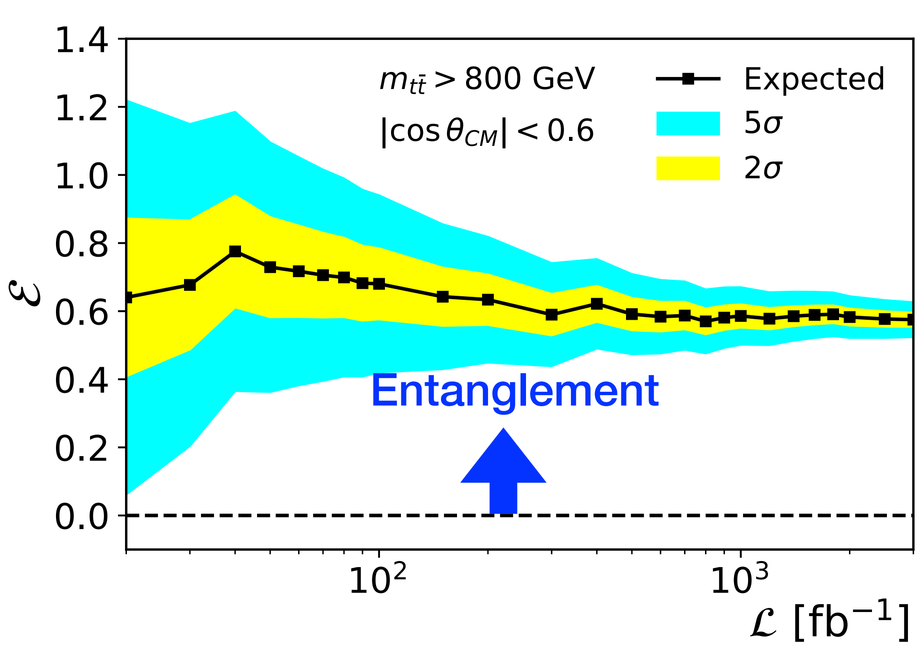
HL-LHC $(L = 3\,\mathrm{ab}^{-1})$ [Severi, Boschi, Maltoni, Sioli (2022)]

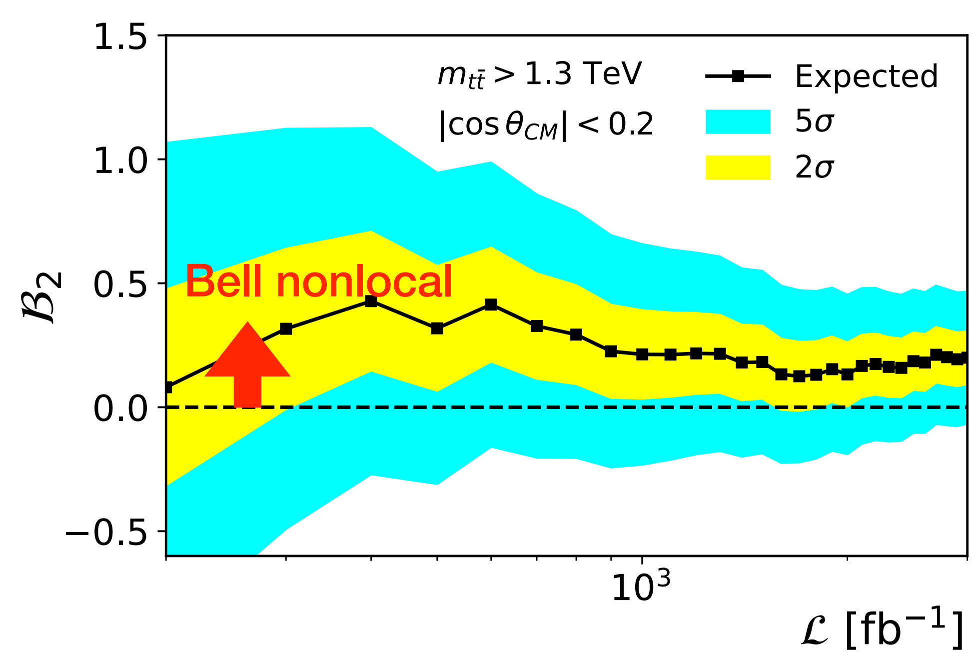
$$|C_{kk} + C_{rr}| - C_{nn} = 1.36 \pm 0.07 > 1 \Rightarrow \text{Entanglement} \gg 5\sigma$$

$$\sqrt{2}S_{\mathrm{CHSH}}/2 = 2.20 \pm 0.1 > 2 \Rightarrow \text{Bell nonlocality} \sim 1.8\sigma$$

MC-sim: semi-leptonic decay, $pp \to t\bar{t} \to (b\ell\nu)(b\underbrace{jj}_{\text{boosted top-tagging}})$

[Dong, Goncalves, Kong, Navarro (2023)]

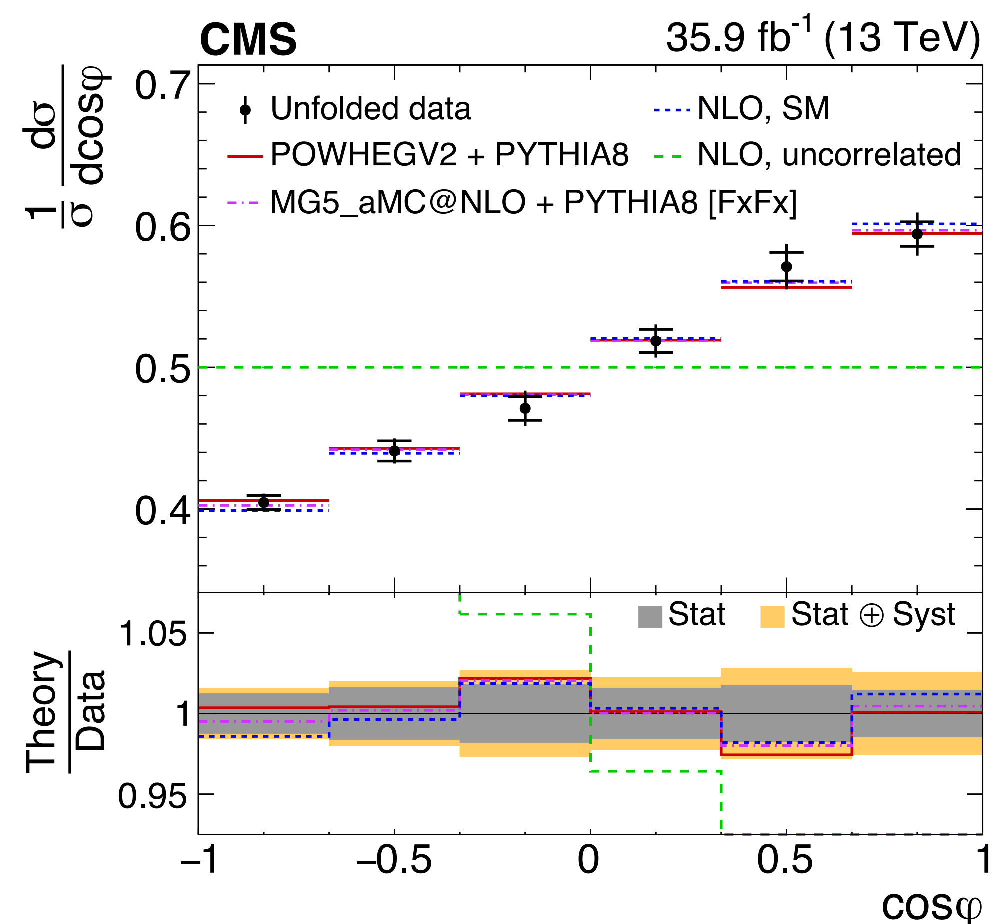




Entanglement in CMS

Recently, D has been measure by CMS in di-leptonic $pp \to t\bar{t}$

[Phys. Rev. D 100, 072002]



$$D \equiv \frac{\mathrm{Tr}[C]}{3} < -\frac{1}{3}$$ entanglement

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1 - D\cos\varphi}{2}$$

CMS result:

$$D = -0.237 \pm 0.011 > -\frac{1}{3}$$ entanglement is not detected

To see the entanglement, selecting certain kinematical regions is crucial.

A dedicated analysis is needed.

$H \to WW^*, ZZ^*$

[Barr (2022)]
[Aguilar-Saavedra ,Bernal, Casas, Moreno (2022)]
[Aguilar-Saavedra (2023)]
[Fabbrichesi, Floreanini, Gabrielli, Marzola (2023)]

- Conceptually less clear since one particle is off-shell.

  $\Rightarrow$ virtual particle with mass shifted: $m_{V*} = f \cdot m_V \quad (0 < f < 1)$

- two qutrits (rather than qubits)

- the final state is pure:

$$\left.\begin{array}{l} |\Psi_{VV*}\rangle \simeq |+-\rangle - \beta|00\rangle + |-+\rangle \\ \\ \beta = 1 + \dfrac{m_H^2 - (1+f)^2 m_V^2}{2 f m_V^2} \sim 1 \end{array}\right\} \Rightarrow \text{(almost) maximally entangled}$$

CGLMP Qutrit inequality

CGLMP function

[Collins Gisin Linden Massar Popescu (2002)]

$$I_3 \equiv P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1)$$
$$-P(A_1 = B_1 - 1) - P(B_1 = A_2) - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1)$$

*) $P(A_i = B_j + k)$ is the probability that A_i and B_j are differ by k mod 3

$$I_3 \le \begin{cases} 2 & \text{Local theories} \\ 1 + \sqrt{11/3} \simeq 2.9149 & \text{Quantum Mechanics} \end{cases}$$

Quantum state tomography

- It is convenient to reconstruct the density matrix from the kinematics, then analysis entanglement and nonlocality

- density matrix is 9 x 9 Hermitian matrix with unit trace. It can be expanded by two sets of Gell-Mann matrices and $8+8+64=80$ real parameters (9^2-1)

$$\rho = \frac{1}{9}(\mathbf{1}\otimes\mathbf{1}) + \frac{1}{3}\sum_{i=1}^{8} a_i(\lambda_i\otimes\mathbf{1}) + \frac{1}{3}\sum_{j=1}^{8} b_j(\mathbf{1}\otimes\lambda_j) + \sum_{i,j=1}^{8} c_{ij}(\lambda_i\otimes\lambda_j)$$

- real parameters a_i, b_j, c_{ij} can be reconstructed from the directions of two charged leptons, $\mathbf{n}_1$ and $\mathbf{n}_2$, using the eight **Wigner P functions**, Φ_i^P

$$a_i = \frac{1}{2}\left\langle \Phi_i^P \mathbf{n_1} \right\rangle_{\text{av}} \qquad b_i = \frac{1}{2}\left\langle \Phi_i^P \mathbf{n_2} \right\rangle_{\text{av}} \qquad c_{ij} = \frac{1}{4}\left\langle (\Phi_i^P \mathbf{n_1})(\Phi_j^P \mathbf{n_2}) \right\rangle_{\text{av}}$$

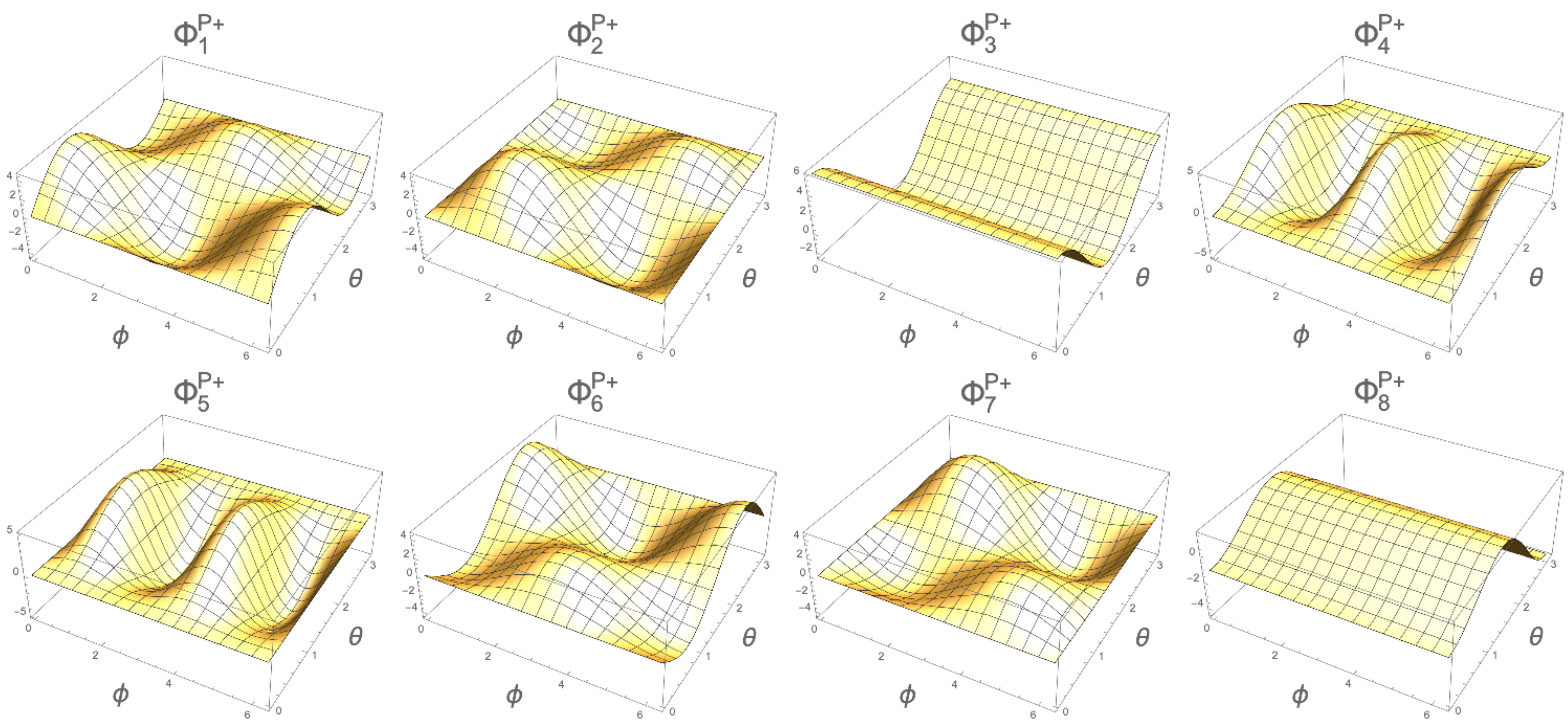
[Ashby-Pickering, Barr, Wierzchucka (2022)]

Quantum state tomography

- Wigner functions for $W^\pm \to \ell^\pm \nu$

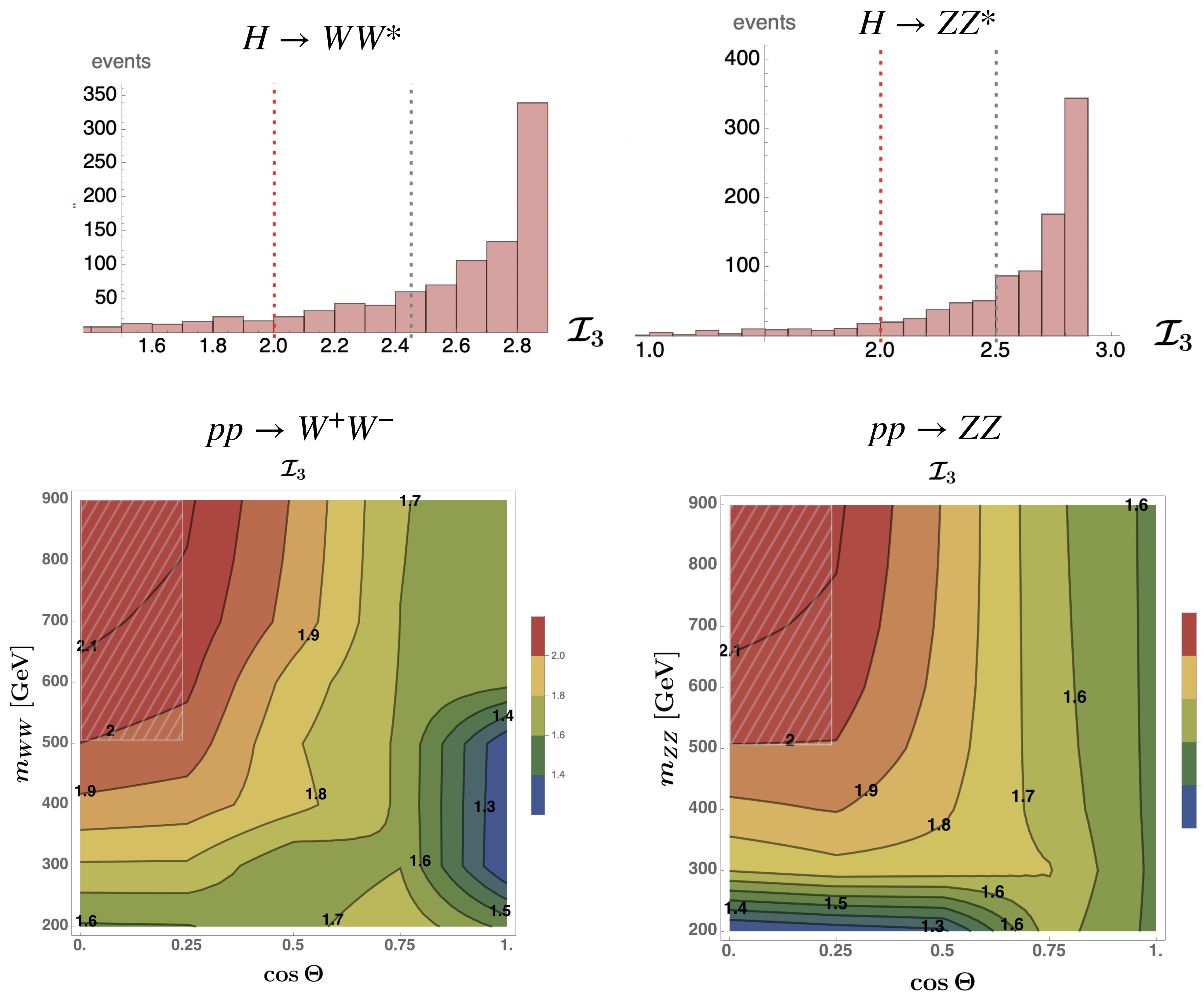
[Ashby-Pickering, Barr, Wierzchucka (2022)]

$$\Phi_1^{P\pm} = \sqrt{2}(5\cos\theta \pm 1)\sin\theta\cos\phi$$
$$\Phi_2^{P\pm} = \sqrt{2}(5\cos\theta \pm 1)\sin\theta\sin\phi$$
$$\Phi_3^{P\pm} = \frac{1}{4}(\pm 4\cos\theta + 15\cos 2\theta + 5)$$
$$\Phi_4^{P\pm} = 5\sin^2\theta\cos 2\phi$$
$$\Phi_5^{P\pm} = 5\sin^2\theta\sin 2\phi$$
$$\Phi_6^{P\pm} = \sqrt{2}(\pm 1 - 5\cos\theta)\sin\theta\cos\phi$$
$$\Phi_7^{P\pm} = \sqrt{2}(\pm 1 - 5\cos\theta)\sin\theta\sin\phi$$
$$\Phi_8^{P\pm} = \frac{1}{4\sqrt{3}}(\pm 12\cos\theta - 15\cos 2\theta - 5)$$



CGLMP function I_3 in optimal measurement axes

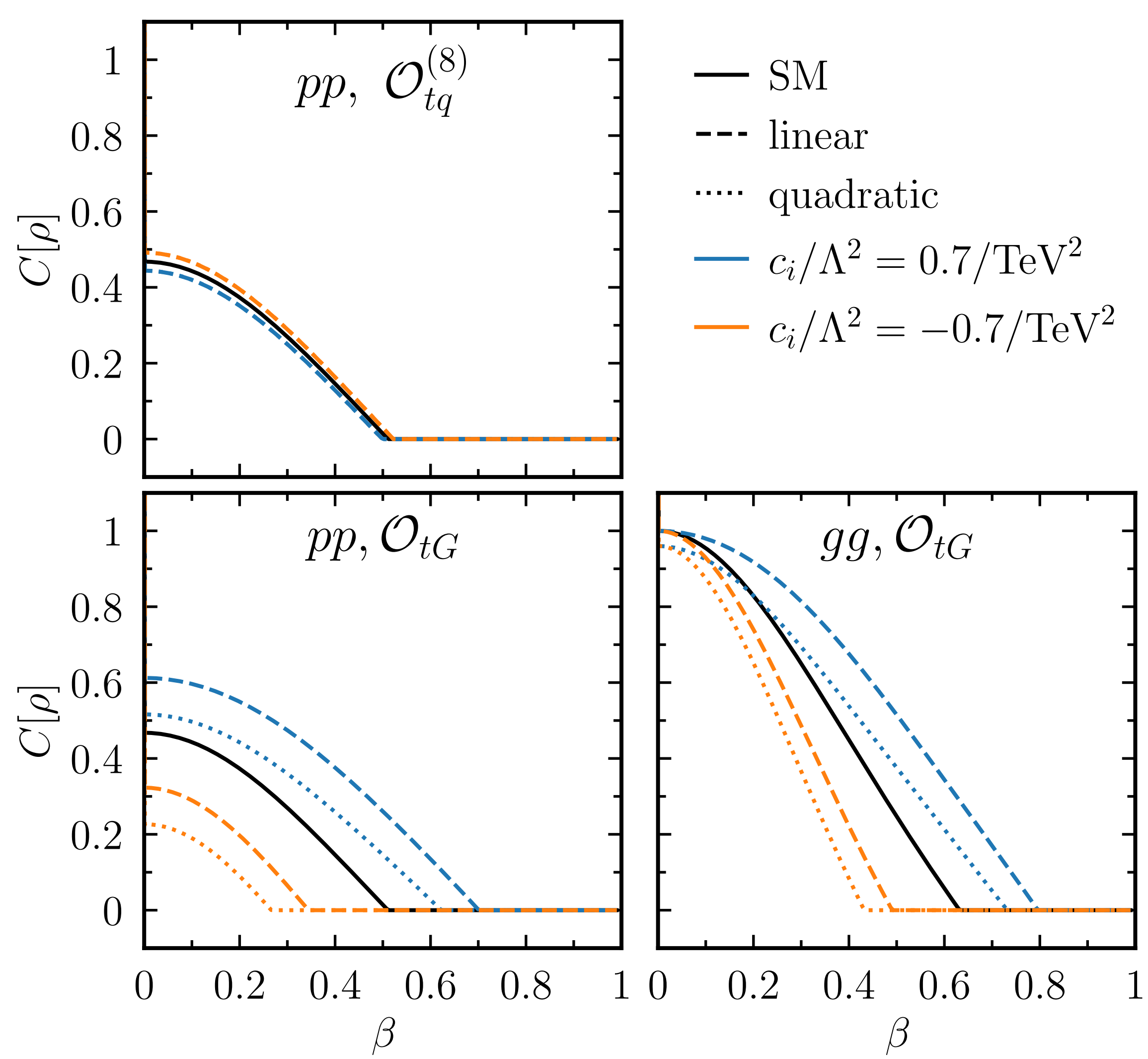
[Fabbrichesi, Floreanini, Gabrielli, Marzola (2023)]



Effect of BSM $pp \to t\bar{t}$

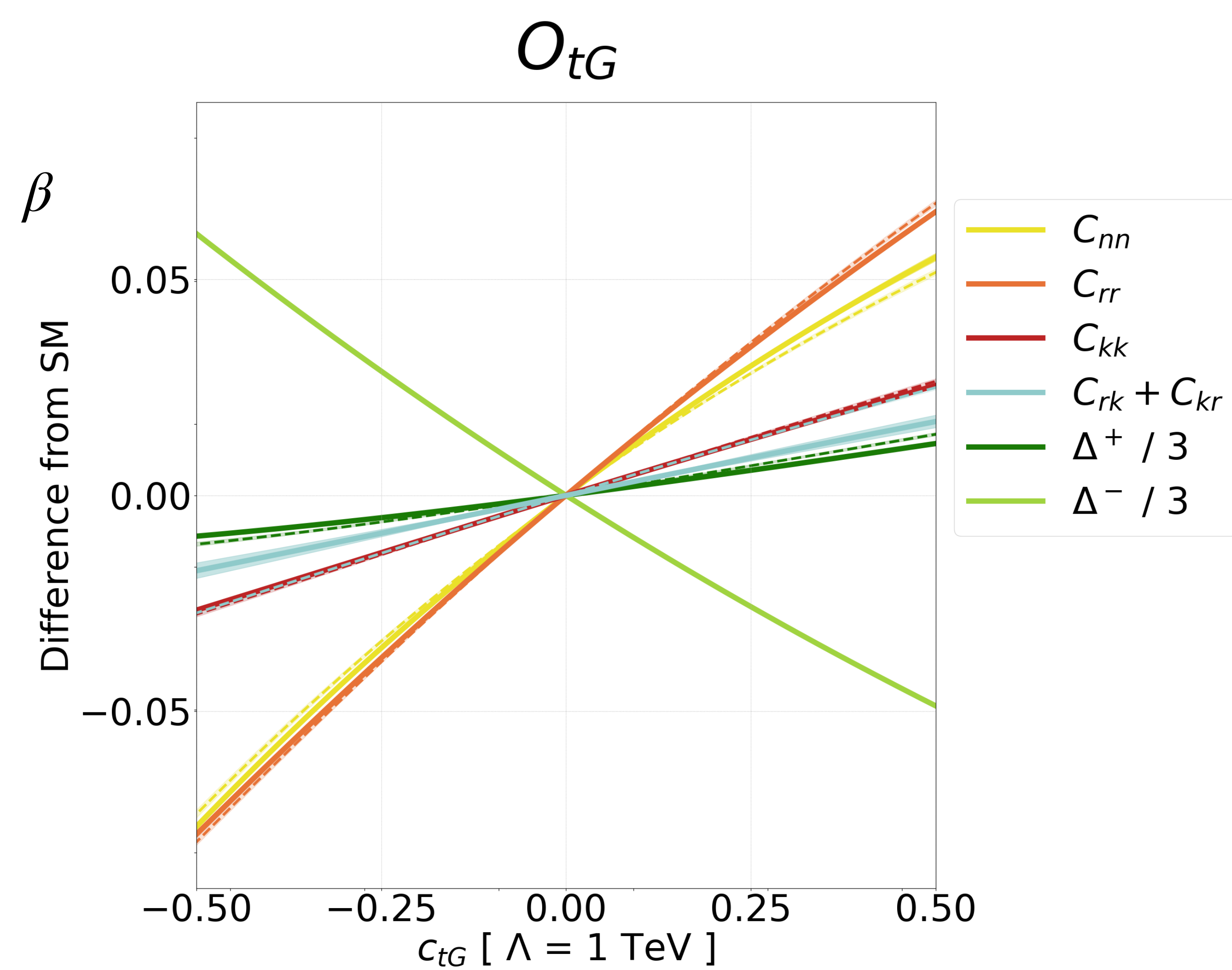
$$\mathcal{O}_{tG} = g_S\, \overline{Q} T_A \tilde{\varphi} \sigma^{\mu\nu} t\, G^A_{\mu\nu} \qquad \mathcal{O}^8_{tq} = \sum_{\mathrm{f}=1}^{2} (\bar{q}_{\mathrm{f}} \gamma_\mu T_A q_{\mathrm{f}})(\bar{t} \gamma^\mu T^A t)$$

[Aoude Madge Maltoni Mantani (2022)]



β: top velocity in the $t\bar{t}$ rest frame

[Severi Vryonidou (2023)]



$$\Delta^{\pm} = \pm(C_{kk} + C_{rr}) - C_{nn} - 1$$

Local **R**eal Hidden Variable theories:

Mermin ineq:

$$P(abc|XYZ) = \sum_\lambda q_\lambda P_\lambda(a|X) P_\lambda(b|Y) P_\lambda(c|Z) \longrightarrow \langle \mathcal{B}_{\mathrm{M}} \rangle_{\mathrm{LR}} \leq 2 \quad \langle \mathcal{B}_{\mathrm{M}} \rangle_{\mathrm{QM}} \leq 4$$

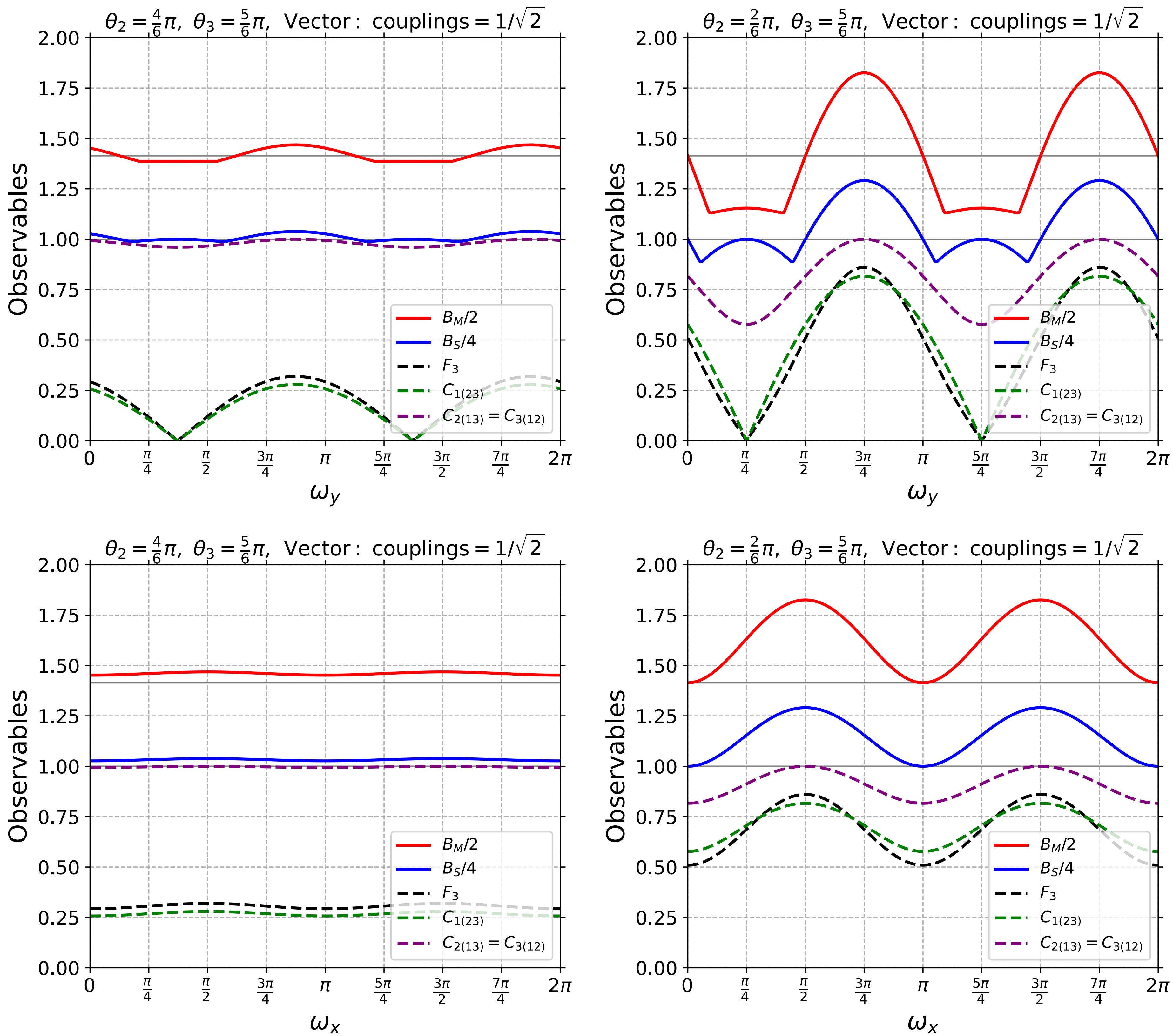
Hybrid (**L**ocal-Nonlocal) **R**eal theories:

$$P(abc|XYZ) = \sum_\lambda q_\lambda P_\lambda(ab|XY) P_\lambda(c|Z) + \sum_\mu q_\mu P_\mu(ac|XZ) P_\mu(b|Y) + \sum_v q_v P_v(bc|YZ) P_v(a|X)$$

$$\longrightarrow \langle \mathcal{B}_{\mathrm{S}} \rangle_{\mathrm{HLR}} \leq 4 \quad \langle \mathcal{B}_{\mathrm{S}} \rangle_{\mathrm{QM}} \leq 4\sqrt{2}$$

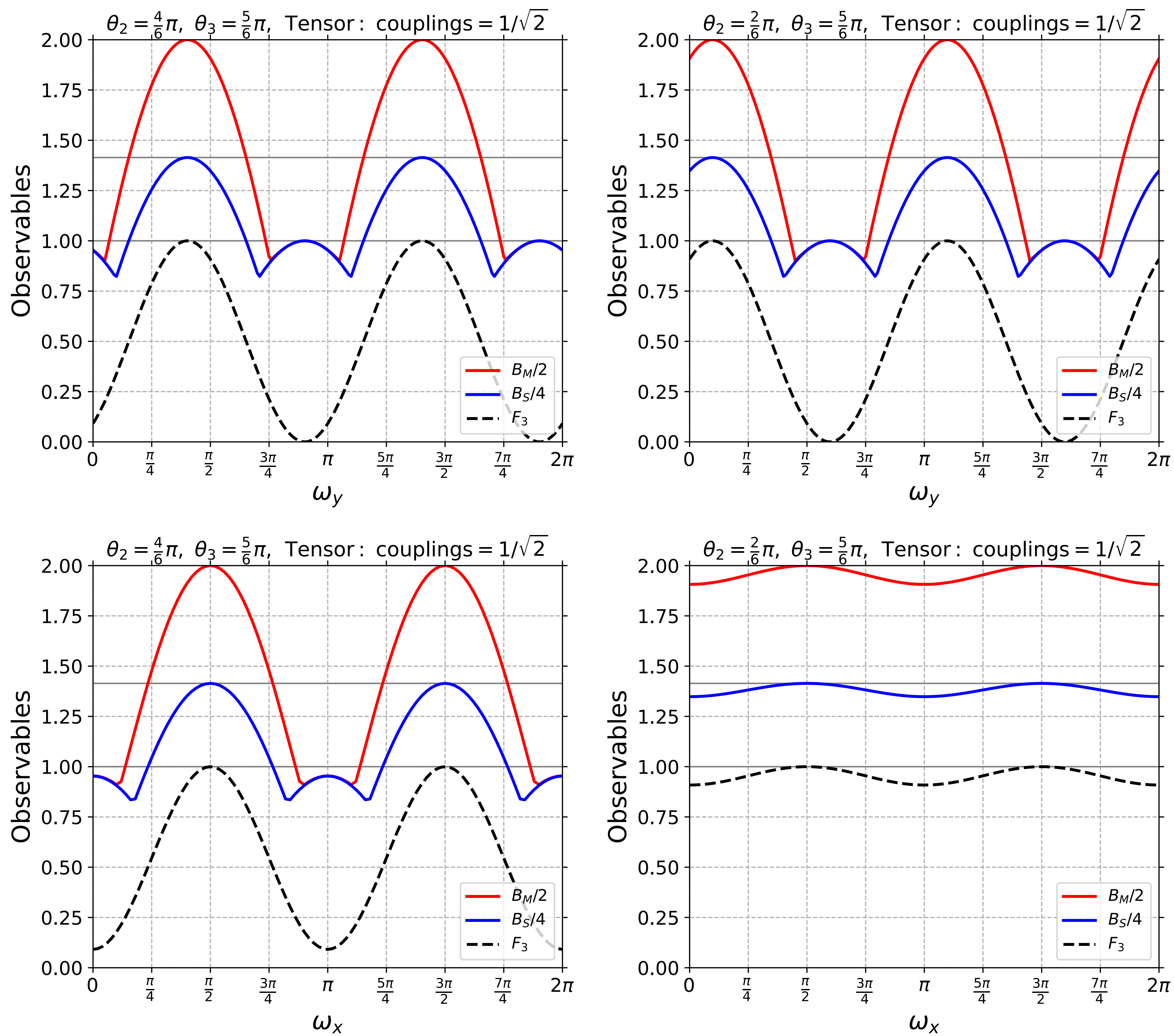
Svetlichny ineq

Nonlocality for Vector



[KS, Spannowsky, Horodecki, *in progress*]

Nonlocality for Tensor



[KS, Spannowsky, Horodecki, *in progress*]