Lattice field theory with Worldlines and Worldsheets

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Der Wissenschaftsfonds.

Why worldlines/worldsheets? And what are they?

Path integrals for quantum field theory

In a (euclidean) Feynman path integral we compute vacuum expectation values as weighted sums over all field configurations.

$$\langle O \rangle \;=\; \frac{1}{Z} \int D[\phi] \; e^{-\frac{1}{\hbar} S[\phi]} \; O[\phi]$$

In the lattice formulation we regularize the Feynman path integral by introducing a finite space time lattice Λ

$$x \in \mathbb{R}^4 \;\to\; x \in \Lambda \subset \mathbb{N}^4 \quad , \quad \phi(x) \;\to\; \phi_x \quad , \quad D[\phi] \;\to\; \prod_{x \in \Lambda} d\phi_x$$

$$S[\phi] \;\to\; \sum_{x \in \Lambda} \Big(m^2 \, |\, \phi_x \,|^2 \; + \lambda \, |\, \phi_x \,|^4 \; - \sum_{\nu} \phi_x^* \, [\, \phi_{x+\hat{\nu}} - 2\phi_x + \phi_{x-\hat{\nu}} \,] \Big)$$

Path integral becomes a very high dimensional integral

$$\langle O \rangle \;=\; \frac{1}{Z} \int D[\phi] \; e^{-S[\phi]} \; O[\phi]$$

Monte Carlo simulation

In a Monte Carlo simulation one generates a finite number of field configurations $\phi^{(j)}, j = 1, 2 \ldots N$ with probability

$$P[\phi^{(j)}] \quad = \quad \frac{1}{Z} e^{-S[\phi^{(j)}]}$$

Vacuum expectation values assume the form of mean values

$$\langle O \rangle \quad = \quad \frac{1}{N} \sum_{j=1}^{N} O[\phi^{(j)}] \; + \; \mathcal{O}(1/\sqrt{N})$$

Complex action problem / sign problem

- In general, lattice field theories with finite chemical potential μ or a topological term have actions $S[\phi]$ with an imaginary part.

$$S[\phi] \;=\; S_R[\phi] \,+\, i\, S_I[\phi]$$

- The Boltzmann factor

$$e^{\,-\,S[\phi]} \;\in\; \mathbb{C}$$

  thus has a complex phase and cannot be used as a probability weight.

- Standard Monte Carlo simulation techniques are not available for a non-perturbative analysis.

  "Complex action problem" or "Sign problem"

- Generic feature of finite density field theories both, on the lattice and in the continuum, for bosonic and fermionic theories.

- In some cases an exact mapping to a worldline/worldsheet representation solves the problem.

What is a worldline/worldsheet representation?

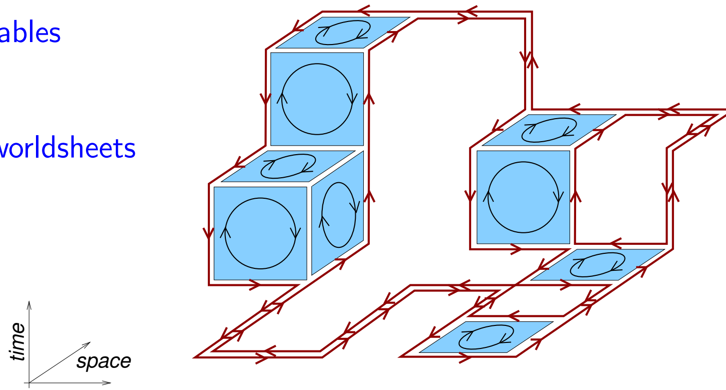
- A worldline/worldsheet representation is a change of variables for the dynamical degrees of freedom.

- The original degrees of freedom, such as $\phi_x \in \mathbb{C}$ or $A_{x,\mu} \in \mathbb{R}$ are replaced by integer-valued occupation numbers

$$\begin{aligned} j_{x,\mu} &\in \mathbb{Z} \quad \text{for matter flux} \\ p_{x,\mu\nu} &\in \mathbb{Z} \quad \text{for gauge flux} \end{aligned}$$

- The worldline/worldsheet variables are subject to constraints.

- Configurations of worldlines/worldsheets come with weight factors.



How does one get to worldlines and worldsheets?

Worldline representation for the charged scalar ϕ^4 field

- Lattice action: $\quad (\phi_x \in \mathbb{C} \, , \quad M^2 = 8 + m^2)$

$$S = \sum_x \left[M^2 \, |\phi_x|^2 \; + \lambda \, |\phi_x|^4 \right] - \sum_{x,\nu} \left[\phi_x^\star \, \phi_{x+\widehat{\nu}} \; + \; \phi_x \, \phi_{x+\widehat{\nu}}^\star \right]$$

- Expand the nearest neighbor terms of e^{-S}:

$$\prod_{x,\nu} \exp\left(\phi_x^\star \, \phi_{x+\widehat{\nu}}\right) \times \exp\left(\phi_x \, \phi_{x+\widehat{\nu}}^\star\right)$$

$$= \prod_{x,\nu} \sum_{k_{x,\nu}=0}^{\infty} \frac{\left(\phi_x^\star \; \phi_{x+\widehat{\nu}}\right)^{k_{x,\nu}}}{k_{x,\nu} \, !} \; \times \sum_{l_{x,\nu}=0}^{\infty} \frac{\left(\phi_x \; \phi_{x+\widehat{\nu}}^\star\right)^{l_{x,\nu}}}{l_{x,\nu} \, !}$$

$$= \sum_{\{k,l\}} \prod_{x,\nu} \frac{1}{k_{x,\nu}! \, l_{x,\nu}!} \; \prod_x \phi_x^{\sum_\nu (l_{x,\nu} + k_{x-\hat{\nu},\nu})} \; \phi_x^{\star \, \sum_\nu (k_{x,\nu} + l_{x-\hat{\nu},\nu})}$$

- The $k_{x,\nu}$ and $l_{x,\nu}$ will turn into the new worldline degrees of freedom.

Worldline representation - integrating out the original fields

- Integral over ϕ_x at site x: (S_j, T_j are sums of the $k_{y,\nu}, l_{y,\nu}$ connected to x)

$$\int_{\mathbb{C}} d\,\phi_x \;\; e^{-M^2|\phi_x|^2 - \lambda|\phi_x|^4} \; (\phi_x)^{S_j} \; (\phi_x^\star)^{T_j}$$

- Polar coordinates $\phi_x = re^{i\theta}$ to separate radial and U(1) parts (symmetry):

$$\int_0^\infty dr \; r^{\,S_j + T_j + 1} \; e^{-M^2 r^2 - \lambda r^4} \int_{-\pi}^{\pi} d\theta \; e^{\,i\theta\,(S_j - T_j)} \;=\; \mathcal{I}(S_j + T_j)\;\delta(S_j - T_j)$$

- At every site there is a weight factor $\mathcal{I}(S_j + T_j)$ and a constraint.

- The constraint $\delta(S_j - T_j)$ enforces vanishing flux of $j_{x,\nu} = k_{x,\nu} - l_{x,\nu}$ at each site.

Worldline representation - final form

- The original partition function is mapped exactly to a sum over configurations of the dual variables $k_{x,\nu}, l_{x,\nu} \in \mathbb{N}_0$

$$Z \;=\; \sum_{\{k,l\}} \mathcal{W}[\,k,l\,]\,\mathcal{C}[\,j\,] \quad \text{with} \quad j_{x,\nu} = k_{x,\nu} - l_{x,\nu}$$

- Weight factor from radial d.o.f. and combinatorics:

$$\mathcal{W}[\,k,l\,] \;=\; \prod_{x,\nu} \frac{1}{k_{x,\nu}!\,l_{x,\nu}!} \;\prod_x \mathcal{I}\Big(\sum_\nu \left[\,k_{x,\nu} + k_{x-\widehat{\nu},\nu} + l_{x,\nu} + l_{x-\widehat{\nu},\nu}\,\right]\Big)$$

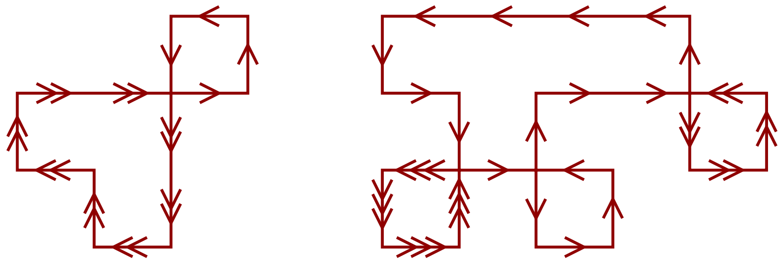
$$\mathcal{I}(n) \;=\; \int_0^\infty dr\; r^{\,n+1}\; e^{-M^2 r^2 - \lambda r^4}$$

- Zero divergence constraint from integrating over the symmetry group

$$\mathcal{C}[\,j\,] \;=\; \prod_x \delta\Big(\sum_\nu \left[\,j_{x,\nu} - j_{x-\widehat{\nu},\nu}\,\right]\Big) \qquad \Leftrightarrow \qquad \forall\, x \;:\; \sum_\nu \left[\,j_{x,\nu} - j_{x-\widehat{\nu},\nu}\,\right] \;=\; 0$$

Admissible configurations are loops:

- Admissible configurations of dual variables are oriented loops of flux:



- Chemical potential μ couples to temporal winding number of the flux $\Rightarrow e^{\mu\,\Omega[j]} \in \mathbb{R}$
- MC simulations directly in terms of worldlines overcome the complex action problem.
- Net particle number is a topological invariant $=$ temporal winding number of flux $\Omega[j]$.
- For gauge degrees of freedom the fluxes live on the plaquettes of the lattice $F_{\mu\nu}(x) \;\rightarrow\; p_{x,\mu\nu} \in \mathbb{Z}$.

Four examples for the use of worldlines and worldsheets

- Non-perturbative study of BEC in the gauged relativistic Bose Gas
  PRL 2013, NPB 2013, PLB 2013, CPC 2013

- Low temperature condensation and scattering parameters
  PRL 2018, PRL 2015

- Breaking of charge conjugation symmetry in 2d U(1) lattice field theories at $\theta = \pi$
  Preprint in preparation, PRL 2020, NPB 2018

- Breaking of the self dual symmetry in QED with electric and magnetic charges
  JHEP 06 2022, JHEP 04 2022, NPB 2019

Example 1: *Gauged relativistic Bose gas / U(1) gauge Higgs model at finite chemical potential* μ

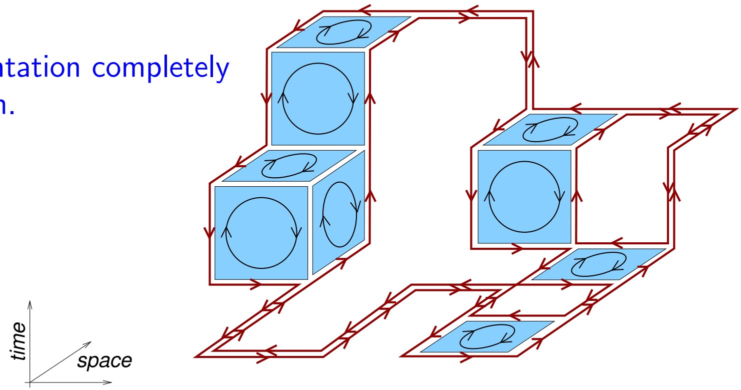
- Relativistic Bose gas with a chemical potential μ

$$S[A,\phi] \;=\; \frac{1}{4e^2}\int d^4x\, F_{\rho\sigma}F_{\rho\sigma} \;+$$
$$\int d^4x \Big[|(\partial_\nu + iA_\nu)\phi|^2 + \mu[\phi^*(\partial_4 + iA_4)\phi - \phi(\partial_4 - iA_4)\phi^*] + (m^2 - \mu^2)\,|\phi|^2 + \lambda\,|\phi|^4 \Big]$$

- The chemical potential leads to an imaginary part of the euclidean action and thus to a complex action problem.

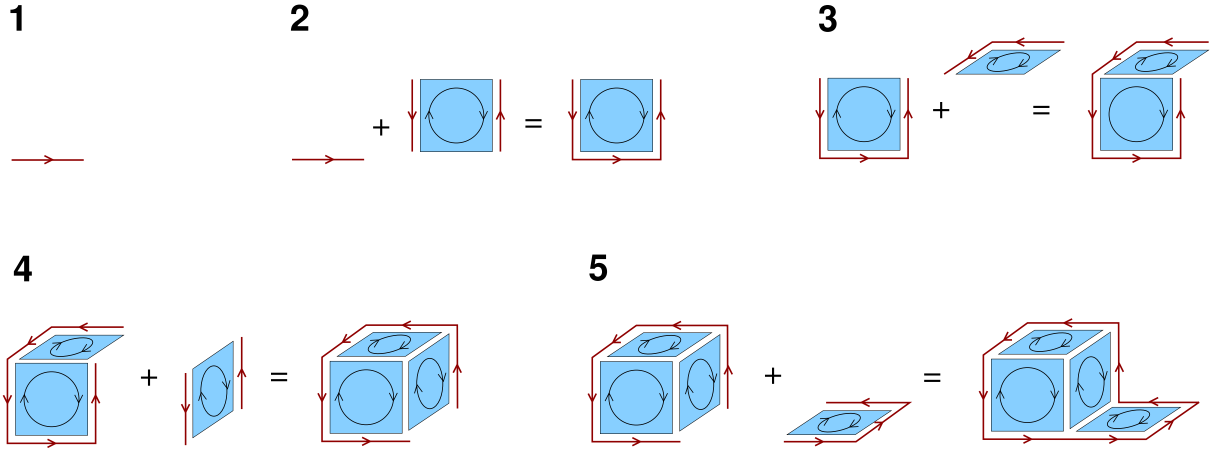
- The worldline/worldsheet representation completely solves the complex action problem.

- Monte Carlo simulation of the worldlines/worldsheets allows to access non-perturbative physics, such as phase transitions.



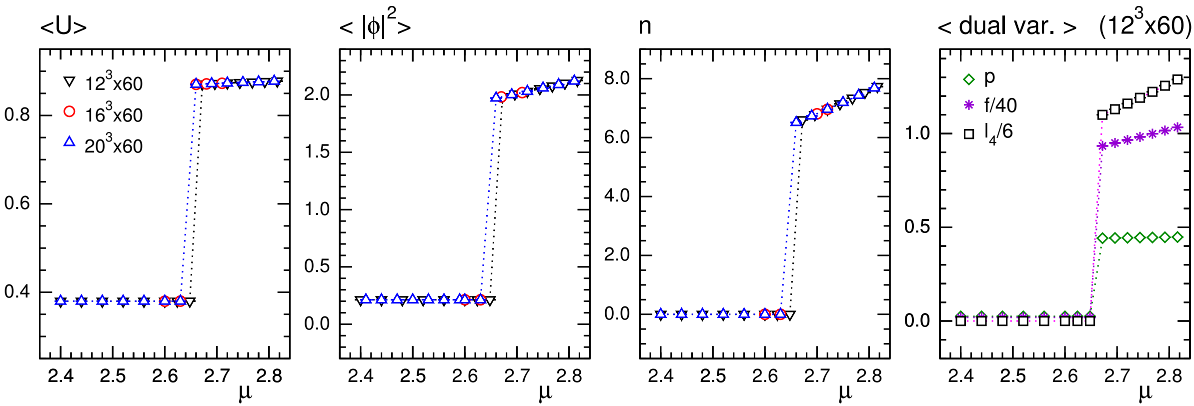
Generalized worm algorithm for gauge Higgs systems

Worm starts by inserting a unit of matter flux. Adding segments transports the defect across the lattice until the defect is healed in a final step.



Y. Delgado Mercado, C. Gattringer, A. Schmidt, Comp. Phys. Comm. 184, 2013

BEC in the confining phase at low T



- In the confining phase the dependence on the chemical potential μ sets in only when μ reaches the mass of the lowest excitation. "Silver Blaze behaviour"

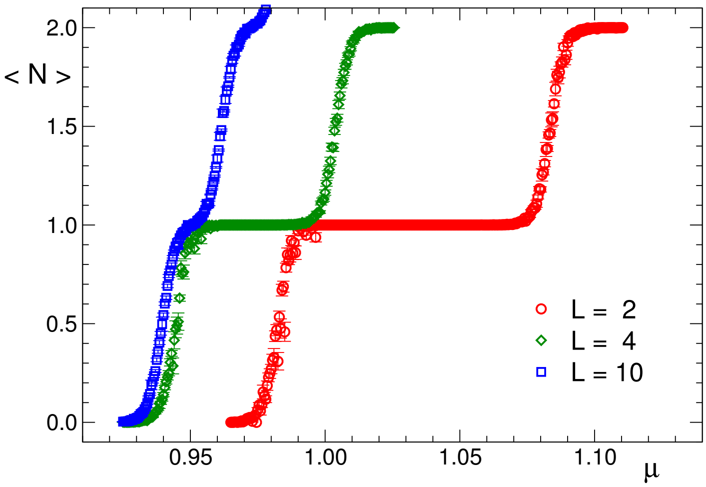
- The corresponding BEC is accompanied by a condensation of the worldline/worldsheet variables.

PRL 2013, NPB 2013, PLB 2013, CPC 2013

Example 2: *Low temperature condensation and scattering parameters*

At very very low temperature one observes "condensation thresholds".

Expectation value $\langle N \rangle$ of the particle number as a function of the chemical potential μ at very low temperature (charged scalar field):



- At critical values $\mu_n(L)$ one observes jumps from $\langle N \rangle = n-1$ to $\langle N \rangle = n$.
- The condensation thresholds $\mu_n(L)$ depend on the spatial extent L.

Connection of condensation thresholds and n-particle energies

- Grand canonical partition sum and grand potential:

$$Z \;=\; \mathsf{Tr}\, e^{-\beta(\hat{H} - \mu \hat{N})} \;=\; e^{-\beta\,\Omega(\mu)}$$

- Low T: In each particle sector Z is governed by the minimal grand potential $\Omega(\mu)$

$$\Omega(\mu) \quad \xrightarrow{T\to 0} \quad \begin{cases} \Omega^{N=0}_{min} \;=\; 0\,, & \mu \;\in\; [0, \mu_1] \\ \Omega^{N=1}_{min} \;=\; m \;-\; 1\mu\,, & \mu \;\in\; [\mu_1, \mu_2] \\ \Omega^{N=2}_{min} \;=\; W_2 - 2\mu\,, & \mu \;\in\; [\mu_2, \mu_3] \\ \Omega^{N=3}_{min} \;=\; W_3 - 3\mu\,, & \mu \;\in\; [\mu_3, \mu_4] \\ \ldots \end{cases}$$

- m: physical mass, W_2: minimal 2-particle energy, W_3: minimal 3-particle energy ...
- Use continuity of $\Omega(\mu)$ to relate the critical μ_n to m and the W_n.

$$m(L) \;=\; \mu_1(L)\,, \quad W_2(L) \;=\; \mu_1(L) + \mu_2(L)\,, \quad \ldots \quad W_n(L) \;=\; \sum_{k=1}^{n} \mu_k(L)$$

Connection of condensation thresholds and n-particle energies

- The multi-particle energies are governed by low energy parameters.

- In particular their finite volume dependence can be related to scattering data.

  (K. Huang, C.N. Yang, M. Lüscher, S.R. Beane, W. Detmold, M.J. Savage, S.R. Sharpe, M.T. Hansen)

$$\mathcal{I} = -8.914, \mathcal{J} = 16.532$$

$$m(L) = m_\infty + \frac{A}{L^{\frac{3}{2}}}\, e^{-L\, m_\infty}$$

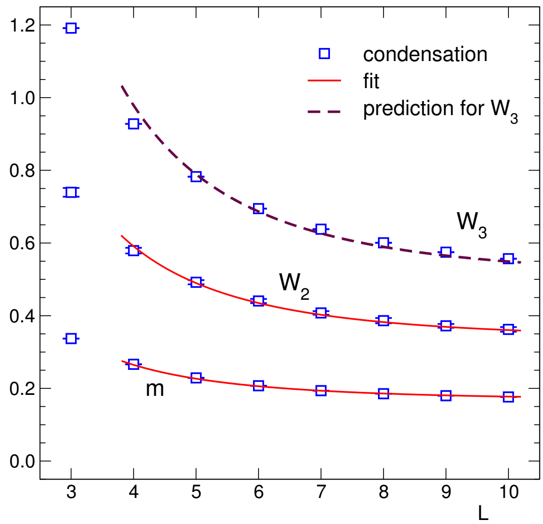
$$W_2(L) = 2m + \frac{4\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 - \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right]$$

$$W_3(L) = 3m + \frac{12\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 + \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right]$$

$$m(L) = \mu_1(L)\,, \quad W_2(L) = \mu_1(L) + \mu_2(L)\,, \quad W_3(L) = \mu_1(L) + \mu_2(L) + \mu_3(L)$$

- We thus expect that one can describe the thresholds $\mu_n(L)$ with scattering data.

Comparison of threshold data with the finite volume relations



- Good agreement: Condensation can indeed be described with scattering data.
- Key technical ingredient: Worldline representation

PRL 2018, PRL 2015

Example 3: *Breaking of C in 2d U(1) lattice field theories at $\theta = \pi$*

- Self-interacting 2d fermions coupled to U(1) gauge fields with topological term:

$$S = \int_{\mathbb{T}^2} d^2x \Big(\overline{\psi} \gamma_\mu D_\mu \psi \; - \; J \, (\overline{\psi}\psi)^2 \; + \; \frac{1}{4e^2} F_{\mu\nu}^2 \Big) \quad , \quad Q \; = \; \int_{\mathbb{T}^2} \frac{d^2x}{2\pi} F_{12} \; \in \; \mathbb{Z}$$

- Charge conjugation:

$$A_\mu \; \rightarrow \; -A_\mu \quad , \quad \psi \; \leftrightarrow \; \overline{\psi} \quad , \quad Q \; \rightarrow \; -Q$$

- The partition sum ....

$$Z \; = \; \int D[A] D[\psi, \overline{\psi}] \; e^{\, -S \, + \, i \, \theta \, Q}$$

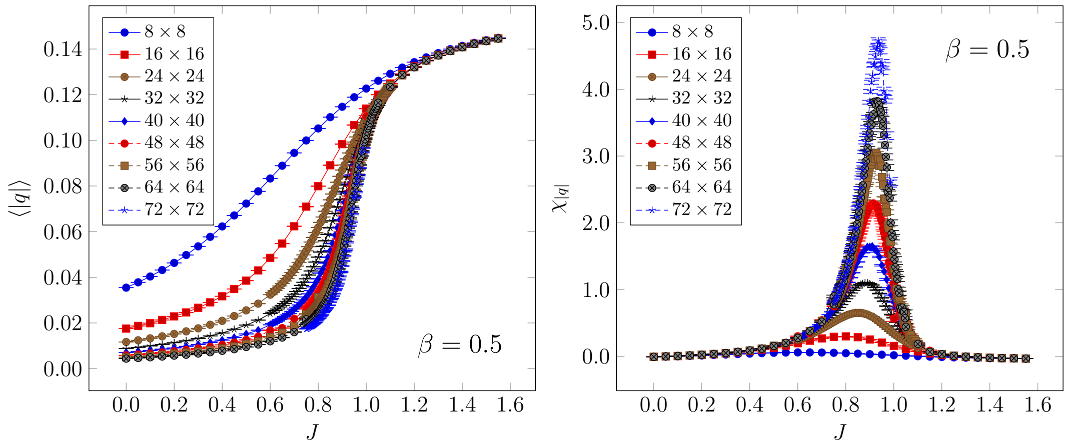
.... is C invariant also for $\theta = \pi$.

- Can charge conjugation invariance be broken spontaneously for $\theta = \pi$?

- Worldline/worldsheet representation solves complex action problem and provides an integer-valued lattice regularization of Q. In this special case one also gets rid of the fermion signs.

Breaking of C for 2d fermions coupled to U(1) gauge fields at $\theta = \pi$

- Topological charge density and susceptibility:



- Spontaneous breaking of charge conjugation symmetry at $\theta = \pi$ with a critical point in the Ising universality class.

Preprint in preparation, PRL 2020, NPB 2018

Example 4: *Breaking of a self-dual symmetry in QED with electric and magnetic charges*

- By introducing magnetic charges the Maxwell equations can be made self-dual.
- Can we construct a lattice regularization of U(1) gauge theory that couples to electrically and magnetically charged matter such that the theory is self-dual?
- What are the properties of such a theory? Can self-duality be broken spontaneously?
- Challenge on the lattice: Discretization introduces artificial magnetic monopoles.
- Monopole problem is solved by using the Villain action with an additional constraint:

$$Z(\beta) \;=\; \int D[A] \sum_{\{n\}} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big)\, e^{-\frac{\beta}{2}\sum_{x,\mu<\nu} F_{x,\mu\nu}F_{x,\mu\nu}}$$

$$F_{x,\mu\nu} \;=\; (dA)_{x,\mu\nu} + 2\pi\, n_{x,\mu\nu} \quad , \quad n_{x,\mu\nu} \in \mathbb{Z} \quad , \quad A_{x,\mu} \in [-\pi,\pi) \quad , \quad \beta \;=\; 1/e^2$$

- Self-dual lattice theory of U(1) gauge fields:

$$Z(\beta) \;=\; Z(\widetilde{\beta}) \qquad \text{with} \quad \widetilde{\beta} \;=\; 1/4\pi^2\beta \;\;\Rightarrow\;\; \beta_{selfdual} \;=\; 1/2\pi$$

Coupling matter in a self-dual way

Pure gauge theory at $\theta = 0$ with constraints generated by integrating $\widetilde{A}^m$

$$Z = \int D[A] \int D[\widetilde{A}^m] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_x \sum_{\mu<\nu} F_{x,\mu\nu}^2} \, e^{\, i \sum_{\tilde{x}} \sum_\mu \widetilde{A}^m_{x,\mu} (\partial \widetilde{n})_{\tilde{x},\mu}}$$

A simple self-dual theory with electric and magnetic matter

$$Z = \int D[A] \int D[\widetilde{A}^m] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_x \sum_{\mu<\nu} F_{x,\mu\nu}^2} \, e^{\, i \sum_{\tilde{x}} \sum_\mu \widetilde{A}^m_{x,\mu} (\partial \widetilde{n})_{\tilde{x},\mu}} \, Z_{J_e}[A] \, \widetilde{Z}_{J_m}[\widetilde{A}^m]$$

Partition sums for electric and magnetic matter fields in a background fields A and $\widetilde{A}^m$

$$Z_{J_e}[A] \; = \; \int D[\varphi^e] \, e^{-S_{J_e}[\varphi^e, A]} \quad , \quad \widetilde{Z}_{J_m}[\widetilde{A}^m] \; = \; \int D[\widetilde{\varphi}^m] \, e^{-S_{J_m}[\widetilde{\varphi}^m, \widetilde{A}^m]}$$

Duality transformation in a nutshell

$$x \leftrightarrow \tilde{x} \quad , \quad A_x \leftrightarrow \widetilde{A}^m_{\tilde{x}} \quad , \quad n_{x,\mu\nu} \to \widetilde{p}_{\tilde{x},\mu\nu} \quad , \quad \beta \to \widetilde{\beta} = 1/4\pi^2\beta \quad , \quad J_e \leftrightarrow J_m$$

The system can be simulated without sign problem when $\widetilde{Z}_{J_m}[\widetilde{A}^m]$ has a worldline representation such that $\widetilde{A}^m$ can be integrated out.

Setup for studying spontaneous breaking of the self-dual symmetry

We consider (here we set $J_e = J_m \equiv J$)

$$Z = \int D[A] \int D[\widetilde{A}^m] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_x \sum_{\mu<\nu} F_{x,\mu\nu}^2} \; e^{\,i\sum_{\tilde{x}}\sum_\mu \widetilde{A}^m_{x,\mu}\,(\partial\widetilde{n})_{\tilde{x},\mu}} \; Z_J[A] \; \widetilde{Z}_J[\widetilde{A}^m]$$

with

$$\begin{aligned} Z_J[A] &= \int D[\varphi] \;\; e^{\,JS[\varphi,A]} \\ S[\varphi, A] &= \frac{1}{2}\sum_{x,\mu}\Big[\varphi_x^*\, e^{iA_{x,\mu}}\,\varphi_{x+\hat{\mu}} \;+\; c.c.\Big] \end{aligned}$$

For all J the system has self-dual symmetry at the self-dual gauge coupling

$$\beta \;=\; \widetilde{\beta} \;=\; \frac{1}{2\pi} \;\equiv\; \beta^*$$

Can the self-dual symmetry be broken as a function of the matter field coupling J ?

Setup for studying spontaneous breaking of the self-dual symmetry

Order parameter for breaking of the self-dual symmetry

$$M_m \equiv s_e - s_m = s[\varphi^e, A] - s[\widetilde{\varphi}^m, A^m] \quad , \quad s = S/V$$

Breaking is signalled by

$$\langle M_m \rangle_{\beta^*, J} \neq 0$$

Worldline representation

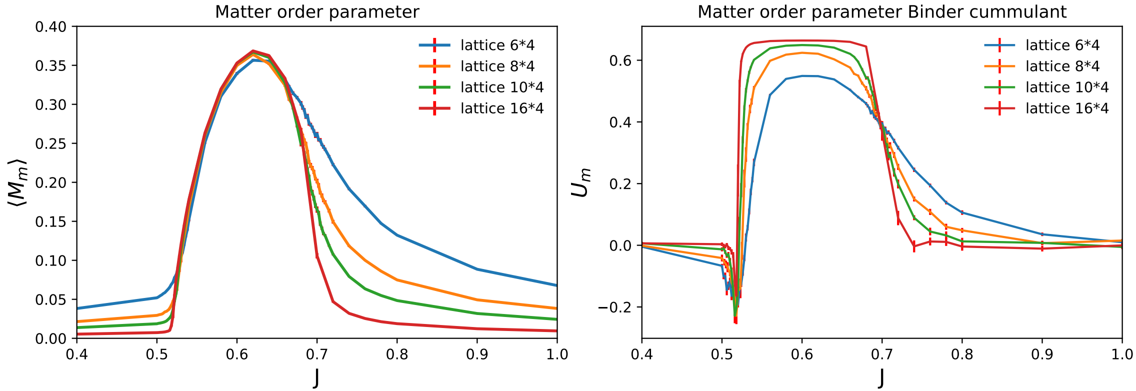
$$Z = \int D[A] \sum_{\{n\}} \int D[\varphi^e] \, e^{-\beta S_g[A,n] + JS[\varphi^e, A]} \prod_{\tilde{x},\mu} I_{(\partial \widetilde{n})_{\tilde{x},\mu}}(J)$$

No remaining constraints for the degrees of freedom $A_{x,\mu}, \varphi^e_x$ and $n_{x,\mu\nu}$

$\Rightarrow$ standard local MC algorithms.

Numerical results

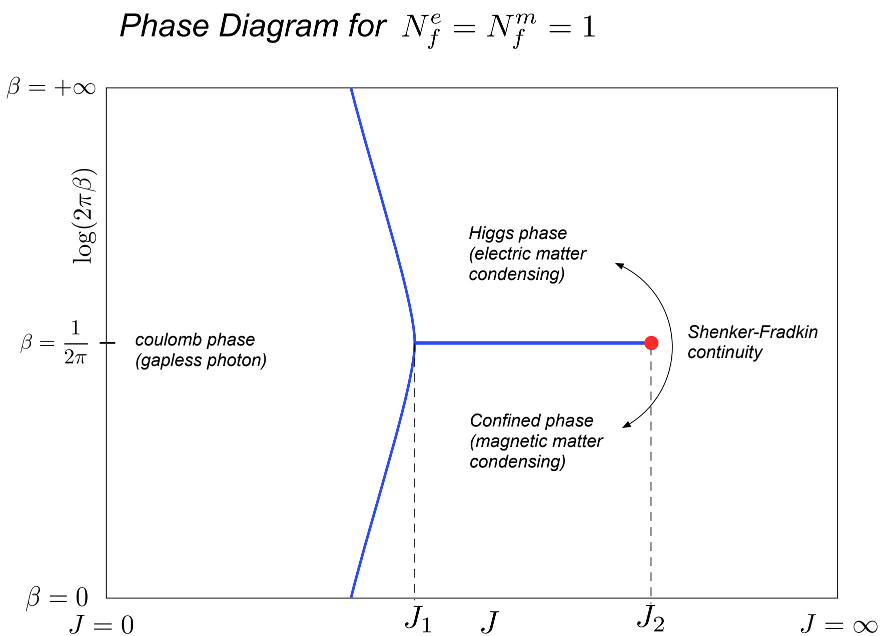
Order parameter and Binder cumulant



Detailed finite size scaling analysis, cross-checked with a second order parameter:

- First order point at $J_1 = 0.518(2)$
- Second order point at $J_2 = 0.700(1)$ with $\nu = 1/2$, $\gamma = 1$ (4d Ising = Gaussian FP)

Conjectured phase diagram:



JHEP 06 2022, JHEP 04 2022, NPB 2019

Conclusions and challenges

- Many field theories have a complex action problem at non-zero chemical potential or when a topological term is coupled.
- For some theories it is possible to exactly rewrite the lattice regularized partition sum in terms of worldlines and worldsheets.
- Monte Carlo simulation in the new form gives access to non-perturbative physics.
- Examples discussed:
  - BEC in the relativistic Bose gas
  - Condensation and scattering data
  - Breaking of C symmetry at $\theta = \pi$
  - Breaking of self dual symmetry in QED with electric and magnetic charges
- Challenges we work on:
  - Fermion worldlines
  - Non-abelian gauge groups