Semi-leptonic $B \to \pi$, $B_s \to K$ decays & Bayesian inference for z-expansion fits

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Lattice meets Continuum, Siegen, DE

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Outline

1. Motivation and the status quo
2. Challenges on the lattice
3. An example: $B_s \to K\ell\nu$ [JTT, RBC/UKQCD PRD 107 (2023) 114512]
4. z-expansions and Bayesian inference
5. Improvements and Summary

Motivation and the status quo

Why $|V_{ub}|$

C[Cabibbo '63]KM[Kobayashi, Maskawa '73]

- Status: [PDG'24]

	d	s	b
u	0.97367(32)	0.22431(85)	$0.00367(15)^\dagger$
c	0.221(4)	0.975(6)	$0.0398(6)^\dagger$
t	0.0086(2)	0.0415(9)	1.010(27)

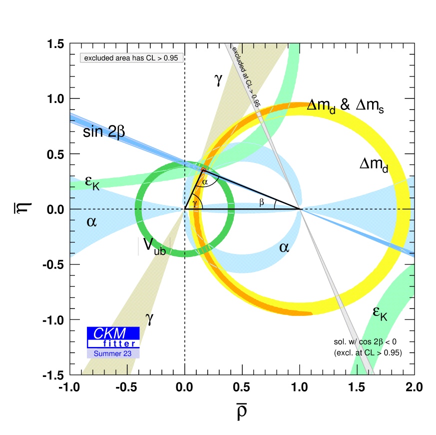
$^\dagger V_{ub}^{\rm inc} = 0.00413(26)\,,\quad V_{cb}^{\rm inc} = 0.0422(5)$

- Uncertainties [%]

$$\begin{pmatrix} 0.03 & 0.4 & 4.1^\dagger \\ 1.8 & 0.6 & 1.5^\dagger \\ 2.3 & 2.2 & 2.7 \end{pmatrix}$$

- V_{ub} exc+inc: 5.2% ($\sqrt{\chi^2} = 1.4$)

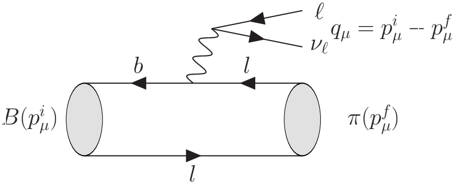
Unitarity Triangle [CKMfitter'05 + web updates]

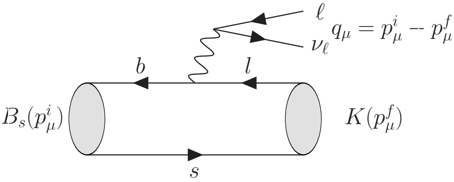


$|V_{ub}|$ least well known

$B \to \pi$ vs $B_s \to K$

$$\frac{\mathrm{d}\Gamma(B_{(s)} \to P\ell\nu_\ell)}{\mathrm{d}q^2} = |V_{ub}|^2\,\mathcal{K}\left[\left(1+\frac{m_\ell^2}{2q^2}\right)\left|f_+(q^2)\right|^2 + \mathcal{K}_2 m_\ell^2\left|f_0(q^2)\right|^2\right]$$

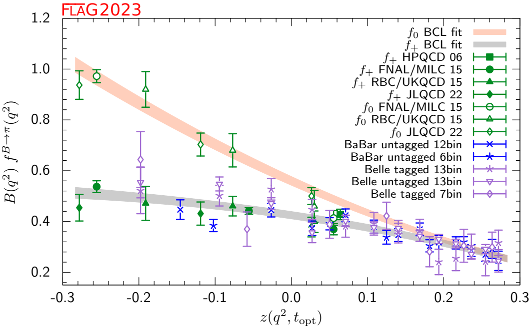




- Only differ in spectator quark:
  $\Rightarrow$ Very similar computation on the lattice
- Complementary channels to extract $|V_{ub}|$
- Most experimental data obtained for $\ell \in \{e, \mu\}$, so $m_\ell \sim 0$
  $\Rightarrow$ $f_+(q^2)$ "more important"

Experimental situation for $b \to u\ell\nu$

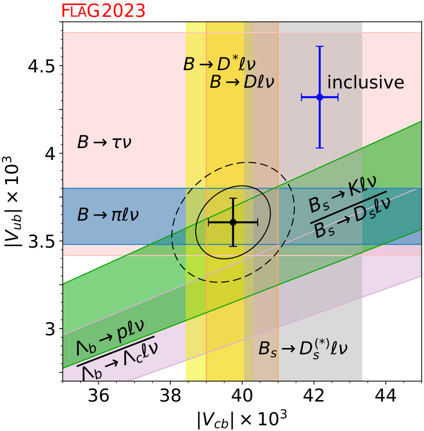
$B \to \pi\ell\nu$



Belle & BaBar multiple bins (↑)
+ Future: Belle II, LHCb

- $B_s \to K\ell\nu$
  LHCb 2 bins for $B_s \to K\ell\nu$ (normalised by $B_s \to D_s$)
  LHCb working on more bins
- $\Lambda_b \to p\ell\nu$
  LHCb $\Lambda_b \to p\ell\nu$ (normalised by $\Lambda_b \to \Lambda_c$)
- $\cdots$

What is the status of V_{ub}? Let's look at FLAG!



Consistency between different determinations ✓(or is it?!)

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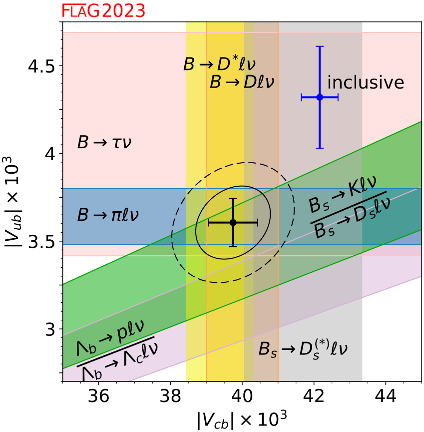

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$B \to \tau\nu$: good agreement! ✓

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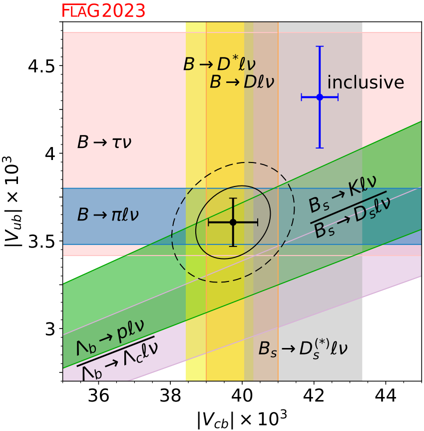

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$\Lambda_b \to p$: Only a single result ✓(?)

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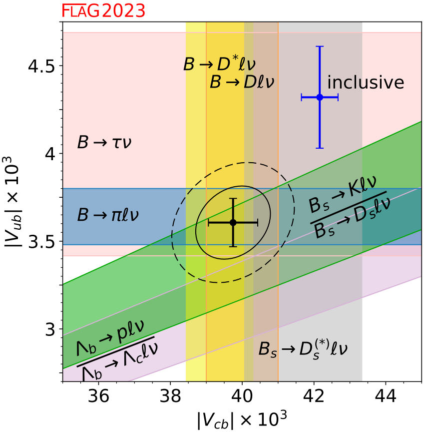

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∃ sys errors ⇒ p only indicative

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$B \to \pi\ell\nu$: $p \sim 2 \times 10^{-5}$ ✗

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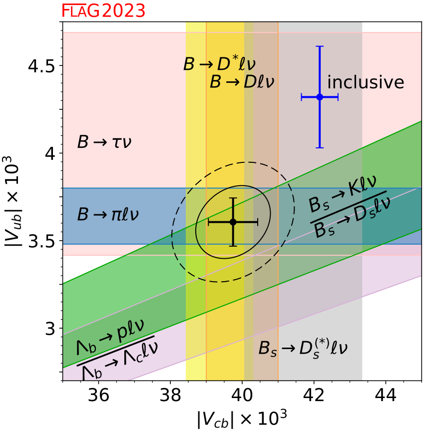
$B \to \pi$ ($N_f = 2+1$)

	Central Values	Correlation Matrix				
a_0^+	0.423 (21)	1	-0.00466	-0.0749	0.402	0.0920
a_1^+	−0.507 (93)	-0.00466	1	0.498	-0.0556	0.659
a_2^+	−0.75 (34)	-0.0749	0.498	1	-0.152	0.677
a_0^0	0.561 (24)	0.402	-0.0556	-0.152	1	-0.548
a_1^0	−1.42 (11)	0.0920	0.659	0.677	-0.548	1

Table 46: Coefficients and correlation matrix for the $N^+ = N^0 = 3$ z-expansion fit of the $B \to \pi$ form factors f_+ and f_0. The coefficient a_2^0 is fixed by the $f_+(q^2=0) = f_0(q^2=0)$ constraint. The chi-square per degree of freedom is $\chi^2/\text{dof} = 43.6/12$ and the errors on the z-parameters have been rescaled by $\sqrt{\chi^2/\text{dof}} = 1.9$. The lattice calculations that enter this fit are taken from FNAL/MILC 15 [58], RBC/UKQCD 15 [59] and JLQCD 22 [60]. The parameterizations are defined in Eqs. (533) and (534).

What is the status of V_{ub}? Let's look at FLAG!



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$B \to \pi\ell\nu$: $p \sim 2 \times 10^{-5}$ ✗

$B_s \to K\ell\nu$: $p \sim 7 \times 10^{-6}$ ✗

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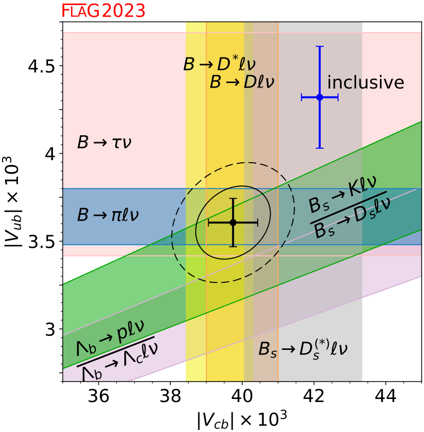
$B_s \to K$ ($N_f = 2+1$)

	Central Values	Correlation Matrix						
a_0^+	0.370(21)	1.	0.2781	−0.3169	−0.3576	0.6130	0.3421	0.2826
a_1^+	−0.68(10)	0.2781	1.	0.3672	0.1117	0.4733	0.8487	0.8141
a_2^+	0.55(48)	−0.3169	0.3672	1.	0.8195	0.3323	0.6614	0.6838
a_3^+	2.11(83)	−0.3576	0.1117	0.8195	1.	0.2350	0.4482	0.4877
a_0^0	0.234(10)	0.6130	0.4733	0.3323	0.2350	1.	0.6544	0.5189
a_1^0	0.135(86)	0.3421	0.8487	0.6614	0.4482	0.6544	1.	0.9440
a_2^0	0.20(35)	0.2826	0.8141	0.6838	0.4877	0.5189	0.9440	1.

Table 48: Coefficients and correlation matrix for the $N^+ = N^0 = 4$ z-expansion of the $B_s \to K$ form factors f_+ and f_0. The coefficient a_3^0 is fixed by the $f_+(q^2=0) = f_0(q^2=0)$ constrain. The chi-square per degree of freedom is $\chi^2/\text{dof} = 3.82$ and the errors on the z-parameters have been rescaled by $\sqrt{\chi^2/\text{dof}} = 1.95$.

(I counted 7 fit parameters and 19 datapoints $\Rightarrow$ 12 dof's)

What is the status of V_{ub}? Let's look at FLAG!



$B \to \tau\nu$: good agreement! ✓

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$B \to \pi\ell\nu$: $p \sim 2 \times 10^{-5}$ ✗

$B_s \to K\ell\nu$: $p \sim 7 \times 10^{-6}$ ✗

$|V_{ub}|\,(B \to \pi)$: $p \sim 3 \times 10^{-5}$ ✗

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$B \to \pi\ell\nu$ ($N_f = 2+1$)

	Central Values	Correlation Matrix					
$\lvert V_{ub}\rvert \times 10^3$	3.64 (16)	1	-0.812	-0.108	0.128	-0.326	-0.151
a_0^+	0.425 (15)	-0.812	1	-0.188	-0.309	0.409	0.00926
a_1^+	−0.441 (39)	-0.108	-0.188	1	-0.498	-0.0343	0.150
a_2^+	−0.52 (13)	0.128	-0.309	-0.498	1	-0.190	0.128
a_0^0	0.560 (17)	-0.326	0.409	-0.0343	-0.190	1	-0.772
a_1^0	−1.346 (53)	-0.151	0.00926	0.150	0.128	-0.772	1

Table 57: $|V_{ub}|$, coefficients for the $N^+ = N^0 = N^T = 3$ z-expansion of the $B \to \pi$ form factors f_+ and f_0, and their correlation matrix. The chi-square per degree of freedom is $\chi^2/\mathrm{dof} = 116.7/62 = 1.88$ and the errors on the fit parameters have been rescaled by $\sqrt{\chi^2/\mathrm{dof}} = 1.37$. The lattice calculations that enter this fit are taken from FNAL/MILC [58], RBC/UKQCD [59] and JLQCD [60]. The experimental inputs are taken from BaBar [161, 162] and Belle [163, 164].

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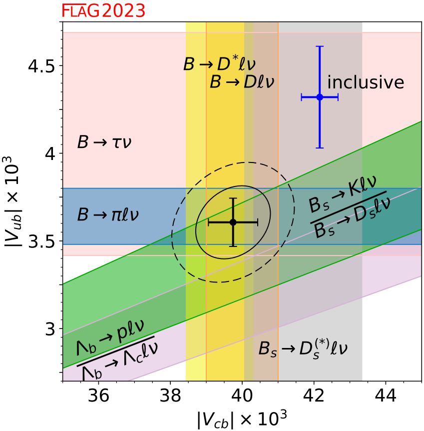
What is the status of V_{ub}? Let's look at FLAG!



$B \to \tau\nu$: good agreement! ✓

$\Lambda_b \to p$: Only a single result ✓(?)

$B \to \pi\ell\nu$: $p \sim 2 \times 10^{-5}$ ✗

$B_s \to K\ell\nu$: $p \sim 7 \times 10^{-6}$ ✗

$|V_{ub}|\,(B \to \pi)$: $p \sim 3 \times 10^{-5}$ ✗

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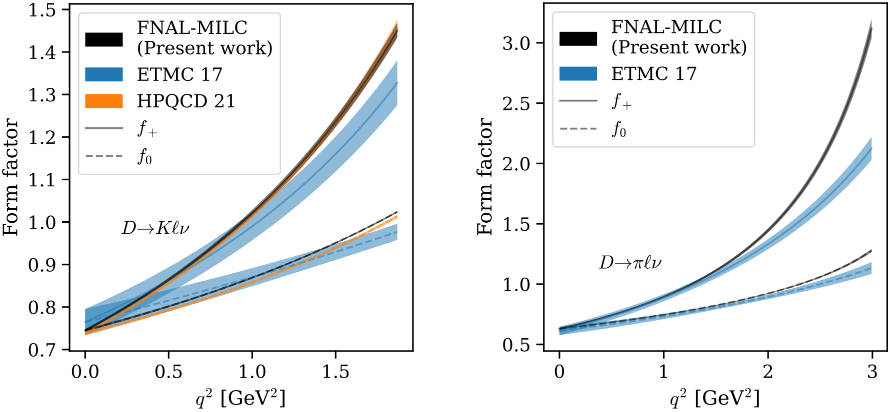
We need to scrutinise this!

What about charm decays?

Not unique to b-decays: Similar tensions for $c \to s$ and $c \to d$

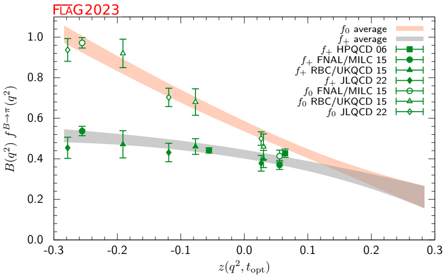
[FNAL/MILC'22]   [FNAL/MILC'22]

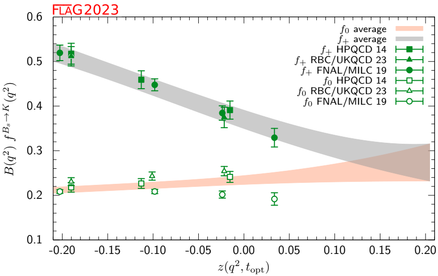


Puzzling: Tensions at large q^2 where data should be most precise.

$\Rightarrow$ We need to resolve these discrepancies

FLAG's summary of $B \to \pi$ and $B_s \to K$





- f_+ looks fine, f_0 shows some tensions
- Most experimental data obtained for $\ell \in \{e, \mu\}$, so $m_\ell \sim 0$ and recall:

$$\frac{\mathrm{d}\Gamma(B_{(s)} \to P\ell\nu_\ell)}{\mathrm{d}q^2} = |V_{ub}|^2 \mathcal{K} \left[\left(1 + \frac{m_\ell^2}{2q^2}\right) \left|f_+(q^2)\right|^2 + \mathcal{K}_2 m_\ell^2 \left|f_0(q^2)\right|^2 \right]$$

Does that mean V_{ub} should be fine? ✗

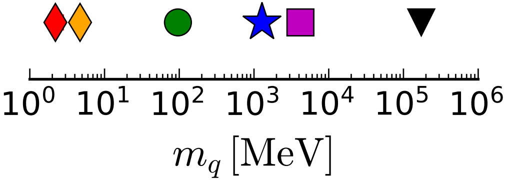
- kinematic extrapolation (z-expansion) stabilised by kinematic constraint $f_0(0) = f_+(0)$, so f_0 does impact CKM determinations!

Challenges on the lattice

Multiple scale problem on the lattice: back of the envelope

Control effects of IR (finite volume) and UV (discretisation) regulators:

$m_\pi L \gtrsim 4$ $\qquad\qquad a^{-1} \gg$ Mass scale of interest



For $m_\pi = m_\pi^{\mathrm{phys}} \sim 140\,\mathrm{MeV}$ and $\overline{m}_b(m_b) \approx 4.2\,\mathrm{GeV}$:

$$L \gtrsim 5.6\,\mathrm{fm} \qquad\qquad a^{-1} \gg 4.2\,\mathrm{GeV} \approx (0.05\,\mathrm{fm})^{-1}$$

Requires $N \equiv L/a \gg 120 \Rightarrow N^3 \times (2N) \gg 4 \times 10^8$ lattice sites.

very expensive to satisfy both constraints simultaneously. . .

. . . needs to be repeated for different values of a.

Currently computationally impossible at physical quark masses!

How to simulate the b-quark?

Effective action for b

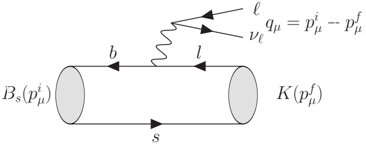
- Can tune to $m_b \sim m_b^{\mathrm{phys}}$
- comes with systematic errors which are hard to estimate/reduce

Relativistic action for b

- Theoretically cleaner and systematically improvable
- $m_b < m_b^{\mathrm{phys}}$: control extrapolation to m_b^{phys}

- relativistic will win in the long term
- for now, settle on a compromise
- very different systematics ⇒ complementary results!

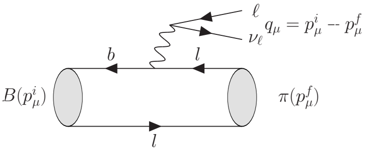
Complication: semileptonic form factors depend on momentum transfer q^2:



$$q^2 = (E_{B_s} - E_K)^2 - (\vec{p}_i - \vec{p}_f)^2$$

Interplay between varying m_b in q^2, M_{B_s} and $(am_b)^n$ highly non-trivial!

Challenges in computing $f_X(q^2)$: example $B \to \pi\ell\nu$



- $q^\mu = p_B^\mu - p_\pi^\mu$
- $M_B \approx 5.28\,\mathrm{GeV}$, $M_\pi \approx 0.14\,\mathrm{GeV}$
- Semileptonic region $q^2 \in [0, q^2_{\mathrm{max}}]$
- $q^2_{\mathrm{max}} \equiv (M_B - M_\pi)^2 \sim 26.4\,\mathrm{GeV}^2$

- physical kinematics in the B rest-frame: $q^2 = 0 \Leftrightarrow |p_\pi|^2 = 6.96\,\mathrm{GeV}^2$
- Assuming $M_\pi L = 4$ and physical pion masses implies:
  $\Rightarrow$ final state momentum of $\vec{p}_\pi \approx \frac{2\pi}{L}(7,7,7)$ to reach $q^2 \sim 0$.
- typical simulations cannot achieve (i.e. control) this
  $\Rightarrow$ compromise in at least one of the following:
  - $M_\pi > M_\pi^{\mathrm{phys}}$ ($\Rightarrow$ need chiral extrapolation)
  - $m_b < m_b^{\mathrm{phys}}$ ($\Rightarrow$ need heavy quark mass extrapolation)
  - $q^2_{\mathrm{min}} \gg 0$ ($\Rightarrow$ need kinematic extrapolation)

From correlators to the physical world

Extrapolations are based on theoretical foundations...

- Extraction of ground state parameters see also → Oliver Bär's talk
- M_π^{phys} (chiral) extrapolation guided by heavy meson chiral perturbation theory (HMχPT)
- m_b^{phys} (heavy quark) extrapolation guided by HQET
- $a \to 0$ (continuum limit) extrapolation guided by Symanzik E.T.

- $q^2 = 0$ (kinematic) extrapolation guided by model independent z-expansion (BGL, BCL)
  - Physical q^2 dependence can be mapped to interval $z(q^2) \in [-z_{\max}, z_{\max}]$ with $0 < z_{\max} \ll 1$
  - BGL expansion: $f_X(z) = \frac{1}{B_X \phi_X} \sum_i a_i z^i$, unitarity bounds $\sum_i |a|_i^2 < 1$.

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... but they are intertwined and difficult

and all of them come with systematic uncertainties - are they controlled?

An example: $B_s \to K\ell\nu$ [JTT, RBC/UKQCD PRD 107 (2023) 114512]

Set up of our calculation and required extrapolations

Simulations

- 3 $a \in [0.07, 0.11]\,\mathrm{fm}$.
- finite volume $L \Rightarrow$ discrete momenta $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}$.
- heavier-than-physical $M_\pi^{\mathrm{sim}} \in [270, 430]\,\mathrm{MeV}$.

Physics

- Need to recover $a \to 0$ limit.
- Need continuous description of $q^2 = M_{B_s}^2 + M_K^2 - 2M_{B_s}E_K$.
- Recover physical light quark masses.

Use HMχPT [NP **B812** 64,NP **840** 54,PRD **67** 054010] for extrapolation:

$$f_X^{B_s\to K}(M_\pi, E_K, a^2) = \frac{\Lambda}{E_K + \Delta_X}\left[c_{X,0}\left(1 + \frac{\delta f(M_\pi^s) - \delta f(M_\pi^p)}{(4\pi f_\pi)^2}\right) + c_{X,1}\frac{\Delta M_\pi^2}{\Lambda^2} + c_{X,2}\frac{E_K}{\Lambda} + c_{X,3}\frac{E_K^2}{\Lambda^2} + c_{X,4}(a\Lambda)^2\right]$$

(+vary fit ansatz + estimate missing/H.O. terms)

Choice of ff basis: (f_0, f_+) vs $(f_\parallel, f_\perp)$

- Interested in matrix elements of the vector current $\langle B_s | \mathcal{V}^\mu | K \rangle$
- Can be decomposed as

$$\langle K|\mathcal{V}^\mu|B_s\rangle = \sqrt{2M_{B_s}}\left[v^\mu f_\parallel(E_K) + p_\perp^\mu f_\perp(E_K)\right]$$

- trivially related to f_+, f_0 via

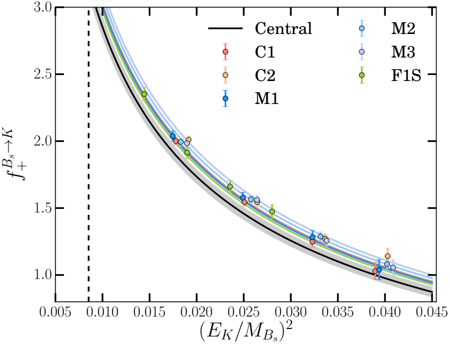
$$f_0(q^2) = \frac{\sqrt{2M_{B_s}}}{M_{B_s}^2 - M_K^2}\left[(M_{B_s} - E_K)f_\parallel(E_K) + (E_K^2 - M_K^2)f_\perp(E_K)\right]$$

$$f_+(q^2) = \frac{1}{\sqrt{2M_{B_s}}}\left[f_\parallel(E_K) + (M_{B_s} - E_K)f_\perp(E_K)\right]$$

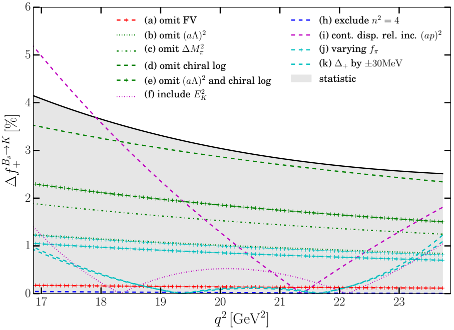
$\Rightarrow$ Convenient: $\mathcal{V}_0$ vs $\mathcal{V}_i$ to isolate $f_\parallel$ and $f_\perp$ from correlators

RBC/UKQCD 23 Fit results f_+

HMχPT fit to lattice data

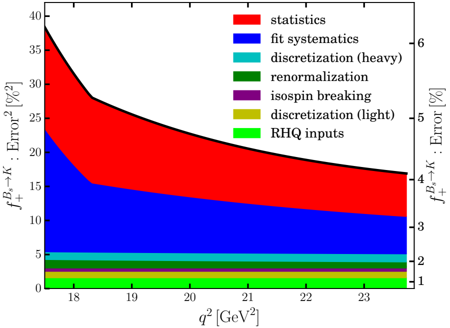


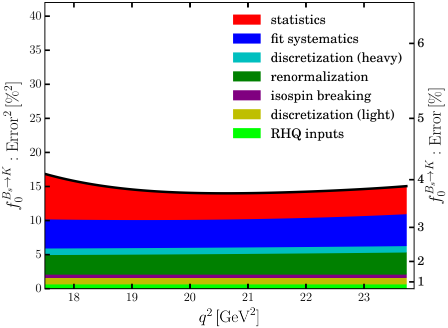
Fit systematics



$\Rightarrow$ take maximal deviation between the chosen fit and fit variation as fit-systematic value.

Assembling the error budget





- Dominated by statistical and fit systematic uncertainties $\Rightarrow$ both improvable!
- Most precise near $q^2_{\rm max}$
- Data covers range $q^2 > 17\,\mathrm{GeV}^2$

Some insights from $B_s \to K$ (1)

$$f_X^{B_s \to K} = \frac{\Lambda}{E_K + \Delta_X} \times \left[\chi(M_\pi^2) + k(E_K) + d((a\Lambda)^2)\right]$$

where Δ_X is the "relevant pole mass"

$$\Delta_+ = M_{B^*(1^-)} - M_{B_s}, \quad M_{B^*(1^-)} = 5.32471\,\mathrm{GeV} \quad \text{(exp.)}$$
$$\Delta_0 = M_{B^*(0^+)} - M_{B_s}, \quad M_{B^*(0^+)} = 5.63\,\mathrm{GeV} \quad \text{(the.)}$$

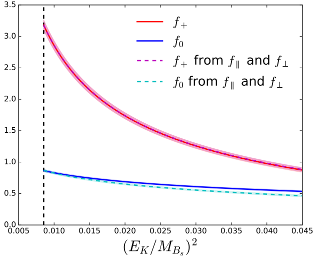
RBC/UKQCD'15 and FNAL/MILC'19 strategy:

1. Assume $f_\parallel$ dominated by f_0 and $f_\perp$ dominated by f_+.
2. HMχPT fit to $f_\parallel$, $f_\perp$ using $\Delta_\parallel \sim \Delta_0$, $\Delta_\perp \sim \Delta_+$
3. converting to f_+, f_0 in the continuum

Is this justified?

And how to deal with poles when $m_h \neq m_b$?

Some insights from $B_s \to K$ (2)

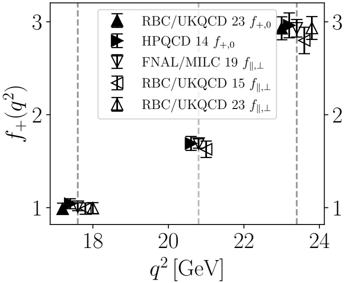


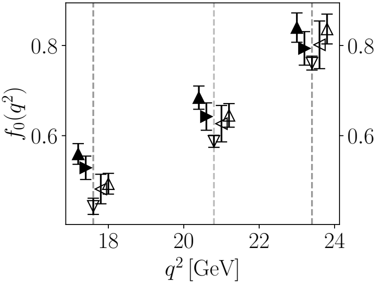
$\leftarrow$ All fine for f_+ (red vs magenta)✓

$\leftarrow$ Several (stat) sigmas difference for f_0 ✗!!

$\leftarrow$ Discrepancy gets worse with increasing energy $\Rightarrow$ easy to miss!

$\downarrow$ picture persists with full error budget

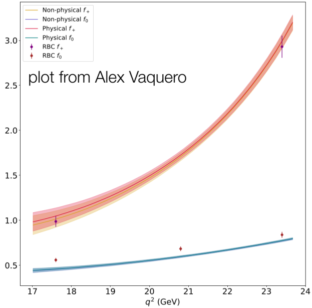




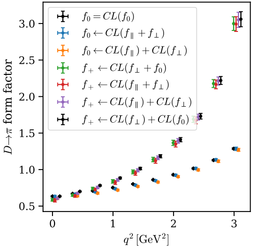
$\Rightarrow$ Not unique to $B_s \to K$, same strategy was used for $B \to \pi$

It remains puzzling

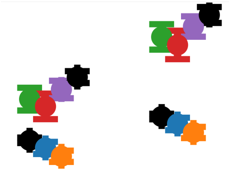
[from A. Kronfeld's talk last week]



[FNAL/MILC, Phys.Rev.D 107 (2023) 9, 094516]



$\downarrow$ (zoom)



- ($\uparrow$) Re-analysis of FNAL/MILC'19 doesn't reproduce what we see.
- But we see an effect in our data (prev. slide).

There is an effect in $D_{(s)} \to \pi/K$

z-expansions and Bayesian inference

Theory inputs

- Lattice results *after* chiral-continuum limit typically provide
  - ffs in restricted kinematical range $q^2 \in [q^2_{\mathrm{min,\,sim}}, q^2_{\mathrm{max}}]$.
  - continuous description of ffs in terms of small number of basis functions (e.g. from HMχPT)
  - $q^2_{\mathrm{max}} \equiv (M_i - M_f)^2$, so particularly relevant for $B \to \pi$ and $B_s \to K$.
  - e.g. for $B_s \to K$ $17\,\mathrm{GeV}^2 \lesssim q^2_{\mathrm{min,\,sim}}$
- Sum rule results typically individual point(s) near $q^2 \approx 0$.
- Ff's satisfy kinematic constraint $f_+(0) = f_0(0)$.

GOAL: Want form factors over full range $[0, q^2_{\mathrm{max}}]$.

Extrapolating over the full kinematic range: z-expansion

- Map $q^2 \in [0, q^2_{\mathrm{max}}]$ to $z \in [z_{\mathrm{min}}, z_{\mathrm{max}}]$ with $|z| < 1$ and branch cut t_*.

$$z(q^2; t_0) = \frac{\sqrt{t_* - q^2} - \sqrt{t_* - t_0}}{\sqrt{t_* - q^2} + \sqrt{t_* - t_0}}$$

- Form factor is a polynomial in z after poles have been removed

e.g. BGL: Boyd, Grinstein, Lebed [PRL **74** 4603]:

$$f_X(q^2) = \left(\prod_{\mathrm{poles}} \frac{1}{B_X(q^2)}\right) \frac{1}{\phi_X(q^2)} \sum_{n\geq 0} a_{X,n} z^n$$

Goal: Determine some "un-truncated" number of coefficients $a_{X,n}$ to obtain model independent parameterisation.

Unitarity constraint

Substituting BGL into the unitarity constraint

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}|B_X(q^2)\phi_X(q^2,t_0)f_X(q^2)|^2 \leq 1\,,$$

gives

$$|\boldsymbol{a}_X|^2 \leq 1$$

(see [Gubernari, Reboud, van Dyk, Virto, JHEP 09 (2022) 133] and [Flynn, Jüttner, JTT, JHEP 12 (2023) 175] for modified versions of this constraint)

- **MUST** be satisfied for any believable fit!
- Provides strong constraint on allowed size of coefficients $a_{X,i}$.
- In practice, any z-expansion fit must necessarily be truncated:

$$B_X\phi_X f_X = \sum_{n=0}^{K_X-1} a_{X,n}z^n$$

- **How can we best make use of this?**

Frequentist z-expansion fit: pro's and con's

Typically only small number of coefficients in HMχPT:

$$f_X^{B_s\to K} = \frac{\Lambda}{E_K + \Delta_X}\left(c_{X,0} + c_{X,1}\frac{E_K}{\Lambda} + c_{X,2}\frac{E_K^2}{\Lambda^2}\right)$$

$\Rightarrow$ limited number of independent data points N_X - typically $\approx$ 2-4 per ff.

$\Rightarrow$ Maximum meaningful truncation K_X of the z-expansion requires

$$\underbrace{N_+ + N_0}_{\text{input data}} - \underbrace{(K_+ + K_0)}_{\text{fit parameters}} + \underbrace{1}_{\text{kinematic constraint}} = N_{\rm dof} \geq 1$$

Goal: Determine $\boldsymbol{a}$, given data $\boldsymbol{f}$, covariance $C_{\boldsymbol{f}}$ and Z (encoding ϕ_X, B_X, kinematic constraint)

$$\chi^2(\boldsymbol{a},\boldsymbol{f}) = [\boldsymbol{f} - Z\boldsymbol{a}]^T\, C_{\boldsymbol{f}}^{-1}\, [\boldsymbol{f} - Z\boldsymbol{a}]^T\ .$$

pro Clear measure to assess quality of fit (p-value).

con No satisfactory way to assess systematic effect of truncation.

con Can only check unitarity constraint *a posteriori*.

Bayesian ansatz: Our *prior knowledge*

Our prior knowledge about probability distribution for

1. data points $\pi_{\boldsymbol{f}}(\boldsymbol{f}|\boldsymbol{f}_p, C_{\boldsymbol{f}_p}) \propto \exp\left(-\frac{1}{2}(\boldsymbol{f}-\boldsymbol{f}_p)^T C_{\boldsymbol{f}_p}^{-1}(\boldsymbol{f}-\boldsymbol{f}_p)\right)$
2. BGL ansatz: $\Theta(\boldsymbol{f},\boldsymbol{a}|Z) \propto \delta\left(|\boldsymbol{f}-Z\boldsymbol{a}|\right)$
3. unitarity: $\pi_{\boldsymbol{a}}(\boldsymbol{a}|\mathrm{unitarity}) \propto \theta(1-|\boldsymbol{a}_+|^2)\,\theta(1-|\boldsymbol{a}_0|^2)$

Marginalising over $\boldsymbol{f}$ (using 1. and 2.) gives

$$\pi_{\boldsymbol{a}}(\boldsymbol{a}|\boldsymbol{f}_p, C_{\boldsymbol{f}_p}) \propto \exp\left(-\frac{1}{2}\chi^2(\boldsymbol{a},\boldsymbol{f}_p)\right)$$

Now combining these

$$\begin{aligned}\pi_{\boldsymbol{a}}(\boldsymbol{a}|\boldsymbol{f}_p, C_{\boldsymbol{f}_p})\pi_{\boldsymbol{a}}(\boldsymbol{a}|\mathrm{unitarity}) &\propto \theta(\boldsymbol{a})\exp\left(-\frac{1}{2}\chi^2(\boldsymbol{a},\boldsymbol{f}_p)\right)\\ &= \theta(\boldsymbol{a})\exp\left(-\frac{1}{2}(\boldsymbol{a}-\tilde{\boldsymbol{a}})^T C_{\tilde{a}}^{-1}(\boldsymbol{a}-\tilde{\boldsymbol{a}})\right)\end{aligned}$$

with $\tilde{\boldsymbol{a}} = \tilde{C}_{\boldsymbol{f}_p}(Z^T C_{\boldsymbol{f}_p}^{-1}\boldsymbol{f}_p)$ and $C_{\tilde{\boldsymbol{a}}}^{-1} = Z^T C_{\boldsymbol{f}_p}^{-1} Z$

Bayesian Inference regulated by unitarity and analyticity

Compute the expectation value $g(\boldsymbol{a})$ in the presence of "*prior knowledge* B" via Bayes' theorem as (with normalisation $\mathcal{Z} = \int d\boldsymbol{a}\, \pi(\boldsymbol{a}|B)$)

$$\begin{aligned} \langle g(\boldsymbol{a})\rangle &= \frac{1}{\mathcal{Z}} \int d\boldsymbol{a}\, g(\boldsymbol{a})\, \pi(\boldsymbol{a}|B)\,, \\ &= \frac{1}{\mathcal{Z}} \int d\boldsymbol{a}\, g(\boldsymbol{a})\, \theta(1-|\boldsymbol{a}_+|^2)\, \theta(1-|\boldsymbol{a}_0|^2) \exp\left(-\frac{1}{2}(\boldsymbol{a}-\tilde{\boldsymbol{a}})^T C_{\tilde{\boldsymbol{a}}}^{-1} (\boldsymbol{a}-\tilde{\boldsymbol{a}})\right) \end{aligned}$$

$\Rightarrow$ Computable via Monte-Carlo integration by drawing from multivariate normal distribution $\mathcal{N}(\tilde{\boldsymbol{a}}, C_{\tilde{\boldsymbol{a}}})$[1].

con No clear measure to assess quality of fit (p-value).

pro Unitarity constraint automatically satisfied

pro Remove truncation error by computing as many terms as relevant

[1] See Flynn, Jüttner, JTT, JHEP 12 (2023) 175 for details of efficient algorithm

Application I: RBC/UKQCD'23 [JTT et al., PRD **107**, 114512]

$(N_+, N_0) = (2, 3)$ independent datapoints as input for z-exp.
Frequentist: limited to $K_+ + K_0 \leq 5$ truncation uncertainty unquantified!
Bayesian: unlimited order $\Rightarrow$ Can investigate effect of truncation! ✓

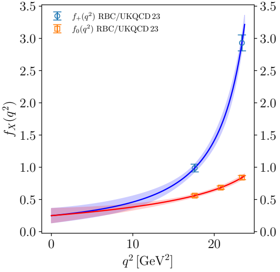
K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$	$a_{0,6}$	$a_{0,7}$
2	2	0.0981(36)	-0.286(14)						
2	3	0.0917(39)	-0.331(19)	-0.211(53)					
3	2	0.0950(37)	-0.263(15)						
3	3	0.0953(43)	-0.254(41)	0.02(13)					
4	4	0.0953(42)	-0.254(42)	0.02(21)	-0.02(60)				
5	5	0.0954(44)	-0.254(41)	0.02(21)	-0.01(55)	-0.00(62)			
6	6	0.0957(42)	-0.251(41)	0.04(21)	-0.01(52)	-0.06(65)	0.07(65)		
7	7	0.0955(44)	-0.250(40)	0.06(20)	0.05(50)	-0.13(72)	0.17(79)	-0.12(69)	
8	8	0.0954(43)	-0.250(41)	0.06(22)	0.06(50)	-0.18(84)	0.2(1.0)	-0.21(99)	0.10(74)
K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	$a_{+,5}$	$a_{+,6}$	$a_{+,7}$
2	2	0.0293(11)	-0.0871(46)						
2	3	0.0249(16)	-0.0999(57)						
3	2	0.0245(16)	-0.0799(50)	0.093(21)					
3	3	0.0245(15)	-0.078(12)	0.101(49)					
4	4	0.0246(17)	-0.077(32)	0.100(68)	-0.03(70)				
5	5	0.0246(17)	-0.074(31)	0.099(70)	-0.08(67)	0.05(70)			
6	6	0.0247(16)	-0.073(32)	0.101(69)	-0.10(69)	0.09(74)	-0.05(71)		
7	7	0.0247(17)	-0.071(33)	0.107(70)	-0.11(72)	0.08(89)	-0.04(89)	0.03(73)	
8	8	0.0248(17)	-0.068(35)	0.102(74)	-0.18(77)	0.2(1.1)	-0.2(1.3)	0.1(1.2)	-0.06(82)

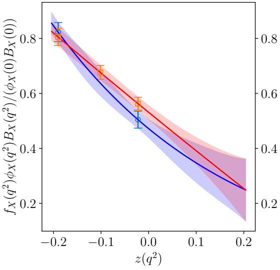
$\Rightarrow$ **Stable results for** $(K_+, K_0) \sim (5, 5)$

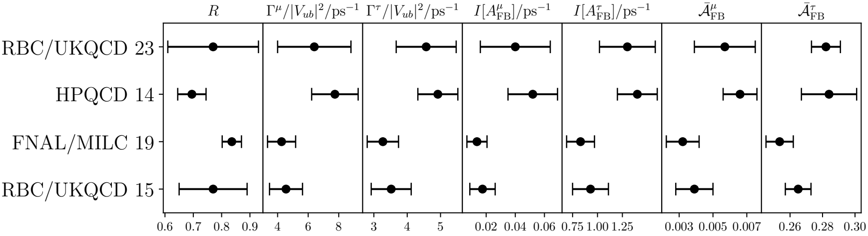
Results I: RBC/UKQCD'23 [JTT et al., PRD **107**, 114512]

- B.I. converges from $(K_+, K_0) \sim (5, 5)$
- $10^3 \times |V_{ub}| = 3.78(61)$
- $R_{B_s \to K} = 0.77(16)$
- $R^{\rm impr}_{B_s \to K} = 1.72(11)$

$\Rightarrow$ more pheno in paper







Status of the Literature for $B_s \to K\ell\nu$

Joint frequentist fit to RBC/UKQCD23, HPQCD14 and FNAL/MILC19:

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	p	χ^2/N_{dof}	N_{dof}
2	2	0.0854(17)	-0.2565(75)	-	-	-	0.00	5.15	14
3	3	0.0864(18)	-0.2379(95)	0.061(28)	-	-	0.00	3.89	12
4	4	0.0887(27)	-0.08(17)	2.2(2.4)	7.0(7.9)	-	0.00	4.53	10
5	5	0.0887(28)	0.07(20)	6.1(3.3)	41.5(19.0)	93.3(44.0)	0.00	5.04	8

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	p	χ^2/N_{dof}	N_{dof}
2	2	0.02641(58)	-0.0824(26)	-	-	-	0.00	5.15	14
3	3	0.02534(73)	-0.0792(31)	0.062(12)	-	-	0.00	3.89	12
4	4	0.02592(97)	-0.033(50)	0.69(69)	2.1(2.3)	-	0.00	4.53	10
5	5	0.0266(10)	0.052(65)	2.21(97)	11.1(5.6)	17.2(15.1)	0.00	5.04	8

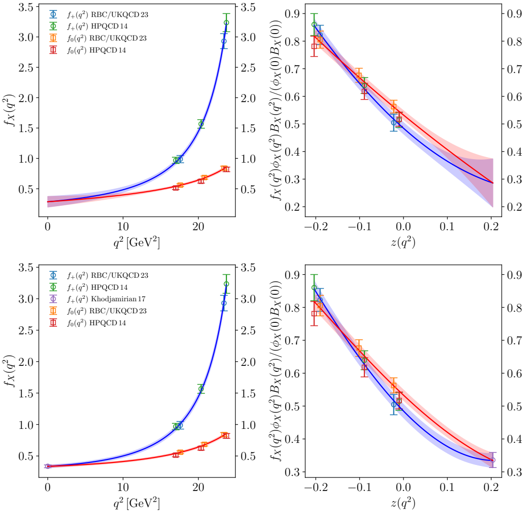
This data is incompatible $\Rightarrow$ should not be fitted simultaneously

Frequentist fit ideally suited to detect this!

Exploit Complementarity of Frequentist and Bayesian!

1. Assess compatability of data first (Frequentist)
2. Perform truncation independent z-expansion to compatible datasets

Results II: Joint analysis to multiple datasets



Joint fit to

- RBC/UKQCD23
- HPQCD14

Joint fit to

- RBC/UKQCD23
- HPQCD14
- Khodjamirian17 (sum rules)

Easy to jointly fit multiple datasets!

$\Rightarrow$ more pheno in [Flynn, Jüttner, JTT, JHEP 12 (2023) 175]

Publically available code!

Code: BFF

Flynn, AJ, Tsang, arXiv:2303.11285

Python3 available via github/Zenodo

https://github.com/andreasjuettner/BFF https://zenodo.org/record/7799543#.ZEezTy8Ro80

```
        }
######################################################################################
# specify input for BGL fit
######################################################################################
input_dict = {
        'decay':        'Btopi',
        'Mi':           pc.mBphys,      # initial-state mass
        'Mo':           pc.mpiphys,     # final-state mass
        'sigma':        .5,             # sigma for prior in algorithm
        'Kp':           4,              # target Kp (BGL truncation) - can be changed later
        'K0':           4,              # target K0 (BGL truncation) - can be changed later
        'tstar':        '29.349570696829012', # value of t*
        't0':           'self.tstar - np.sqrt(self.tstar*(self.tstar-self.tm))', # definition of t0
        'chip':         pc.chip_Btopi,  # susceptibility fp
        'chi0':         pc.chi0_Btopi,  # susceptibility f0
        'mpolep':       [pc.mBstar],    # fplus pole
        'mpole0':       [],             # fzero pole (no pole for BstoK)
        'N'    :        N,              # number of desired samples
        'outer_p':      [3./2,'48*np.pi',3,2], # specs for outer function fp
        'outer_0':      [3./2,'16*np.pi/(self.tp*self.tm)',1,1], # specs for outer function f0
        'seed':         123,            # RNG seed
```

```
input_data = {
        'RBCUKQCD 23 lat':
        {
         'data type':   'ff',
         'label'        :'RBC/UKQCD 23',
         'Np':          2,
         'N0':          3,
         'qsqp':        np.array([17.60,23.40]),
         'qsq0':        np.array([17.60,20.80,23.40]),
         'fp':          fparray,
         'f0':          f0array,
         'Cff':         cov_array
        }
}
```

[taken from A.Jüttner's talks at LHCb Implications 2023]

Improvements and Summary

Literature overview $B \to \pi$, $B_s \to K$

	heavy	light	$N_{\rm ens}$	N_a	$M_\pi^{\rm min}/MeV$
$B \to \pi$					
HPQCD'06	NRQCD	asqtad	6	2	400
RBC/UKQCD'15	RHQ	DWF	5	2	300
FNAL/MILC'15	Fermilab	asqtad	12	**4**	255
JLQCD'22	**DWF**[2]	DWF	11	3	230
$B_s \to K$					
HPQCD'14	NRQCD	asqtad	5	2	260
FNAL/MILC'19	Fermilab	asqtad	6	3	255
RBC/UKQCD'23	RHQ	DWF	6	3	270

- work in progress by several collaborations
- typically improving over **continuum limit, i.e.** N_a, $M_\pi^{\rm min}$
- expect more fully-relativistic results to come

[2] $m_h^{\rm max} \approx 2.44 m_c$

Example: Taming discretisation effects via massive NPR?

Strategy:

1. renormalise $\mathcal{O}_{\rm bare}$ in *regularisation independent* scheme
2. take the continuum limit
3. perturbatively match from this scheme to $\overline{\rm MS}$

Caveat: Typical schemes (RI/(S)MOM) defined in massless limit of QCD

- mass independent renormalisation constants
- introduces discretisation effects scaling with $(am_q)^n$.
- on typical lattices $am_c \sim 0.2$ $am_b \lesssim 1$. Large cut-off effects!

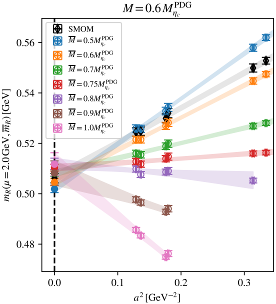
for HQ: $a \to 0$ **often limit hardest to control:**

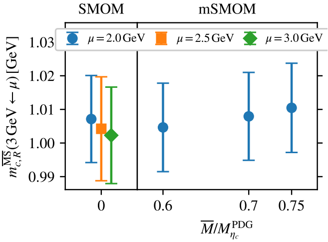
Way out: RI/mSMOM (defined at finite $\overline{m} \geq 0$)? [Boyle et al., 2016]

ADVANTAGE: Different masses at which the scheme is defined.
Tunable! - different approaches to $a \to 0$?
Possible to choose this to reduce cut-off effects?

JTT et al.'24: First numerical implementation of mSMOM.

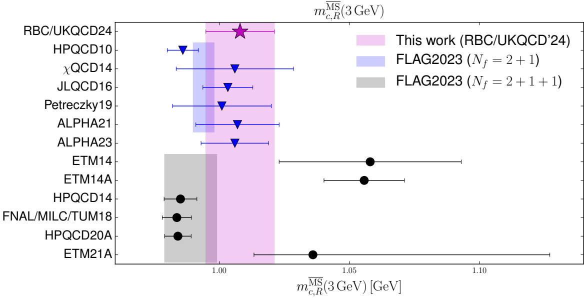
modified approach to the continuum [JTT et al., PRD **110**, (2024) 5, 054512]





← continuum limit results still in different schemes! Values cannot be directly compared from plot.

- Very different CL approaches
- Free choice of tunable parameter



⇒ **Many possible applications in flavour and beyond!**

Summary

The status quo is not great:

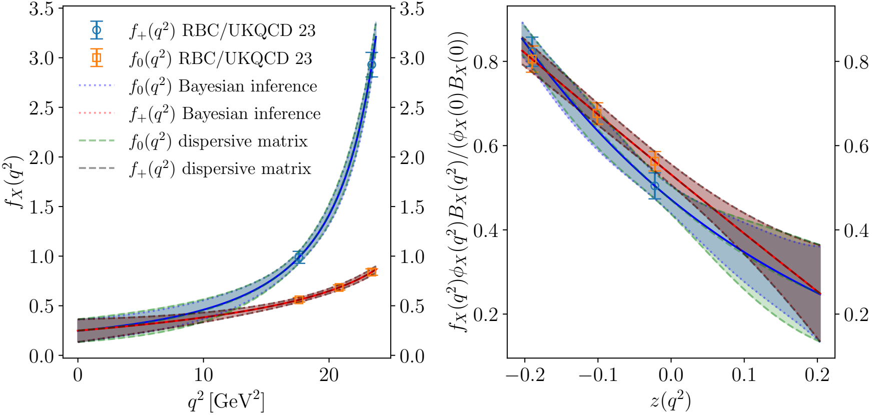
Theory calculations of the same quantities should not disagree!

BUT:

- These are hard and involved calculations.
- Existing literature is **highly complementary**.
- **Many works in progress** $\Rightarrow$ might resolve tensions/provide understanding.
- Suggestion of **benchmark quantities** to scrutinise less complicated intermediate results [JTT, Della Morte, EPJ-ST 233 (2024) 2, 253-27].
- Expect **improvement and scrutiny** of all aspects of these calculations (HQET-inspired vs fully relativistic; excited states; chiral-continuum extrapolation; z-expansions,...).

ADDITIONAL SLIDES

Results III: Bayesian Inference vs Dispersive Matrix Method



- BI and DM produce equivalent results
- BI simpler to implement + easily to combine with multiple data sets
- BI provides numerical value of the coefficients