Matching Perturbation Theory and Lattice

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Lattice Meets Continuum, Siegen, 30/Sep/24

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Lattice & Continuum

- QCD has asymptotic freedom:
  - Factorise short distance from long distance physics
  - presence of UV divergences: Factorisation is scheme dependent
  - a and $\epsilon = (4 - d)/2$ as Lattice / Continuum UV regulator
  - Flavour physics: $\overline{\mathrm{MS}}$ scheme
  - Need schemes, that can be implemented with both methods

Renorm non-singlet $O_\Gamma(x) = \bar\psi(x)\Gamma\psi(x)$

x-space

$$\lim_{a\to 0}\langle O^X_\Gamma(x) O^X_\Gamma(0)\rangle|_{x^2=x_0^2} = \langle O_\Gamma(x_0) O_\Gamma(0)\rangle^{\text{free}}_{\text{cont}}$$

- $O^X_\Gamma(x, x_0) = Z^X_\Gamma(x_0) O_\Gamma(x)$ and $a \ll x_0 \ll \Lambda^{-1}_{\text{QCD}}$

g. flow

$$Z_\chi \bar\chi(t,x)\Gamma\chi(t,x) \overset{t\to 0}{\sim} \zeta_1(t) O_\Gamma(x) + \mathcal{O}(t)$$

- Lattice: a $\to$ 0 then t $\to$ 0
- Continuum: modified feynman rules (exponential)

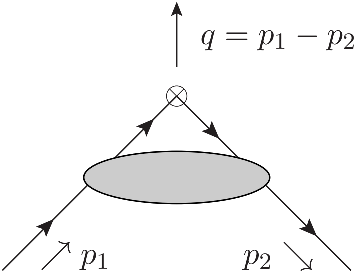
sMOM

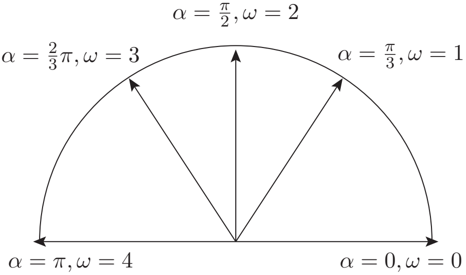
$$\Lambda_\Gamma = S^{-2}\int d^4x_1 d^4x_2 e^{i(p_1x_1 - p_2x_2)}\langle T\psi(x_1) O_\Gamma(0)\psi(x_2)\rangle$$

- Project Λ: $\lambda_{\Gamma,B}(p_1^2, p_2^2, (p_1 - p_2)^2) Z_q^{-1} Z_\Gamma \to \text{tree}$

Interpolating MOM

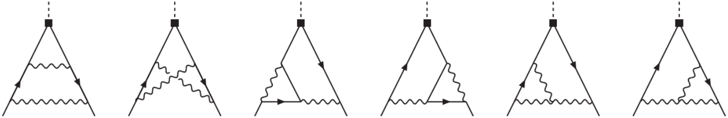
- Ward Identity for non-singlet bilinear: $Z_P = Z_m^{-1}$
- fixed by: $\lambda_R(p_1^2, p_2^2, (p_1 - p_2)^2) = Z_q^{-1} Z_P \mathrm{tr}\,[\Lambda\gamma_5] = 12$
- for $q^2 = (p_1 - p_2)^2 = 0$ contribution from pion pole
- use $p_1^2 = p_2^2 = -\mu^2$ & $(p_1 - p_2)^2 = -\omega\mu^2$





Perturbation Theory

$$C_m^{(\gamma)} = Z_m^{\overline{\mathrm{MS}}}/Z_m^{(\gamma)}(\omega) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_m^{(\gamma,1)} + \frac{\alpha_s^2(\mu)}{(4\pi)^2} C_m^{(\gamma,2)} + \ldots$$



ω	$C_m^{(\gamma,1)}$	$C_m^{(\gamma,2)}$	$C_m^{(\gamma,3)}$
.5	-3.248	-89.07 + 7.571 N_f	
1	-1.979	-55.032 + 6.162 N_f	-2086 -362.6 N_f + 6.7220 N_f^2
2	-0.098	-6.829 + 4.072 N_f	
4	2.575	62.576 + 1.102 N_f	

NNLO result for $\omega = 0 \ldots 4$ 1004.3997

N[3LO] result for $\omega = 1$ 2002.12758

Interpolating Momentum kinematics 2112.11140

- $p_1 = (\mu,0,0,0)$ & $p_2 = \mu(\cos\alpha, \sin\alpha,0,0)$ on Lattice with twisted boundary conditions [$p_1 = (2\pi l/L,0,0,0)$]
- $G_x(p) = \sum_y D^{-1}(x,y)e^{ip\cdot(y-x)}$ Momentum source propagator
  (From solving $\sum_x D(y,x)\tilde{G}_x(p) = e^{ip\cdot y}$)
- Amputed Green's function for $O_\Gamma = \bar{\psi}\Gamma\psi$
  $\Pi_\Gamma = \langle G^{-1}(-p_2)\rangle \sum_x \langle G_x(-p_2)\Gamma G_x(p_1)\rangle\langle G^{-1}(p_1)\rangle$
- Renormalisation conditions on projected result

Lattice Input

Landau gauge fixed 2+1 domain-wall configuration

- 24^3: a^{-1} = 1.785(5)GeV and $Z_V = Z_A = 0.71651(46)$
- 32^3: a^{-1} = 2.383(9)GeV and $Z_V = Z_A = 0.74475(12)$

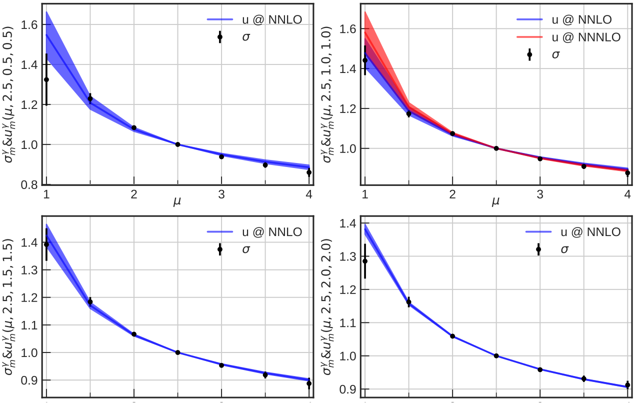
$$Z_q(\mu,\omega) = Z_V \lim_{m\to 0} [\Lambda_V]_{\mathrm{IMOM}} \qquad Z_m(\mu,\omega) = \frac{1}{Z_V} \lim_{m\to 0} \left[\frac{\Lambda_S}{\Lambda_V}\right]_{\mathrm{IMOM}}$$

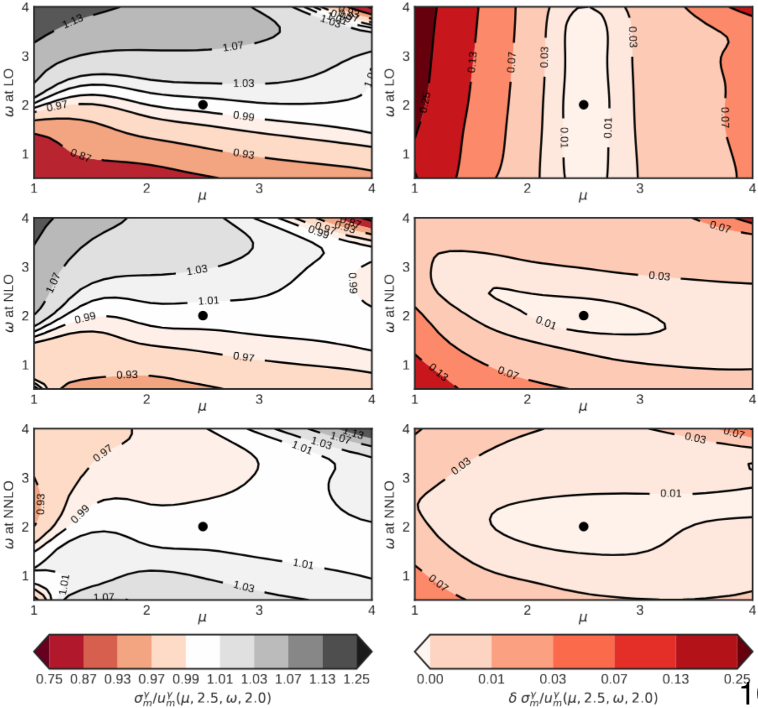
with projectors:

$$\Lambda_V^{(\gamma_\mu)} = \frac{1}{48}\mathrm{Tr}[\gamma_\mu \Pi_{V^\mu}] \qquad \Lambda_V^{(\not{q})} = \frac{q^\mu}{12q^2}\mathrm{Tr}[\Pi_{V^\mu}]$$

Lattice & Continuum 2112.11140

- Lattice: $\sigma_i^\gamma(\mu,\mu_0,\omega,\omega_0) = \lim_{a^2\to 0}\lim_{m\to 0}\frac{Z_i^\gamma(a,\mu,\omega)}{Z_i^\gamma(a,\mu_0,\omega_0)}$
- Continuum: $u_i^\gamma(\mu,\mu_0,\omega,\omega_0) = \lim_{\epsilon\to 0}\frac{Z_i^\gamma(\epsilon,\mu,\omega)}{Z_i^\gamma(\epsilon,\mu_0,\omega_0)}$ + RGE

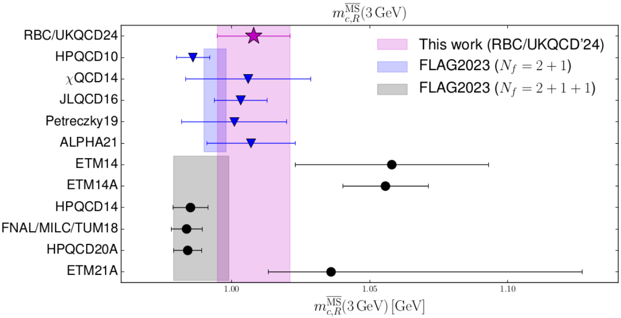




ω at LO
ω at NLO
ω at NNLO
μ
0.75 0.87 0.93 0.97 0.99 1.01 1.03 1.07 1.13 1.25
$\sigma_m^\gamma/u_m^\gamma(\mu, 2.5, \omega, 2.0)$
0.00 0.01 0.03 0.07 0.13 0.25
$\delta\ \sigma_m^\gamma/u_m^\gamma(\mu, 2.5, \omega, 2.0)$

Massive SMOM

- Determine charm quark mass in $\overline{\mathrm{MS}}$
- For massive quark at 1-loop 2407.18700



Bilinears in Gauge invariant schemes

- x-space $\lim_{a\to 0}\langle O^X_\Gamma(x) O^X_\Gamma(0)\rangle|_{x^2=x_0^2} = \langle O_\Gamma(x_0) O_\Gamma(0)\rangle^{\text{free}}_{\text{cont}}$
  - 2-loops for NLO hep-lat/0406019
  - higher orders available
- Gradient flow (Calc. setup: 1905.00882)
  - Solve GFF perturbatively: Feynman rules for flowed fields
  - $Z_\chi \bar{\chi}(t,x)\Gamma\chi(t,x) \overset{t\to 0}{\sim} \zeta_1(t) O_\Gamma(x) + \mathcal{O}(t)$
  - NNLO renormalisation 2311.16799
    - Talks by: Harlander, Lange

WET

- Precise matrix elements needed:
  - Precision tests of the SM
  - Constrain New Physics
- Precision requires NNLO
- Two examples:
  - ε_K
  - charged current decays

Charged Current decays

- $K_{\ell 2}$ and $K_{\ell 3}$ extraction of $\lambda = |V_{us}|$: $\Gamma(K^0 \to \pi^- \ell^+ \nu_\ell(\gamma)) = \frac{G_F^2 m_K^5}{128\pi^3} |V_{us}|^2 S_{EW} |f_+^{K^0\pi^-}(0)|^2 I_{K^0\ell}^{(0)} (1 + \delta_{EM}^{K^0\ell} + \delta_{SU(2)}^{K^0\pi^-})$
  - QED: χPT Seng et.al.'2019,Cirigliano et.al.'23 and Lattice Carrasco et.al.'15,DiCarlo et.al.'19
- $|V_{ud}|$, extracted from nuclear β decays Hardy,Towner'20,
  - $|V_{ud}|\tau_n (1 + 3\lambda^2)(1 + \Delta_f)(1 + \Delta_R) = 5283.321(5)s$
  - λ, Δ_f & Δ_R ratio of (axial-)vector coupling, phase space & radiative corrections
- EW corrections in W-Mass scheme [Marciano, Sirlin]
- EFT Approach Gorbahn et.al.'22,Cirigliano et.al.'23
- $\Delta_{CKM} \equiv 1 - |V_{ud}|^2 - |V_{us}|^2 - \mathcal{O}(|V_{ub}|^2) = 0.$

(Effective) Interaction

- $\mathcal{H}(x) = 4\,\frac{G_F}{\sqrt{2}} C_O\; V_{ud}^* O(x)$
- $O(x) = \left(\bar{d}(x)\gamma^\mu P_L u(x)\right)\left(\bar{\nu}_l(x)\gamma_\mu P_L l(x)\right)$
- SD in W-Mass scheme:
  $\frac{1}{k^2} \to \frac{1}{k^2 - M_W^2} - \frac{M_W^2}{k^2 - M_W^2}\frac{1}{k^2} = \gamma_> - \gamma_<$
  - UV poles → Absorbed into G_F from muon decay
  - Combining with SU(3) current algebra → QCD corrections to S_{EW}
  - No scale separation and α^2 log and $1/s_w{}^2$ effects
- EFT: scale separation, $\mathcal{O}(\alpha^2)$, match to Lattice or χ PT

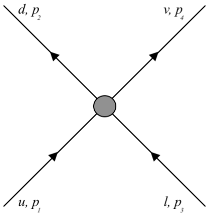
Renormalisation Group Improvement

- RGE: $\mu \frac{d}{d\mu} C(\mu) = \gamma C(\mu)$
- $\gamma = \frac{\alpha(\mu)}{4\pi}\gamma_e + \frac{\alpha^2(\mu)}{(4\pi)^2}\gamma_{ee} + \frac{\alpha}{4\pi}\frac{\alpha_s(\mu)}{4\pi}\left(\gamma_{es} + \frac{\alpha_s(\mu)}{4\pi}\gamma_{ess}\right)$
- Solution: $C(\mu) = J(\mu)u(\mu)u^{-1}(\mu_0)J^{-1}(\mu_0)C(\mu_0)$
  - $u(\mu) = \alpha(\mu)^{\frac{\gamma_e}{2\beta_0}}\alpha_s(\mu)^{-\frac{\alpha}{4\pi}\frac{\gamma_{es}}{2\beta_{0,s}}} \rightarrow (\alpha \ \mathrm{ln})^n$ and $\alpha \ (\alpha_s \ \mathrm{ln})^n$
  - $J(\mu) = 1 + \frac{\alpha(\mu)}{4\pi}\delta J_e + \frac{\alpha}{4\pi}\frac{\alpha_s(\mu)}{4\pi}\delta J_s$
    - $\delta J_e = \frac{\gamma_{ee}}{2\beta_e} - \frac{\beta_{e,1}\gamma_e}{2\beta_{e,0}^2} \rightarrow \alpha \ (\alpha \ \mathrm{ln})^n$
    - $\delta J_s = -\frac{\gamma_{ess}}{2\beta_{s,0}} + \frac{\beta_{s,1}\gamma_{es}}{2\beta_{s,0}^2} \rightarrow \alpha \ \alpha_s \ (\alpha_s \ \mathrm{ln})^n$

Scheme independence

- Scheme dependence through use of naive dimensional regularisation and choice of
  - $E = (\bar{d}\gamma^\mu\gamma^\nu\gamma^\lambda P_L u)(\bar{\nu}_l\gamma_\mu\gamma_\nu\gamma_\lambda P_L l) - 4(4 - a\epsilon - b\epsilon^2)$
- Scheme independent quantities
  - Wilson coefficient: $\hat{C} = u^{-1}(\mu)J^{-1}(\mu)C(\mu)$
  - Flavour decoupling: $\hat{M}_{f\downarrow} = u_{f-1}^{-1}(\mu)J_{f-1}^{-1}(\mu)J_f(\mu)u_f(\mu)$
  - Matrix element: $\hat{m} = m(\mu)J(\mu)u(\mu)$

Lattice Renormalisation



- off-shell renormalisation conditions
  - $\mathrm{RI}^{(\prime)} - \mathrm{MOM}$: $p_1 = p_2 = p_3 = p_4 = p, \quad p^2 = -\mu^2$
  - $\mathrm{RI} - \mathrm{SMOM}$:
    $p_1 = p_3, \quad p_2 = p_4, \quad p_1^2 = p_2^2 = -\mu^2, \quad p_1 \cdot p_2 = -\frac{1}{2}\mu^2$
  - Choose projectors so that $Z_O = 1 + \mathcal{O}(\alpha)$ 2209.05289

Choice of Projector

- $\sigma^A \equiv \frac{1}{4\,p^2}\mathrm{Tr}(S_A^{-1}(p)\not{p}) \overset{A=RI}{=} 1, \quad \lambda^A \equiv \Lambda^A_{\alpha\beta\gamma\delta}\mathcal{P}^{\alpha\beta\gamma\delta}$

- $\Lambda^b = \Lambda^{b,\mu}(p) \otimes \gamma_\mu P_L + \mathcal{O}(\alpha)$, only 2 form factors in RIMOM
  $\Lambda^{b,\mu}(p) = F_1(p)\gamma^\mu P_L + F_2(p)p^\mu \not{p}/p^2 P_L$

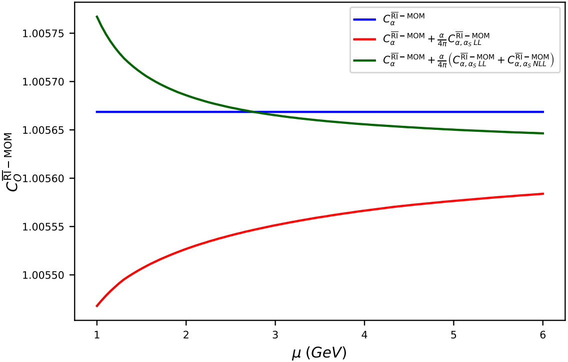
- Choose $\mathcal{P} = -\frac{1}{12\,p^2}(\not{p}P_R \otimes \not{p}P_R + p^2/2\gamma^\nu P_R \otimes \gamma_\nu P_R)$

  - Projects out $F_1(p) \rightarrow$ no pure QCD corrections

- $C_O^{\overline{\mathrm{MS}}\rightarrow RI} = \lambda^{\overline{\mathrm{MS}}}\left(\sigma_u^{\overline{\mathrm{MS}}}\sigma_d^{\overline{\mathrm{MS}}}\sigma_\ell^{\overline{\mathrm{MS}}}\right)^{1/2}$

$\overline{RI}$ and $\overline{MS}$ Wilson coefficients

Including 2-loop EW matching and 3-loop RGE [MG, SJ, Moretti, EM]



Small impact for V_{us}, but larger for V_{ud} for $\square_{\gamma W}$ + NLO

CP violation in $K \to \pi\pi$

- Experimental definition using $\eta_{ij} = \dfrac{\langle \pi^i \pi^j | K_L \rangle}{\langle \pi^i \pi^j | K_S \rangle}$

$$\epsilon_K = (2\eta_{+-} + \eta_{00})/3\ , \qquad \epsilon' = (\eta_{+-} - \eta_{00})/3$$

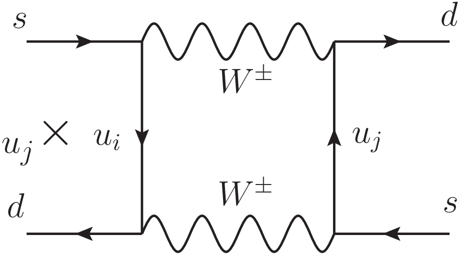
- ϵ_K theory expression $\epsilon_K \simeq \dfrac{\langle (\pi\pi)_{I=0} | K_L \rangle}{\langle (\pi\pi)_{I=0} | K_S \rangle} =$

$$e^{i\phi_\epsilon} \sin\phi_\epsilon \frac{1}{2} \arg\left(\frac{-M_{12}}{\Gamma_{12}} \right) = e^{i\phi_\epsilon} \sin\phi_\epsilon \left(\frac{\mathrm{Im}(M_{12})^{Dis}}{\Delta M_K} + \xi \right)$$

$$\langle K^0 | H^{|\Delta S|=2} | \bar{K}^0 \rangle \to \mathrm{Im}(M_{12})^{Dis}\ ,\ \frac{\mathrm{Im}\langle (\pi\pi)_{I=0} | K^0 \rangle}{\mathrm{Re}\langle (\pi\pi)_{I=0} | K^0 \rangle} \to \xi\ ,\ \phi_\epsilon \equiv \arctan \frac{\Delta M_K}{\Delta \Gamma_K / 2}$$

Kaon Mixing: CKM Structure

$$\sum_{u_i,u_j}^{u,c,t} \lambda_{u_i}\lambda_{u_j} \times$$



	Im	Re	$\mathcal{O}$
λ_t^2	$\sim \lambda^{10}$	$\sim \lambda^{10}$	m_t^2/M_W^2
$\lambda_c\,\lambda_t$	$\sim \lambda^6$	$\sim \lambda^6$	$m_c^2/M_W^2 \ln(m_t/m_c)$
λ_c^2	$\sim \lambda^6$	$\sim \lambda^2$	m_c^2/M_W^2
$\lambda_u\,\lambda_t$	$\sim \lambda^6$	$\sim \lambda^6$	$m_c^2/M_W^2 \ln(m_t/m_c)$
λ_u^2	0	$\sim \lambda^2$	m_c^2/M_W^2

Where $\lambda_i = V_{id}V_{is}^*$, $\lambda \equiv |V_{us}| \sim 0.2$ and we eliminated either:
$\lambda_u = -\lambda_c - \lambda_t$ or $\lambda_c = -\lambda_u - \lambda_t$.

$\Delta S = 2$ Hamiltonian - Phase (In)Dependence

- Recall $\epsilon_K \propto \arg(-M_{12}/\Gamma_{12})$

- Trick: pull out λ_u^* and $(\lambda_u^*)^2$ from $H^{\Delta S=1}$ and $H^{\Delta S=2}$:

- Rephaseing invariant: $\lambda_i \lambda_j^* = V_{id} V_{is}^* V_{jd}^* V_{js}$
- One Operator: $Q_{S2} = (\bar{s}_L \gamma_\mu d_L) \otimes (\bar{s}_L \gamma^\mu d_L)$

$$\mathcal{H}_{f=3}^{\Delta S=2} = \frac{G_F^2 M_W^2}{4\pi^2 (\lambda_u^*)^2} Q_{S2} \Big\{ f_1 C_1(\mu) + iJ\, [f_2\, C_2(\mu) + f_3\, C_3(\mu)] \Big\} + \text{h.c.}$$

- $f_1 = |\lambda_u|^4$, $f_2 = 2\mathrm{Re}(\lambda_t \lambda_u^*)$ and $f_3 = |\lambda_u|^2$

ImM_{12} without ΔM_K pollution

- Using CKM unitarity and the PDG convention we can also write (as used in Lattice [Christ et.al.]):

$$\mathcal{H}^{\Delta=2}_{f=3} = \frac{G_F^2 M_W^2}{4\pi^2}\left[\lambda_u^2 C^{uu}_{S2}(\mu) + \lambda_t^2 C^{tt}_{S2}(\mu) + \lambda_u\lambda_t C^{ut}_{S2}(\mu)\right]Q_{S2} + \text{h.c.}$$

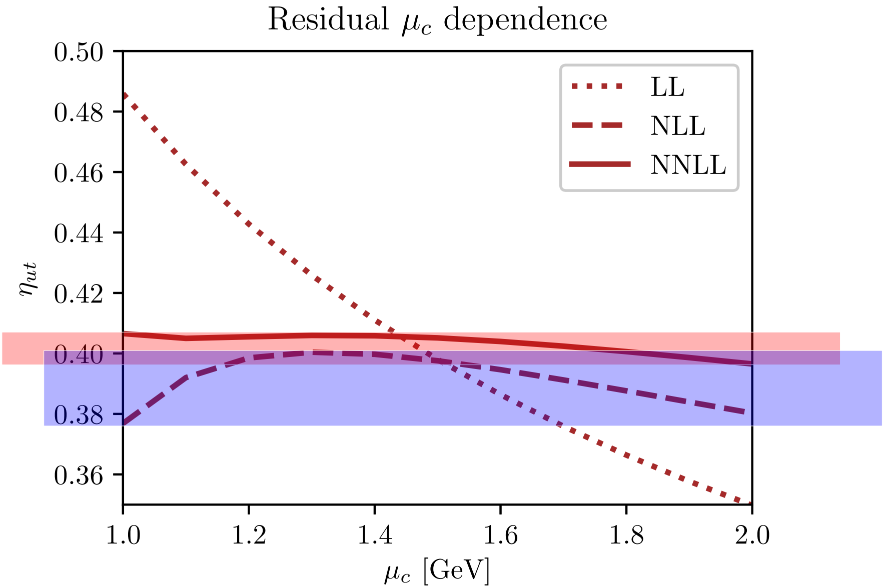
- Now real ReM_{12} and ImM_{12} are disentangled

$$C^{uu}_{S2} \equiv \mathcal{C}_1, \quad C^{tt}_{S2} \equiv \mathcal{C}_2, \quad C^{ut}_{S2} \equiv \mathcal{C}_3$$

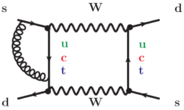
$$\begin{aligned}\mathcal{C}_3 \;&\leftarrow\; (A_{tu} - A_{tc} + A_{cc} - A_{cu}) \leftarrow \\ &\leftarrow\; (A_{uu} - 2A_{cu} + A_{cc}) - (A_{tc} - A_{tu} + A_{uu} - A_{cu})\end{aligned}$$

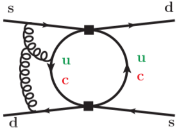
- NNLO QCD corrections Brod et.al.'10,Brod et.al.'11 to C^{ct}_{S2} absorbed into η_{ut} Brod et.al.'19

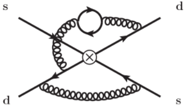
Residual scale dependence

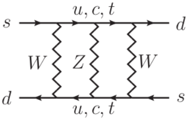


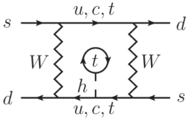
Further Improvements

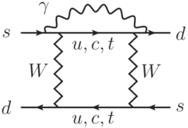


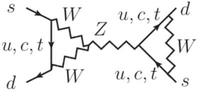








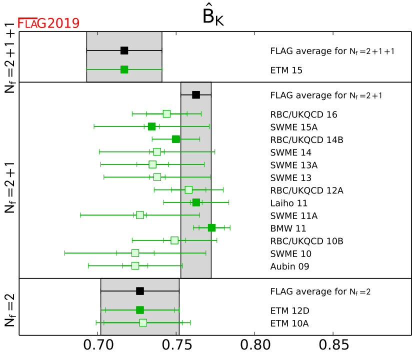




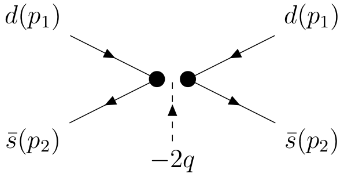
$$|\epsilon_K| = 2.170\,(65)_{\text{pert.}}(76)_{\text{non-pert.}}(153)_{\text{param.}} \times 10^{-3}$$

- $\hat{B}_K = \dfrac{3}{2f_K^2 M_K^2}\langle \bar{K}^0 | Q^{|\Delta S=2|} | K^0 \rangle u^{-1}(\mu_{\rm had})$ from Lattice

$\hat{B}_K$

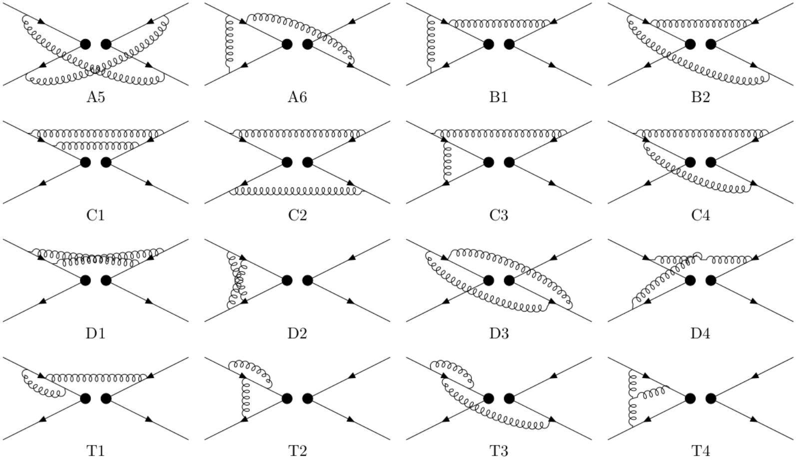


E.g. RBC UKQCD uses SMOM kinematics



- Define momentum-space subtraction schemes
- Projected renormalised Green's function $P_{(\gamma_\mu)}(\Lambda_R) \to$
- $Z_{Q_{S2}}^{(\gamma_\mu,\gamma_\mu)} = \left(Z_q^{(\gamma_\mu)}\right)^2 \dfrac{1}{P_{(\gamma_\mu)}(\Lambda_B)}$
- $Z_{Q_{S2}}^{(\gamma_\mu,\gamma_\mu)} / Z_{Q_{S2}}^{\overline{\mathrm{MS}}}$ converts between Lattice and continuum

SMOM $\hat{B}_K$ @ NNLO



- Use projectors to find $\Lambda^{ijkl}_{\alpha\beta\gamma\delta}$ at 2-loops
- Integrals reduce to scalar off-shell 4-point functions

Combine with Lattice values

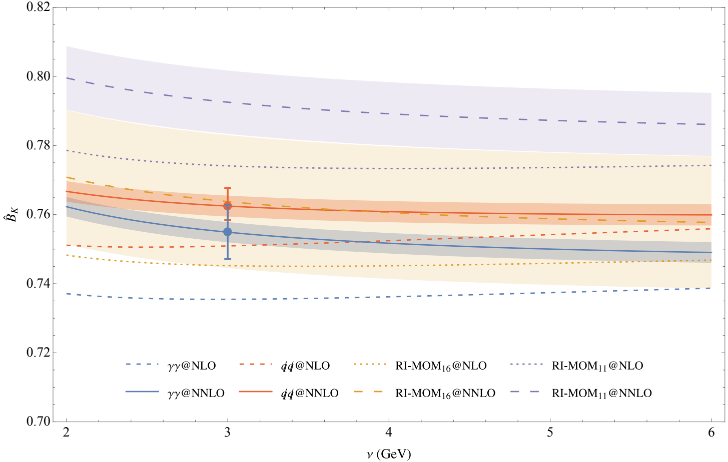
- The result should be independent of the matching scale

$$B_K^{(X,Y)}(|p|)C_{B_K}^{(X,Y)}(|p|,\mu)u^{-1}(\mu)u(\mu_0)$$

- study scale variation setting $\mu_0 = |p|$

- Compute $\hat{B}_K$ for f=3,4 for SMOM, RIMOM, RI'MOM

$\hat{B}_K$ at NNLO

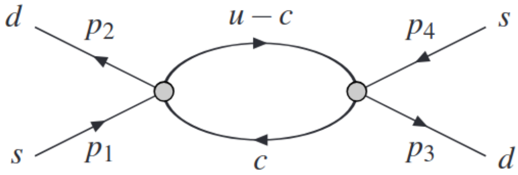


Numerics of NNLO result by [MG, Kvedaraitė, Jäger]

Gauge invariant schemes

- x-scheme
  - (GIRS) Gauge-invariant renormalization scheme
  - GIRS: 2406.08065 NLO conversion factor, including NP operators
  - Spectator effects in b-hadrons 2212.09275
- gradiant flow
  - NNLO weak effective Hamiltonian 22201.08618
  - Matrix elements for Mixing and Lifetime 2310.18059

Long distance contributions



- $p_1 = (\mu, \mu, 0, 0)/\sqrt{2}$, $p_2 = (\mu, 0, \mu, 0)/\sqrt{2}$, ...
- Lattice: Subtract $Q_{S2}(1/a)$ with $X^{Lat}(1/a,\mu)$
- Same continuum → subtract Q_{S2} with $Y^{MS}(1/a, \mu)$
- Proof of principle (unphysical masses) 2309.01193:
  - $\varepsilon_K^{LD}(\mu_{RI} = 2.11 GeV) = 0.199(0.078)e^{i\phi}10^{-3}$
    - add $-0.085e^{i\phi}10^{-3}$ for $\overline{MS} conversion$

Double insertions

- x-space too many insertions
- Double insertions probably simplest in SMOM schemes
- Similar renormalisation for LD contributions to $K \to \pi\nu\bar{\nu}$
  - Near-physical pions: 1910.10644: small effect
  - Use 'old' χPT estimate for $\Delta P_{c,u} = 0.04 \pm 0.02$
- 2311.02923 using UTfit CKM parameters:

$$
\begin{aligned}
\mathrm{BR}(K^+ \to \pi^+\nu\bar{\nu}) &= 8.38(17)_{SD}(25)_{LD}(40)_{para.} \times 10^{-11}\,, \\
\mathrm{BR}_{NA62}(K^+ \to \pi^+\nu\bar{\nu}) &= 13.0^{+3.3}_{-2.9} \times 10^{-11} \\
\mathrm{BR}_{Brookhaven}(K^+ \to \pi^+\nu\bar{\nu}) &= 17.3^{+11.5}_{-10.5} \times 10^{-11}
\end{aligned}
$$

Conclusion

- Fantastic progress in recent years
  - Lattice $\leftrightarrow$ NNLO continuum
- Progress is needed for precision tests of SM
- Different schemes are available
  - Application case dependent
  - Allows cross checks