3rd Edition Lattice meets Continuum - Siegen

October 3, 2024

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UNIVERSITY OF ROME
School of Mathematics, Physical and Natural Sciences

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ROMA TOR VERGATA

Extended Twisted Mass Collaboration

Alessandro De Santis

Inclusive charm and tau decays

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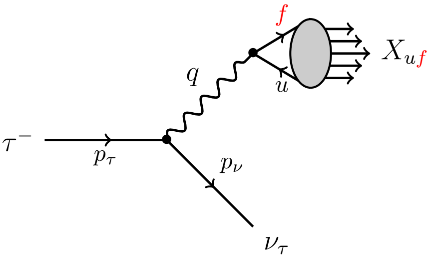


PHYSICAL REVIEW D 108, 074513 (2023)

Editors' Suggestion

Inclusive hadronic decay rate of the τ lepton from lattice QCD

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(Extended Twisted Mass Collaboration)

PHYSICAL REVIEW LETTERS 132, 261901 (2024)

Inclusive Hadronic Decay Rate of the τ Lepton from Lattice QCD: The $\bar{u}s$ Flavor Channel and the Cabibbo Angle

Constantia Alexandrou,[1,2] Simone Bacchio,[2] Alessandro De Santis,[3] Antonio Evangelista,[3] Jacob Finkenrath,[4] Roberto Frezzotti,[3] Giuseppe Gagliardi,[5] Marco Garofalo,[6] Bartosz Kostrzewa,[7] Vittorio Lubicz,[8] Simone Romiti,[6] Francesco Sanfilippo,[5] Silvano Simula,[5] Nazario Tantalo,[3] Carsten Urbach,[6] and Urs Wenger[9]

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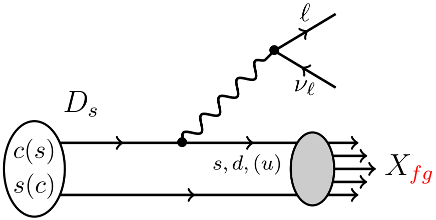

- Christiane Groß's talk on Wednesday
- Proceeding Lattice 2024 (in preparation)
- Paper (in preparation)
- Preliminary results

Inclusive tau decays

Motivations

- Inclusive hadronic τ decays are phenomenologically interesting probes of both the leptonic and hadronic flavour sectors of the Standard Model and allow for unitary checks of the CKM matrix

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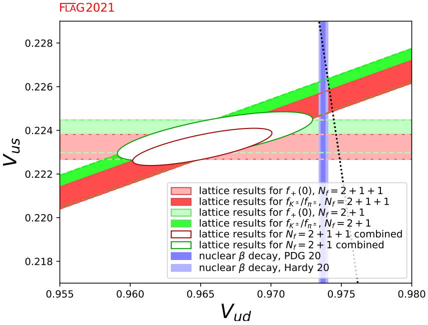

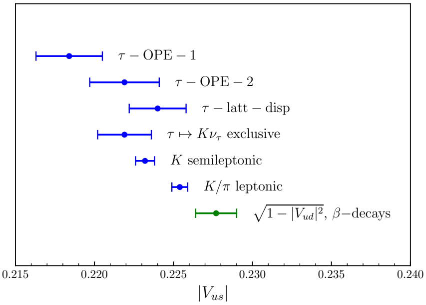

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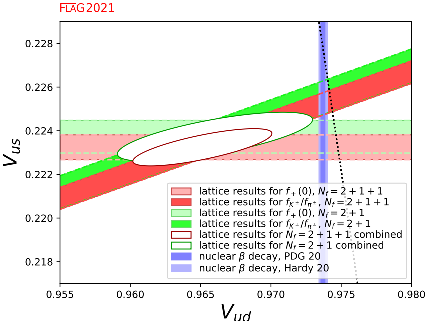

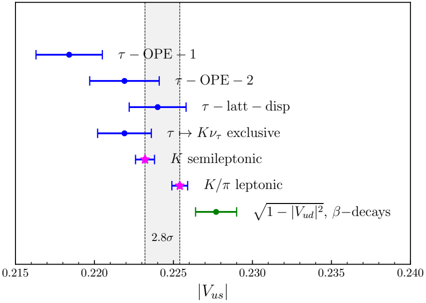

There are important tensions

- between semileptonic kaon decays and the ratio of the leptonic decay rates of kaons and pions

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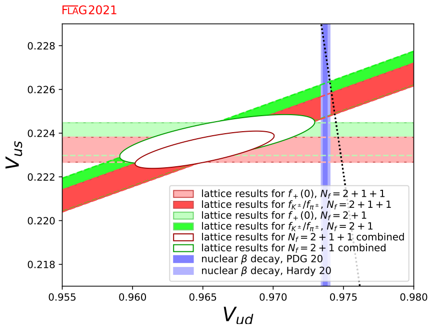

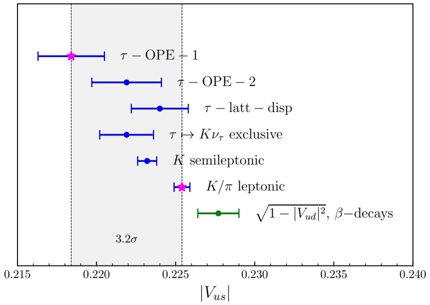

There are important tensions

- ▷ between OPE and K/π

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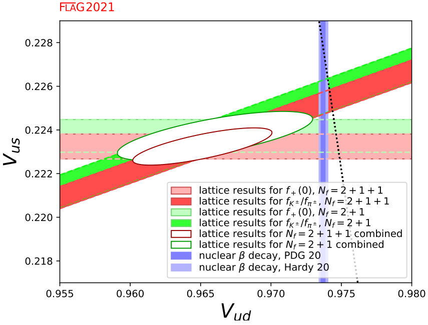

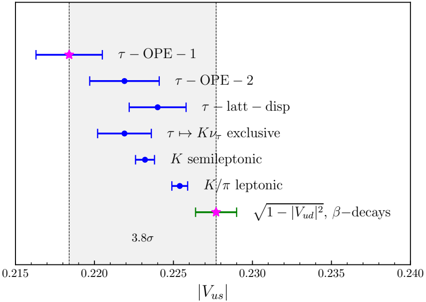

There are important tensions

- between OPE and unitarity $V_{us} \simeq \sqrt{1-|V_{ud}|^2}$ with $|V_{ud}| = 0.97373(31)$ ($\beta-$decays)

A direct lattice computation of the inclusive $\tau \mapsto X_{uf}\nu_\tau$ has been reckoned unfeasible for several years

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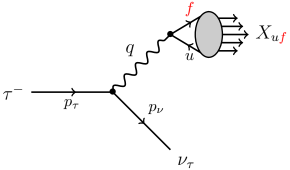
we did it

Kinematics

Amplitude in the Fermi effective theory

$$\mathcal{A}(\tau \mapsto X_{uf}\nu_\tau) = \frac{G_F\sqrt{S_{\rm EW}}}{\sqrt{2}} V_{uf} \langle \nu_\tau | J^\alpha_{\nu_\tau \tau}(0) | \tau \rangle \langle X_{uf} | J^{\alpha\dagger}_{uf}(0) | 0 \rangle$$



Electroweak V-A currents

$$J^\alpha_{\nu_\tau \tau} = \bar{\nu}_\tau \gamma^\alpha (1-\gamma_5)\tau \qquad J^\alpha_{uf} = \bar{u}\gamma^\alpha(1-\gamma_5) f$$

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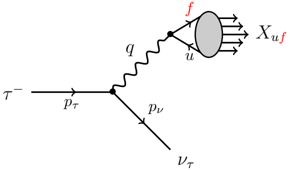

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The inclusive decay rate is

$$\Gamma^{(\tau)}_{uf} = \Gamma(\tau \mapsto X_{uf}\nu_\tau) = \frac{G_F^2 S_{\rm EW}|V_{uf}|^2}{4m_\tau}\int \frac{{\rm d}^3 p_\nu}{(2\pi)^3 2E_\nu} L_{\alpha\beta}(p_\tau, p_\nu)\rho^{\alpha\beta}_{uf}(q), \qquad q = p_\tau - p_\nu$$

with ρ **spectral density** of two hadronic weak currents

$$\rho^{\alpha\beta}_{uf}(q) = \langle 0 | J^\alpha_{uf}(0)(2\pi)^4\delta^4(\mathbb{P}-q)J^{\beta\dagger}_{uf}(0)|0\rangle = (q^\alpha q^\beta - q^2 g^{\alpha\beta})\rho_{\rm T}(q^2) + q^\alpha q^\beta \rho_{\rm L}(q^2)$$

In terms of the dimensionless variable $s = \frac{q^2}{m_\tau^2}$

$$\Gamma_{uf}^{(\tau)} = \frac{G_F^2 S_{\rm EW} |V_{uf}|^2 m_\tau^5}{32\pi^2} \int_0^1 {\rm d}s (1-s)^2 \left[(1+2s) \rho_{\rm T}(s) + \rho_{\rm L}(s) \right],$$

Normalizing by $\Gamma(\tau \mapsto e \bar{\nu}_e \nu_\tau) = \frac{G_F^2 m_\tau^2}{192\pi^3}$

$$R_{uf}^{(\tau)} \equiv \frac{\Gamma(\tau \mapsto X_{uf} \nu_\tau)}{\Gamma(\tau \mapsto e \bar{\nu}_e \nu_\tau)} = 6\pi S_{\rm EW} |V_{uf}|^2 \int_0^1 {\rm d}s (1-s)^2 \left[(1+2s) \rho_{\rm T}(s) + \rho_{\rm L}(s) \right]$$

with closure of the phase-space at $q^2 = m_\tau^2$

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We need the non-perturbative hadronic spectral functions $\rho_{\rm T}$ and ρ_L

Spectral densities from Euclidean correlators

$\rho_{uf}^{\alpha\beta}$ is the spectral density associated with the **two-point Euclidean correlation function**

$$C^{\alpha\beta}(t,\boldsymbol{q}) = \int \mathrm{d}^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \langle 0| J^{\alpha}_{uf}(x) J^{\beta\dagger}_{uf}(0) |0\rangle = \int_0^\infty \frac{\mathrm{d}E}{2\pi} e^{-Et} \rho^{\alpha\beta}_{uf}(E,\boldsymbol{q})$$

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This is challenging
We know how to overcome the problem

We first extend the upper integration bound to infinity

$$R_{uf}^{(\tau)} = 12\pi S_{\mathrm{EW}} \frac{|V_{uf}|^2}{m_\tau^3} \int_0^\infty \mathrm{d}E \left[K_{\mathrm{T}}\left(\frac{E}{m_\tau}\right) E^2 \rho_{\mathrm{T}}(E^2) + K_{\mathrm{L}}\left(\frac{E}{m_\tau}\right) E^2 \rho_{\mathrm{L}}(E^2) \right] \qquad \sqrt{s} = \frac{E}{m_\tau}$$

by introducing the convolution kernels

$$K_{\mathrm{L}}(x) = \frac{1}{x}(1 - x^2)^2 \Theta(1 - x)$$

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We're almost there, to apply our method we need to regularize the non-analiticity at $x = 1$

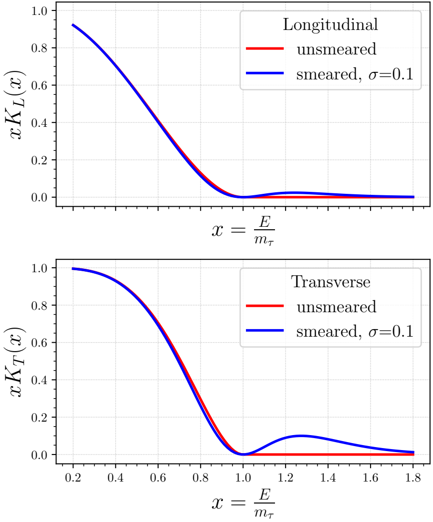
The smeared $R_{uf}^{(\tau)}(\sigma)$

- The convolution kernels are replaced by smeared versions

$$\Theta(1-x) \mapsto \Theta_{\sigma}(1-x) = \frac{1}{1+e^{-(1-x)/\sigma}}$$

- Smearing is needed to deal with $\mathcal{C}^{\infty}$ kernels
- Axiomatically, spectral densities are distributions and **must** be smeared
- $\sigma \mapsto 0$ must be done after $L \mapsto \infty$

$$R_{uf}^{(\tau)} = \lim_{\sigma \mapsto 0} \lim_{a \mapsto 0} \lim_{L \mapsto \infty} R_{uf}^{(\tau)}(\sigma)$$



The HLT method [Hansen M.,Lupo A.,Tantalo N. (2019)]

Our target is the convolution of the spectral density with a given kernel

$$\rho[K](E) = \int_0^\infty \mathrm{d}\omega\, K(E;\omega)\rho(\omega)$$

Stone–Weierstrass **theorem**: Schwartz kernels admit exact polynomial expressions. Choosing the exponential basis

$$K(E;\omega) = \sum_{n=1}^{\infty} g_K(n) e^{-na\omega}$$

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The exact (**model independent**) solution is given by

$$\rho[K](E) = \sum_{n=1}^{\infty} g_K(n) C(an)$$

Of course in Lattice QCD

- $C(an)$ is affected by statistical noise
- The number of time slices is finite: $n = 0, 1, 2, \cdots, T$

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$$\rho[K](E) \simeq \sum_{n=1}^{T} g^\star_K(n) C(an) + \delta\rho$$

Up to systematic and statistical errors, a linear estimator for our solution is then

$$\frac{R^{(\tau,\mathrm{I})}_{uf}(\sigma)}{|V_{uf}|^2} = \frac{24\pi^2 S_{\mathrm{EW}}}{m_\tau^3} \sum_{n=1}^{T} g^\star_{\mathrm{I}}(n) C_{\mathrm{I}}(an) \qquad K^\sigma_{\mathrm{I}}(E) \simeq \sum_{n=1}^{T} g^\star_{\mathrm{I}}(n) e^{-naE} \qquad \mathrm{I} = \mathrm{L}, \mathrm{T}$$

$$W[\boldsymbol{g}] = (1 - \lambda)\frac{A_\alpha[\boldsymbol{g}]}{A_\alpha[\mathbf{0}]} + \lambda B[\boldsymbol{g}]$$

Compute $\boldsymbol{g}$ by searching for optimal balance between systematic and statistical error

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$$B[\boldsymbol{g}] = \boldsymbol{g}^T \cdot \mathrm{Cov}[C] \cdot \boldsymbol{g}$$

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$$A_\alpha[\boldsymbol{g}] = \int_{E_{\mathrm{min}}}^{E_{\mathrm{max}}} \mathrm{d}E\, e^{\alpha a E} \left[\sum_{n=1}^{T} g(n) e^{-anE} - K_{\mathrm{I}}\left(\frac{E}{m_\tau}\right) \right]^2$$

- A_α family of weighted L_2-norms
- $E_{\mathrm{min}} > m_{\mathrm{lightest}}$
- $E_{\mathrm{max}} = r_{\mathrm{max}}/a$ UV cutoff

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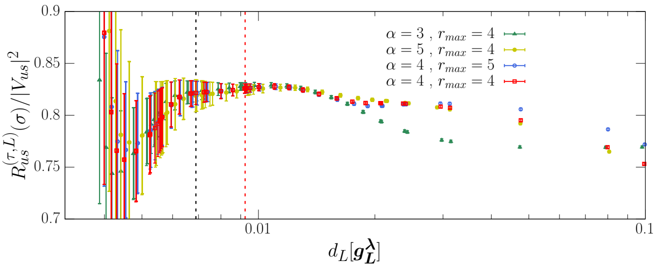
Algorithmic parameters to tune: $\{\lambda, \alpha, r_{\max}\}$ at fixed $\{T, E_{\min}\}$

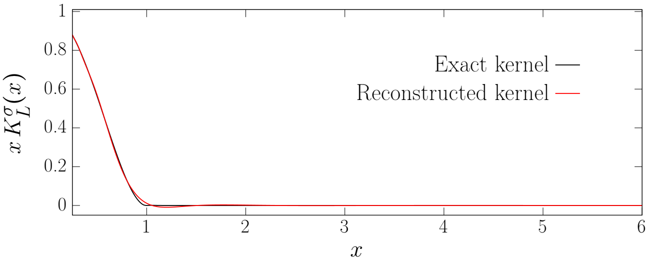
Stability analysis to find the optimal $\boldsymbol{g}$'s

- Quantify the quality of the approximation

$$d[\boldsymbol{g}^{\boldsymbol{\lambda}}] = \sqrt{\frac{A_0[\boldsymbol{g}^{\lambda}]}{A_0[\mathbf{0}]}}$$

- Search for statistically dominated regime (plateau)
- Independence of the algorithmic parameters
- High-quality kernel approximation
- Residual systematic error estimated by considering two reference points





ETMC: lattice setup and ensembles

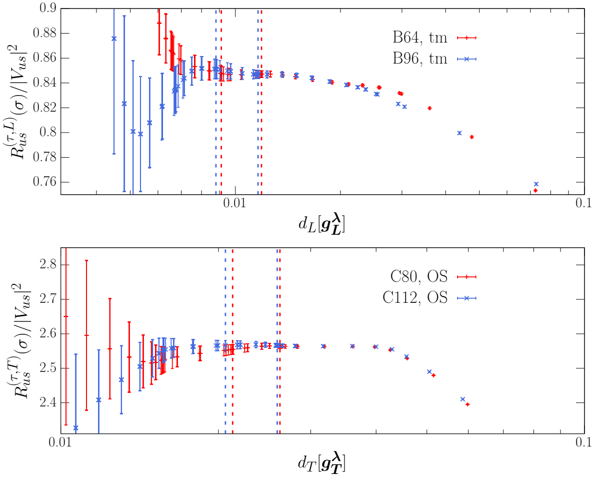
[PRD 107, 074506 (2023) PRD 98, 054518 (2018) PRD 104, 074520 (2021)]

ID	$L^3 \times T$	a [fm]	L [fm]	$\tau_{(us)}$	$\tau_{(ud)}$	D_s
B48	$48^3 \times 96$	0.07951	3.82	✗	✗	✓
B64	$64^3 \times 128$	0.07951	5.09	✓	✓	✓
B96	$96^3 \times 192$	0.07951	7.63	✓	✓	✓
C80	$80^3 \times 160$	0.06816	5.45	✓	✓	✓
C112	$112^3 \times 224$	0.06816	7.63	✓	✗	✗
D96	$96^3 \times 92$	0.05688	5.46	✓	✓	✓
E112	$112^3 \times 224$	0.04891	5.47	✓	✗	✓

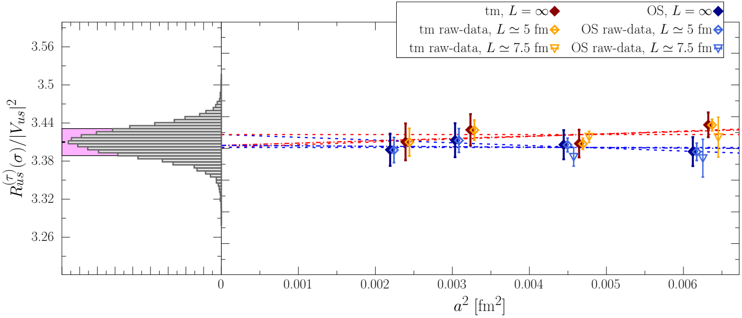
- $N_f = 2+1+1$ dynamical Wilson-clover twisted-mass quarks ($\mathcal{O}(a)$ improvement)
- $M_\pi \sim 140$ MeV
- Two regularizations for currents: twisted mass (tm) and Osterwalder-Seiler (OS)

Finite Size Effects (FSE): $\lim_{\sigma\mapsto 0}\lim_{a\mapsto 0}\lim_{L\mapsto\infty}$

- Check for independence of the volume on two different ensembles and quote the spread as systematic error
- FSE are relevant in less than 1% of the cases: $L\mapsto\infty$ limit achieved
- FSE are $\mathcal{O}(e^{-\sigma L})$ for smeared quantities



Lattice meets continuum: $\lim_{\sigma\mapsto 0} \lim_{a\mapsto 0} \lim_{L\mapsto\infty}$



- Constant, linear and quadratic fits by requiring the **same continuum limit** for tm and OS
- Fits combined by using Bayesian Akaike Information Criterion

$$\omega_k \sim \exp\{-(\chi^2_k + 2N^k_{\text{param}} - N^k_{\text{points}})/2\} \qquad \sigma^2 = \sigma^2_{\text{stat}} + \sum_{k=1}^{N_{\text{fit}}} \omega_k (x_k - \bar{x})^2 \qquad \bar{x} = \sum_{k=1}^{N_{\text{fit}}} \omega_k x_k$$

Limit of vanishing smearing parameter: $\lim_{\sigma\mapsto 0}\lim_{a\mapsto 0}\lim_{L\mapsto\infty}$

Piloted by asymptotic expansion of $\Delta\rho(\sigma) = \int_0^\infty \mathrm{d}x\,(1-x)^n\left[\Theta_\sigma(1-x) - \Theta(1-x)\right]\rho(x)$

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$\boxed{\rho(x)\ \text{ regular at }\ x=1}$

- $n = 0, 1$: $\Delta\rho(\sigma) = \mathcal{O}\left(\sigma^2\right) +$ even powers
- $n = 2, 3$: $\Delta\rho(\sigma) = \mathcal{O}\left(\sigma^4\right) +$ even powers

$\boxed{\rho(x) = Z\delta(1-x)}$

- $n = 0$: $\Delta\rho(\sigma) = \frac{1}{2}Z$
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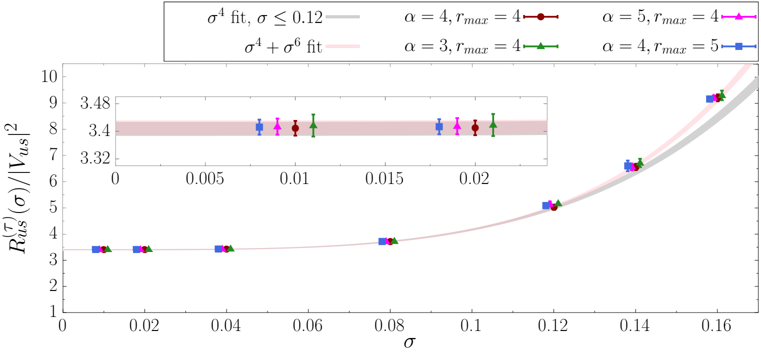

Results: us channel [PRL 132 (2024) 26, 261901]

We found $\boxed{R^{\tau}_{us}/|V_{us}|^2 = 3.407(22)}$ (0.6%) and by using $R_{us} = 0.1632(27)$ HFLAV

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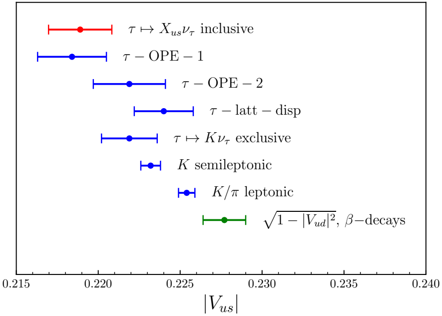

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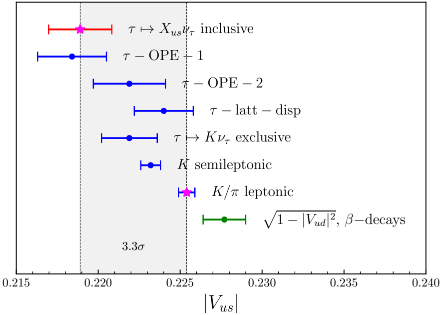

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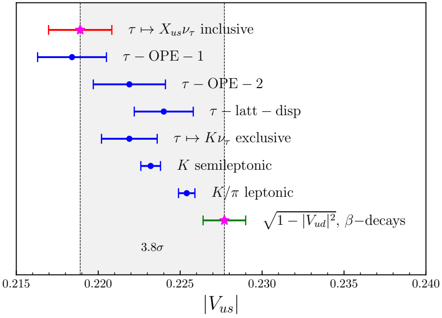
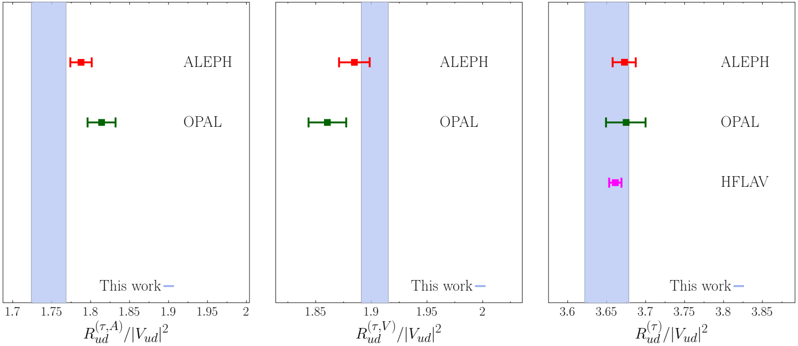


▷ Our result confirms OPE determinations and tension with unitarity from $\beta-$decays

We miss isospin breaking corrections (planned) which are important at this level of precision.

Even though, a $\mathcal{O}\left(\sim 5\%\right)$ effect would be needed to reconcile with K/π ...

Results: ud channel [PRD 108 (2023) 7, 074513]



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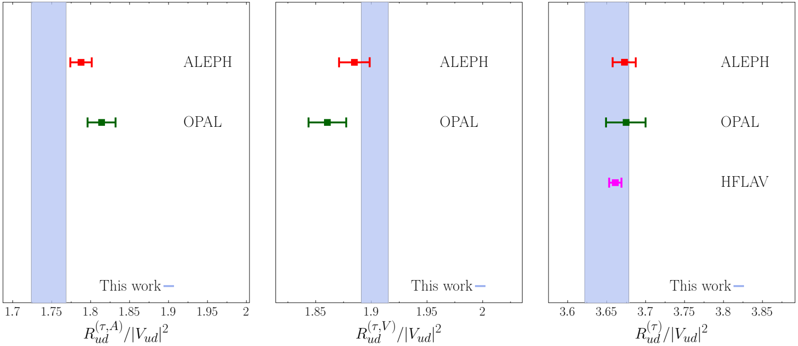

- ALEPH and OPAL are divided by $|V_{ud}| = 0.97373(31)$ from superallowed nuclear $\beta-$decays
- Alternatively: use the HFLAV average $R_{ud} = 3.471(7)$ to get

$$|V_{ud}| = 0.9752(39)$$

Inclusive charm decays

Motivations

- A systematic study of inclusive semi-leptonic decay of heavy mesons has started in the last years
  A. Barone's talk, JHEP 07 (2023) 145, R. Kellermann's Lattice2024 talk

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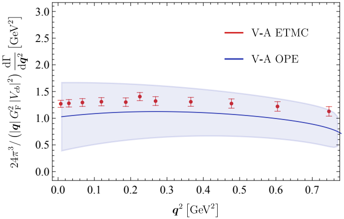

Exploratory study of the $H \mapsto X\ell\nu$ decay done in JHEP 07 (2022) 083 via HLT, but

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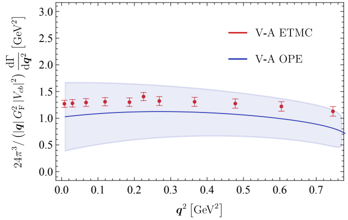

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- Unphysical ensemble
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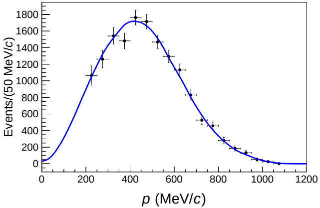
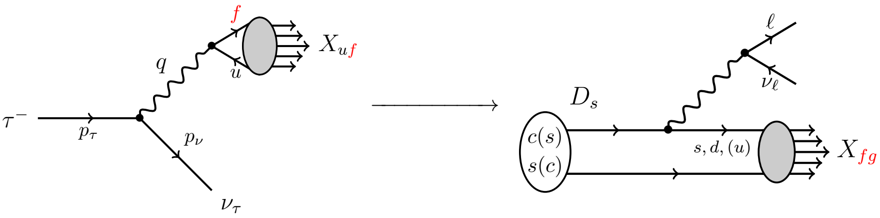
- Comparison only with OPE

The experimental precision of $D_s \mapsto X\ell\nu$ data, achievable from the lattice, offers the opportunity to do a **complete phenomenologically relevant calculation** and at the same time to **validate the method**

$$\Gamma_{\rm semi-lep.} = 8.27(21) \times 10^{-14}\ \text{GeV}\ (2.5\%)\ \text{BES-III}$$



Conceptually similar to τ decays
[P. Gambino and S. Hashimoto (2020), S. Hashimoto (2017)]



$$\Gamma^{(D_s)}_{\text{semi-lep.}} = G_F^2 \left(|V_{cd}|^2 \Gamma^{(D_s)}_{cd} + |V_{cs}|^2 \Gamma^{(D_s)}_{cs} + |V_{us}|^2 \Gamma^{(D_s)}_{su} + \text{disc.} \right)$$

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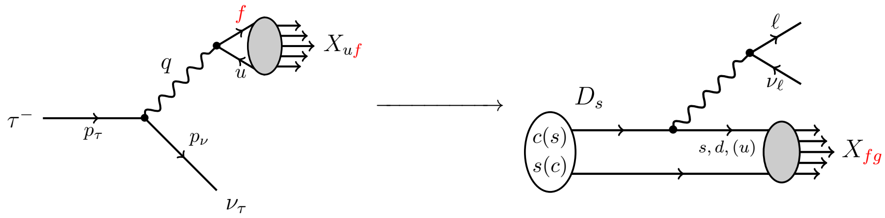

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- The kinematics is determined by a three-body phase space: **differential decay rate**

$$\Gamma^{(\tau)}_{uf} \propto \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2E_\nu} L_{\alpha\beta}(p_\tau, p_\nu) \rho^{\alpha\beta}_{uf}(q) \longrightarrow \Gamma^{(D_s)}_{fg} = \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2E_\nu} \frac{\mathbf{d^3 p_\ell}}{\mathbf{(2\pi)^3 2E_\ell}} L_{\alpha\beta}(p_\ell, p_\nu) H^{\alpha\beta}_{fg}(q)$$

- The meson now shows up in the external states of the spectral density

$$\rho^{\alpha\beta}_{uf}(q) = \langle 0| J^{\alpha}_{uf}(0)(2\pi)^4\delta^4(\mathbb{P}-q)J^{\beta\dagger}_{uf}(0)|0\rangle \longrightarrow H^{\alpha\beta}_{fg}(q) = \frac{(2\pi)^4}{2m_{D_s}}\langle D_s| J^{\alpha}_{fg}(0)\delta^4(\mathbb{P}-q)J^{\beta\dagger}_{fg}(0)|D_s\rangle$$

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- The spectral density can be still associated with an Euclidean correlation function

$$C^{\alpha\beta}(t,\boldsymbol{q}) = \int_0^\infty \frac{\mathrm{d}E}{2\pi} e^{-Et}\rho^{\alpha\beta}_{uf}(E,\boldsymbol{q}) \longrightarrow M^{\alpha\beta}_{fg}(t,\boldsymbol{q}) = \int_0^\infty \mathrm{d}E\, e^{-Et} H^{\alpha\beta}_{fg}(E,\boldsymbol{q})$$

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▷ but this is given by a **four-point function**

$$C^{\alpha\beta}(t,\boldsymbol{q}) \longrightarrow M^{\alpha\beta}_{fg}(t_2-t_1,\boldsymbol{q}) = \lim_{\substack{t_{\mathrm{snk}}\mapsto+\infty \\ t_{\mathrm{src}}\mapsto-\infty}} \frac{C^{\alpha\beta}_{4\mathrm{pt}}(t_{\mathrm{snk}},t_2,t_1,t_{\mathrm{src}};\boldsymbol{q})}{C_{2\mathrm{pt}}(t_{\mathrm{snk}}-t_2)C_{2\mathrm{pt}}(t_1-t_{\mathrm{src}})}$$

After algebra and regularizations

$$24\pi^3 \frac{\mathrm{d}\Gamma_{fg}^{(D_s)}}{\mathrm{d}\boldsymbol{q}^2} = \lim_{\sigma\mapsto 0} \sum_{k=0}^{2} |\boldsymbol{q}|^{3-k} \int_0^\infty \mathrm{d}E (E^{\max} - E)^k \theta_\sigma (E^{\max} - E) Z_{k,fg}^{(0)}$$

- ▷ $Z_{k,fg}^{(0,1)}$ are linear combinations of $H_{fg}^{\alpha\beta}(E,\boldsymbol{q}) \mapsto$ direct application of HLT

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- $Z_{k,fg}^{(0,1)}$ are linear combinations of $H_{fg}^{\alpha\beta}(E, \boldsymbol{q}) \mapsto$ direct application of HLT

Analogously for the first lepton moment

$$48\pi^3 \frac{\mathrm{d}M_{fg}^{(D_s)(1)}}{\mathrm{d}\boldsymbol{q}^2} = \lim_{\sigma\mapsto 0} \sum_{k=0}^{3} |\boldsymbol{q}|^{4-k} \int_0^\infty \mathrm{d}E (E^{\max} - E)^k \theta_\sigma(E^{\max} - E) Z_{k,fg}^{(1)}$$

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$$24\pi^3 \frac{d\Gamma^{(D_s)}_{fg}}{d\boldsymbol{q}^2} = \lim_{\sigma\mapsto 0} \sum_{k=0}^{2} |\boldsymbol{q}|^{3-k} \int_0^\infty dE (E^{\max} - E)^k \theta_\sigma(E^{\max} - E) Z^{(0)}_{k,fg}$$

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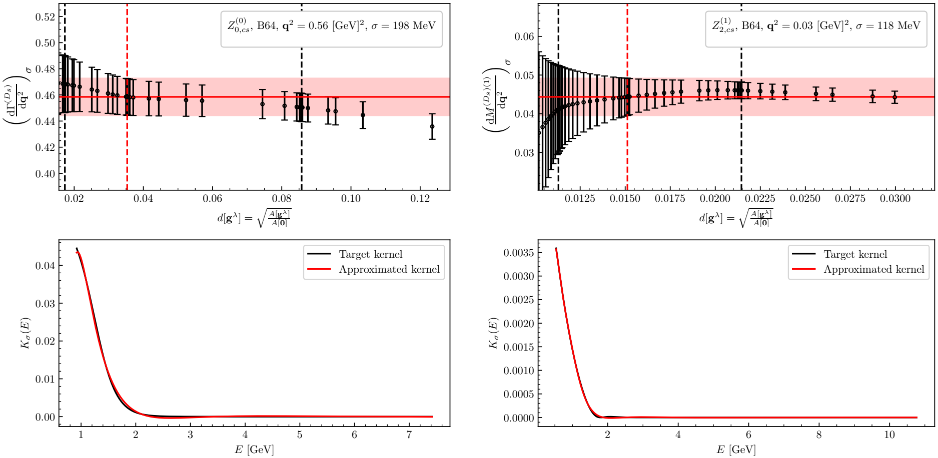
Comments:

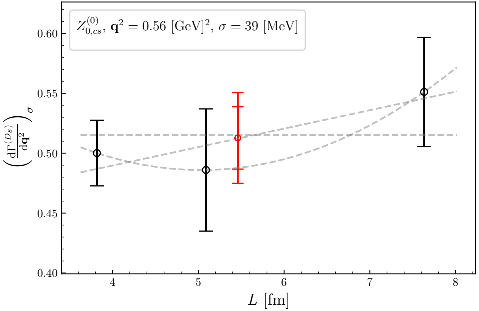
- k: this organization is useful to employ the asymptotic expansion for small σ
- $E^{\max}(\boldsymbol{q}^2) = m_{D_s} - \sqrt{\boldsymbol{q}^2}$

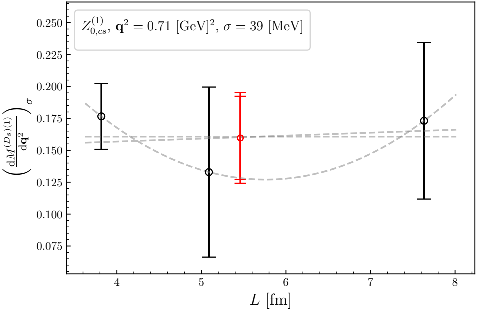
HLT stability analysis



Good control over stability regimes

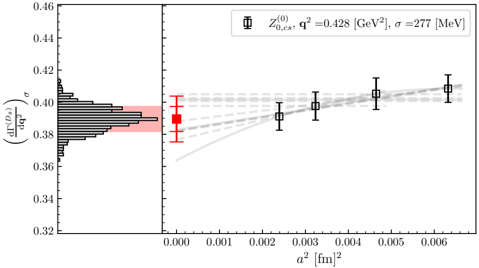
Finite Size Effects

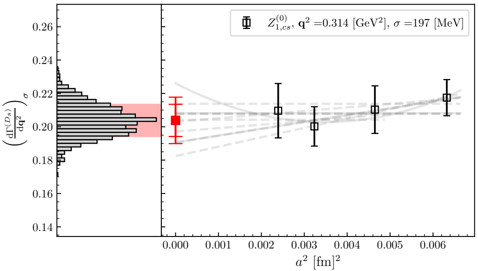




- Interpolation at reference volume $L \sim 5.46$ fm for ensemble B
- Three volumes to assess volume dependence
- Very mild dependence on the volume in all the cases: $L \mapsto \infty$ achieved
- FSE systematic error from maximum spread between different volumes

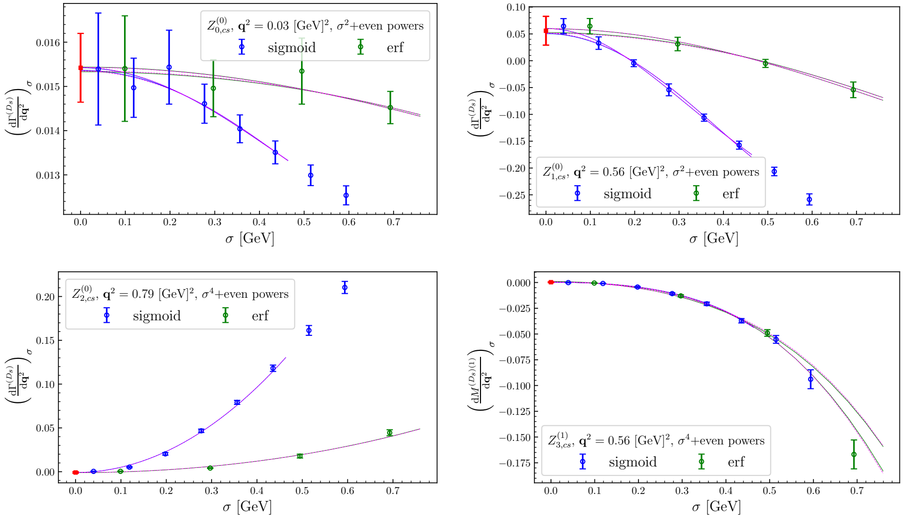
Lattice meets continuum





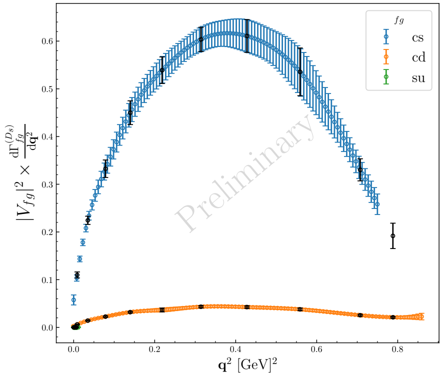
- ▷ Bayesian Akaike IC with constant, linear and quadratic fits
- ▷ Relevant lattice artifacts absent in most of the cases

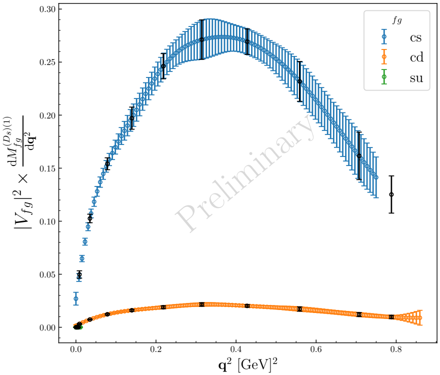
Limit of vanishing smearing parameter



The theoretical **asymptotic expansion** for small σ captures well the data behaviour

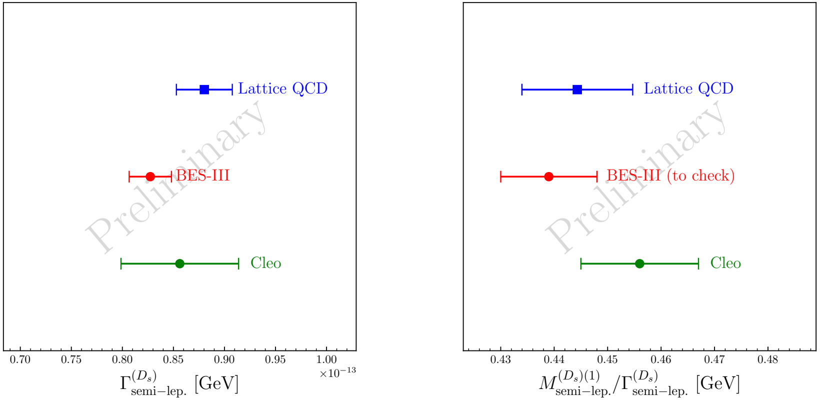
Preliminary results of the differential rates





- Spline cubic interpolation + trapezoid integration with boundaries determined from the lattice
- cd channel is Cabibbo suppressed, su channel has a microscopic phase space
- Disconnected contribution (expected to be small) in production
- Non-vanishing decay rate at the end-point due to isolated pole

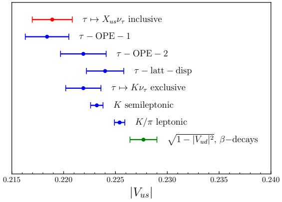
Preliminary results of the integrated differential rates

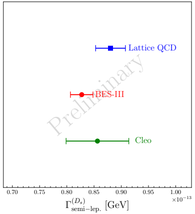


- Achieved the required accuracy on the lattice to compare with experiments
- Nice agreement with experimental data
- The HLT method is successful also in this case

Conclusions

- **Inclusive processes from Lattice QCD are no longer impracticable**
- Inclusive tau and charm decays are only two classes of phenomenologically interesting processes
- Extension to B physics on top of next-to-do list
- Many other applications: R-ratio, general scattering amplitudes, Meson spectroscopy, electroweak amplitudes, glueball spectrum, Sphaleron Rate, ecc.





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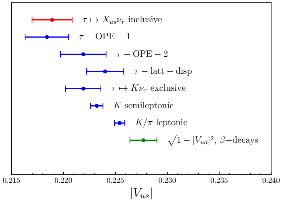
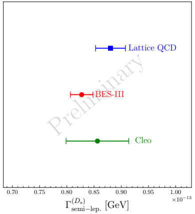




Thank you for the attention!!!

Backup

Definition of $Z_k^{(0)}$ (decay rate)

$$Z_0^{(0)} \equiv Y_2 + Y_3 - 2Y_4 \qquad Z_1^{(0)} \equiv 2(Y_3 - 2Y_1 - Y_4) \qquad Z_2^{(0)} \equiv Y_3 - 2Y_1$$

Form factors decomposition of the hadronic tensor

$$\begin{aligned} m_{D_s}^3 H^{\mu\nu}(p, p_x) = g^{\mu\nu} m_{D_s}^2 h_1 + p^\mu p^\nu h_2 + (p - p_X)^\mu (p - p_X)^\nu h_3 \\ +[p^\mu (p - p_X)^\nu + (p - p_X)^\mu p^\nu] h_4 - i\varepsilon^{\mu\nu\alpha\beta} p_\alpha (p - p_X)_\beta h_5 \end{aligned}$$

$$Y_1 = -m_{D_s} \sum_{ij} \hat{n}^i \hat{n}^j H^{ij} = h_1$$

$$Y_2 = m_{D_s} H^{00} = h_1 + h_2 + \left(1 - \frac{q_0}{m_{D_s}}\right)^2 h_3 + 2\left(1 - \frac{q_0}{m_{D_s}}\right) h_4$$

$$Y_3 = m_{D_s} \sum_{ij} \hat{q}^i \hat{q}^j H^{ij} = -h_1 m_{D_s}^2 + |\mathbf{q}|^2 h_3$$

$$Y_4 = -m_{D_s} \sum_{i} \hat{q}^i H^{0i} = \left(1 - \frac{q_0}{m_{D_s}}\right) |\mathbf{q}| h_3 + |\mathbf{q}| h_4$$

$$Y_5 = \frac{i m_{D_s}}{2} \sum_{ijk} \varepsilon^{ijk} \hat{q}^k H^{ij} = |\mathbf{q}| h_5$$

$$\hat{n}^2 = 1$$
$$\hat{n} \cdot \mathbf{q} = 0$$
$$\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$$

Definition of $Z_k^{(1)}$ (first lepton moment)

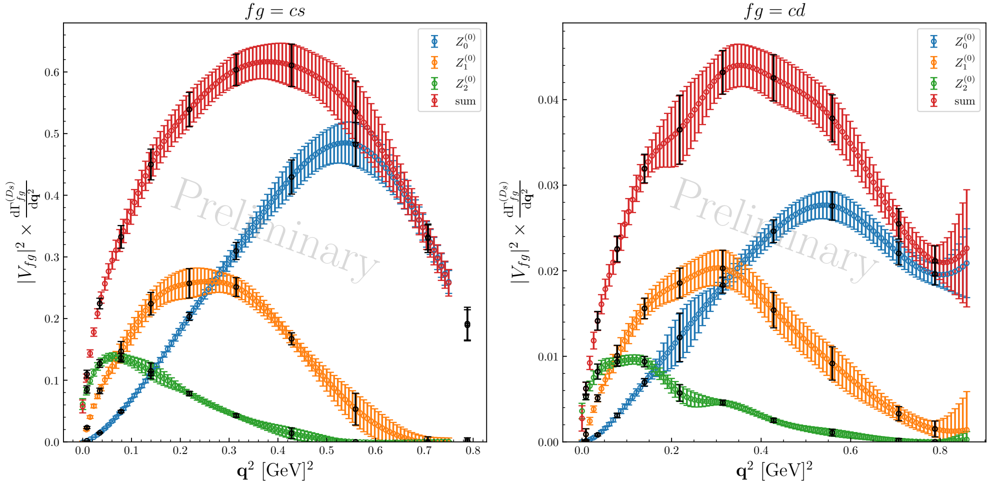
$$Z_0^{(1)} = Y_2 + Y_3 - 2Y_4$$

$$Z_1^{(1)} = -4Y_1 + Y_2 + 3Y_3 - 4Y_4 + 2Y_5$$

$$Z_2^{(1)} = -6Y_1 + 3Y_3 - 2Y_4 + Y_5$$

$$Z_3^{(1)} = -2Y_1 + Y_3$$

Partial contributions to decay rate



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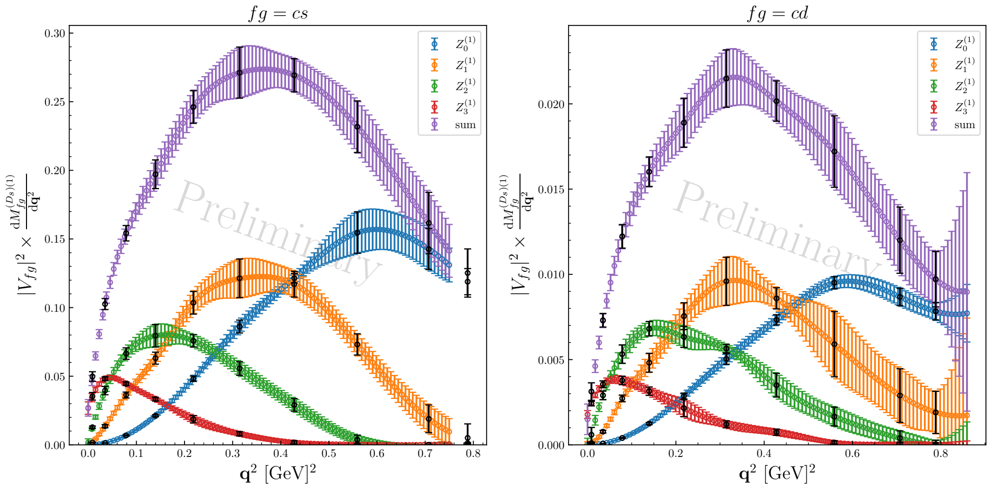
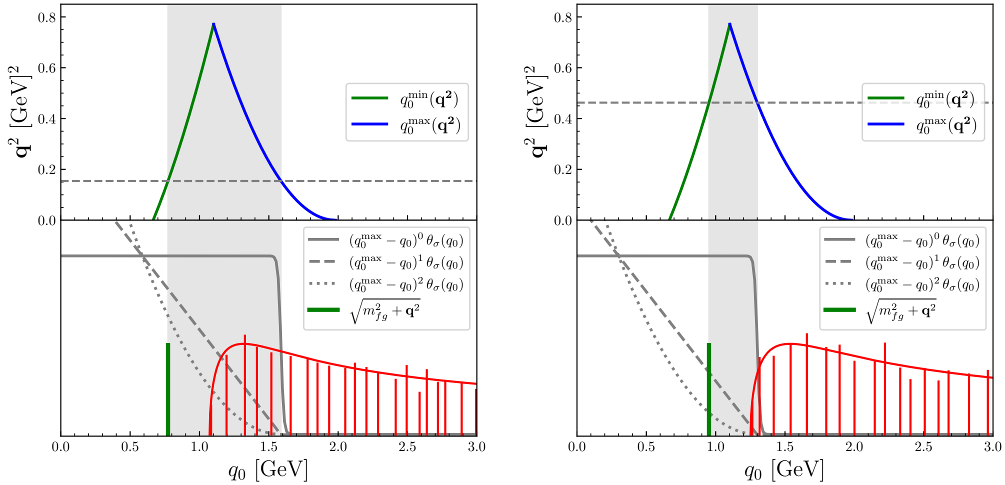


The final hadron phase-space

$$q_0 \in \left[\sqrt{m_{fg}^2 + \mathbf{q}^2}, m_{D_s} - |\mathbf{q}|\right]$$ m_{fg}^2 lightest mass in the spectrum



Decay rate at the end-point

- $q_0 \in \left[\sqrt{m_{fg}^2 + \mathbf{q}^2}, m_{D_s} - |\mathbf{q}|\right]$
- $\boldsymbol{q}^2 \in \left[0, \frac{(m_{D_s} - r_{fg}^2)^2}{4}\right]$, $r_{fg} = \frac{m_{fg}}{m_{D_s}}$
- The spectral function has an isolated pole, separated by the multi-particle states, and by construction at $\boldsymbol{q}^2_{\max}$ this coincides with the end-point: $Z^{(n)}_{0,fg}$ non-vanishing