Factorial Growth of Perturbation Theory and Inclusive Semileptonic B Decays

Andreas S. Kronfeld
Fermilab & IAS TU München

III[rd] Lattice meets Continuum
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Siegen

Inclusive Semileptonic B Decay Rate

- Decay rate:

$$\Gamma(B \to X_c \ell \nu) = \frac{1}{M_B} \,\mathrm{Im} \int \mathrm{d}^4x \, \langle B | T\{\mathcal{H}_{\mathrm{eff}}(x)\mathcal{H}_{\mathrm{eff}}(0)\} | B \rangle$$

- Momentum $p \sim m_b$ dominates, so $x \sim 1/m_b$ allows operator-product expansion, leading to ($\rho = m_c/m_b$)

$$\Gamma(B \to X_c \ell \nu) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} m_b^5 g(\rho) \left[1 + \sum_{l=0} c_l \alpha_s^{l+1} - \frac{\mu_\pi^2}{2m_b^2} - g_G(\rho) \frac{\mu_G^2}{2m_b^2} \cdots \right]$$

where m_b is the pole mass. Known as the heavy-quark expansion (HQE).

- Natural HQE(T) expansion parameter is the pole mass (aka on-shell mass).

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Relation Between Meson Mass and Quark Mass

meson mass

"brown muck"

$$M_B = m_b + \bar{\Lambda} + \mathrm{O}(1/m_b)$$

heavy quark "pole" mass

$\bar{m}_b = m_{b,\overline{\mathrm{MS}}}(\bar{m}_b)$

$$m_b = \bar{m}_b \left(1 + \sum_{l=0} r_l \alpha_s^{l+1}(\bar{m})\right)$$

$$r_l = \{1, 2.44, 8.83, 41.53\} r_0$$

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Prototypical α_s Determination

- Consider an “effective charge” with a single hard scale:

physical quantity

power correction

$$\mathscr{R}(Q) = R(Q) + C_p \frac{\Lambda^p}{Q^p}$$

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power $p \leftrightarrow$ factorial growth

perturbative series in mass-independent scheme

$$R(Q) = \sum_{l=0} r_l(\mu/Q)\,\alpha_s(\mu)^{l+1}$$

- Perturbative part and power correction inseparable.

Factorial Growth

- Even in quantum mechanics, high orders of perturbation theory grow factorially [*e.g.*, Bender & Wu 1971, 1973].
- Also in QFT [*e.g.*, Gross & Neveu 1974, Lautrup 1977].
- In pQCD, r_l grow factorially (known for a long time):

$$r_l \sim R_0^{(p)} \left(\frac{2\beta_0}{p} \right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \equiv R_l^{(p)}$$

for $l \gg 1$. Here $b = \beta_1/2\beta_0^2 \overset{n_f=3}{=} 32/81 \approx 0.4$.

- Does $r_l = \{1, 1.38, 5.46, 26.7\}$ start growing by $l = 3$?

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- Does $r_l = \{1, 1.38, 5.46, 26.7\}$ start growing by $l = 3$?

Summary of Math in arXiv:2310.151137 [in JHEP]

- Use some simple steps and the RGE (which connects μ independence of $R(Q)$ to Q dependence of $R(Q)$—
  - obtain a more slowly growing set of coefficients, $f_k^{(p)}$.
- Invert an infinite matrix (lower triangular).

$$\mathbf{Q}_{\rm g}^{(p)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau^2 pb & -2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^2 & -2\tau^2 pb & -3\tau & 1 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^3 & -2\tau(\tau pb)^2 & -3\tau^2 pb & -4\tau & 1 & 0 & 0 & \cdots \\ -\tau(\tau pb)^4 & -2\tau(\tau pb)^3 & -3\tau(\tau pb)^2 & -4\tau^2 pb & -5\tau & 1 & 0 & \cdots \\ -\tau(\tau pb)^5 & -2\tau(\tau pb)^4 & -3\tau(\tau pb)^3 & -4\tau(\tau pb)^2 & -5\tau^2 pb & -6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \Leftrightarrow \mathbf{Q}_{\rm g}^{(p)\,-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau^2\frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ \tau^3\frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\tau^2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3\tau & 1 & 0 & 0 & 0 & \cdots \\ \tau^4\frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\tau^3\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\tau^2\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4\tau & 1 & 0 & 0 & \cdots \\ \tau^5\frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\tau^4\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\tau^3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\tau^2\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5\tau & 1 & 0 & \cdots \\ \tau^6\frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\tau^5\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\tau^4\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\tau^3\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\tau^2\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

- Simplify and clarify “minimal renormalon subtraction (MRS)” of arXiv:1701.00347 and arXiv:1712.04983 [Komijani].

Main New Result

- Exact result ("=" not "~"):

$$r_l = \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)} + f_l^{(p)}$$

- In some problems, the $f_k^{(p)}$ grow, but more slowly (same formula, p'>p).
  - leads to generalization to cascade of powers.

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This form used to re-sum.

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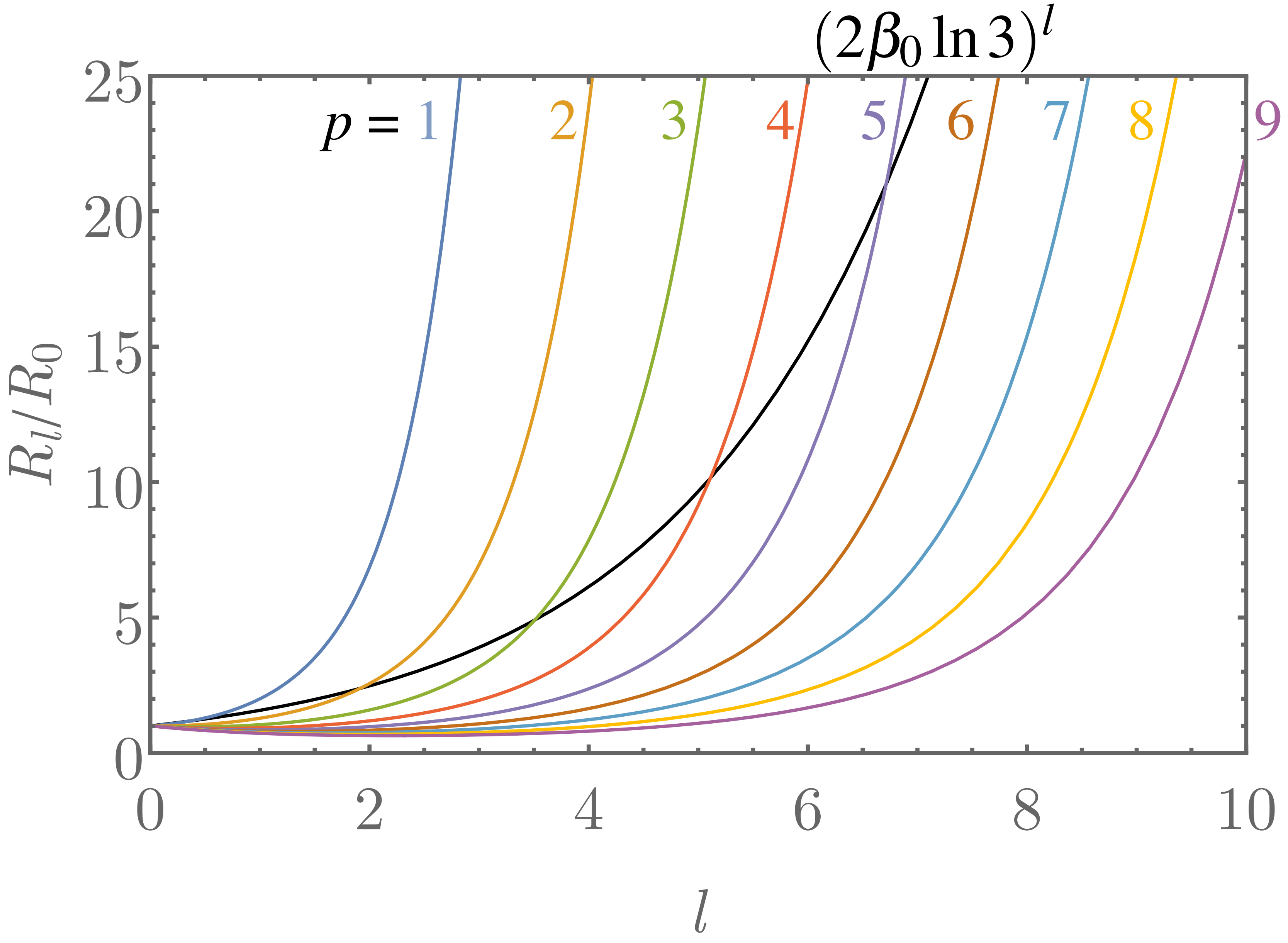
  - leads to generalization to cascade of powers.

Growth ↔ Power

- Larger $p \Rightarrow$ growth takes over at larger l.



Perturbative Series

- In practice, the r_l are in the literature for $l < L$.
- The f_l, $l < L$, are obtained from them, and the formula returns these r_l (as it must).
- For $l \geq L$, the formula tells us (formally) the largest part:

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use the approximate formula for the uncalculated terms.

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- For $l \geq L$, the formula tells us (formally) the largest part:

$$r_l \approx \underbrace{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}_{\text{well-known growth}} \underbrace{\sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}_{\text{Komijani } R_0 \text{ (truncated)}}$$

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Recap & Compendium

- That means

$$\sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_s^{l+1}$$

  with

$$R_l^{(p)} \equiv R_0^{(p)} \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}$$

$$R_0^{(p)} \equiv \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}$$

- Justified because the retained terms are formally larger than the ones omitted.

Rearrange and Borel Sum

- We have

$$R(Q) = \sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_s^{l+1}$$

$$= \underbrace{\sum_{l=0}^{L-1} \left(r_l - R_l^{(p)} \right) \alpha_s^{l+1}}_{R_{\mathrm{RS}}^{(p)}(Q)} + \underbrace{\sum_{l=0}^{\infty} R_l^{(p)} \alpha_s^{l+1}}_{R_{\mathrm{B}}^{(p)}(Q)}$$

- The "renormalon subtracted" part and the "Borel" part.

- The R_l from above yield divergent sum for R_{B}, but we're not done yet: use Borel summation to assign meaning.

Borel Summation

- Using the integral representation of $\Gamma(l+1)$:

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \sum_{l=0}^{\infty} \left[\frac{\Gamma(l+1+pb)}{\Gamma(1+pb)\Gamma(l+1)} \int_0^{\infty} \left(\frac{2\beta_0 t}{p} \right)^l \mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)} \mathrm{d}t \right]$$

$$\to R_0^{(p)} \int_0^{\infty} \frac{\mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t$$

Mathematica knows the sum

where 2nd line comes from (illegally) swapping Σ and $\int$.

- Branch point in integrand at $t = p/2\beta_0$, dubbed "renormalon singularity" ['t Hooft 1979].
- (Alternatively, use integral representation of $\Gamma(l+1+pb)$.)

Borel Summation

- Split integration in two [BKKV, arXiv:1712.04983]:

Mathematica knows the integrals

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \int_0^{p/2\beta_0} \frac{\mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t$$

$$+ R_0^{(p)} \int_{p/2\beta_0}^{\infty} \frac{\mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t$$

where ± on 2nd line comes from choice of contour.

- Without loss, absorb the second line into the power correction in $\mathscr{R}(Q)$:
  - heavy-light hadron mass, $\bar{\Lambda} \to \bar{\Lambda}_{\mathrm{MRS}}$.

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$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{I}\left(pb, 1/2\beta_0\alpha_{\mathrm{g}}(Q)\right)$$

$$- R_0^{(p)} \mathrm{e}^{\pm \mathrm{i} pb\pi} \frac{p^{1+pb}}{2^{1+pb}\beta_0} \Gamma(-pb) \left[\frac{\mathrm{e}^{-1/[2\beta_0\alpha_{\mathrm{g}}(Q)]}}{[\beta_0\alpha_{\mathrm{g}}(Q)]^b} \right]^p$$

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Mathematica knows the integrals

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Assignment

- Thus, we now define

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{J}(pb, p/2\beta_0 \alpha_{\mathrm{g}}(Q))$$

$$\mathscr{J}(c,y) = \mathrm{e}^{-y}\Gamma(-c)\gamma^\star(-c,-y)$$

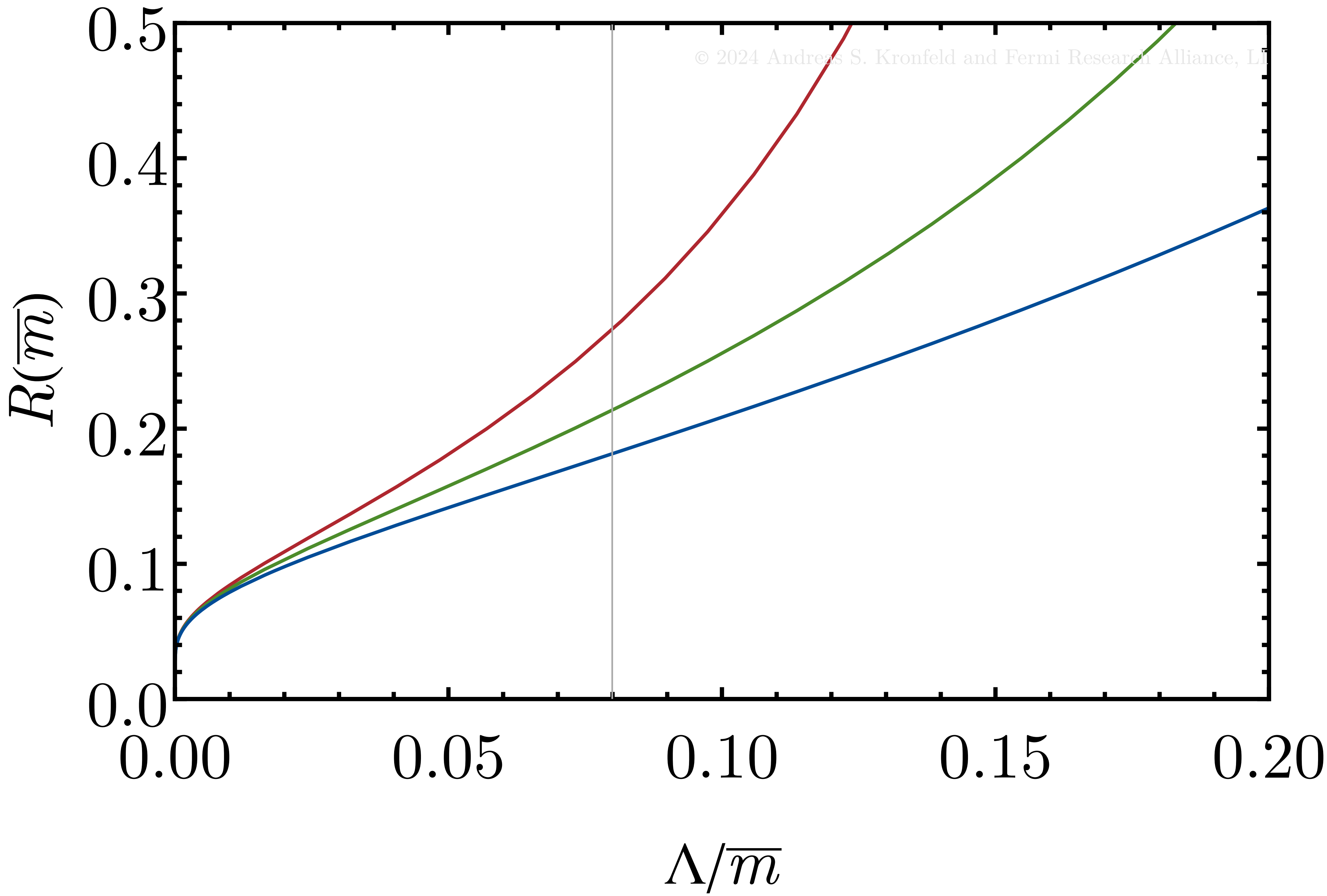
where $\gamma^\star(a,x)$ is an analytic function of both a and x:

limiting function of the incomplete gamma function

  - convergent expansion in $x = -1/2\beta_0\alpha_{\mathrm{g}}$;
  - asymptotic expansion in α_{g} regenerates the starting point; the dropped term is $\mathrm{O}(\mathrm{e}^{-p/2\beta_0\alpha_{\mathrm{g}}})$.
  - So $m_{\mathrm{MRS}} = \bar{m}(1 + R_{\mathrm{RS}}(\bar{m}) + R_{\mathrm{B}}(\bar{m}))$ solves perturbative pole condition just as well as "the" pole mass.

Pole Mass's Horrible Series ($p = 1$)

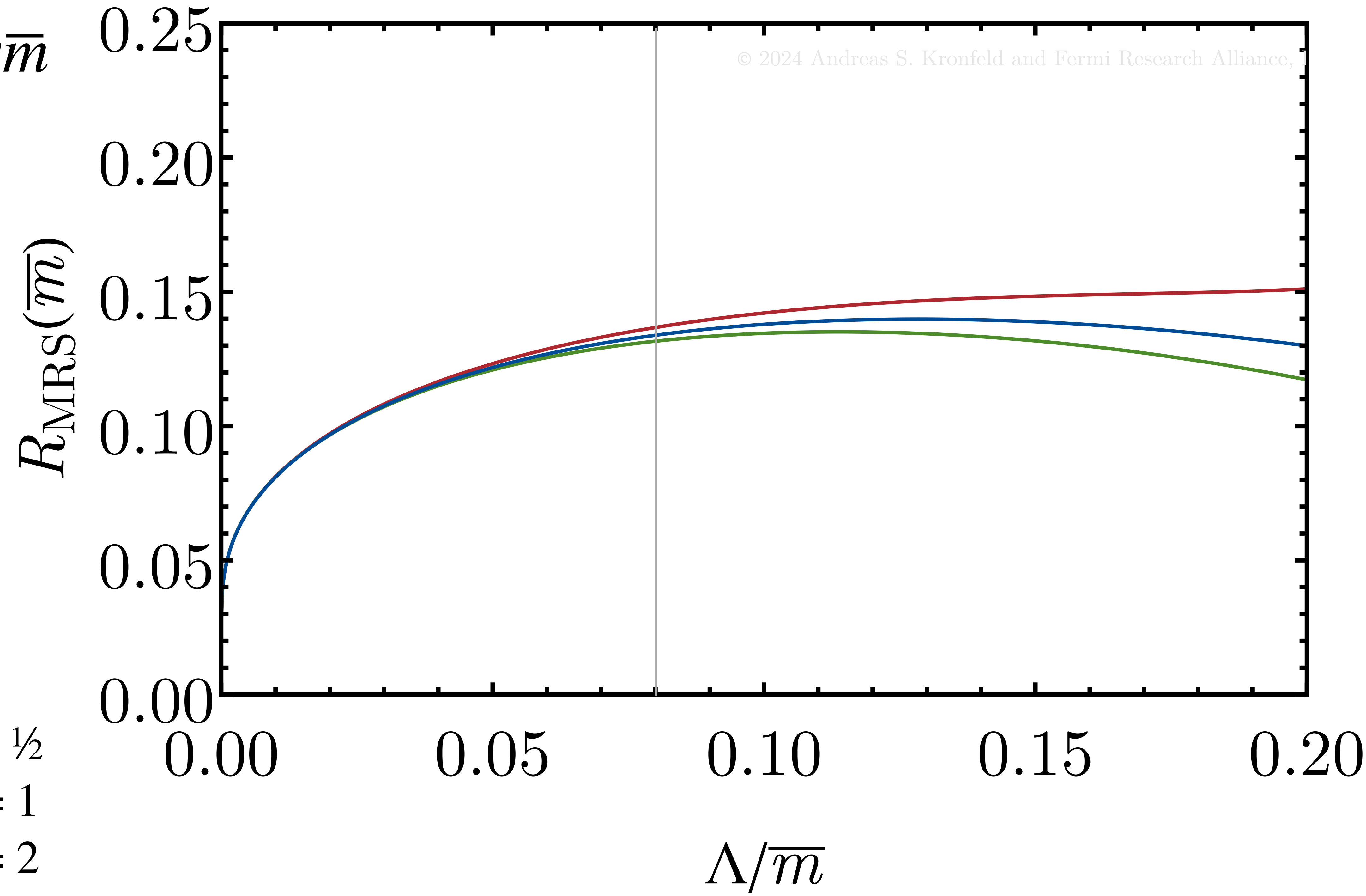
$\mu = s\overline{m}$



$s = ½$
$s = 1$
$s = 2$

Pole Mass's MRS Series

$\mu = s\overline{m}$



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Convergence

- With $s = 1$, R_{RS} term is very small:

$$m_{b,\text{MRS}}/\bar{m}_b = (1.157, 1.133, 1.131, 1.132, 1.132)$$

$$m_{t,\text{MRS}}/\bar{m}_t = (1.0687, 1.0576, 1.0573, 1.0574, 1.0574)$$

with 4-loop R_0 and neglecting mass effects of lighter quarks.

- Varying $s \neq 1$, R_{RS} and R_{B} terms' scale dependence cancels so total is s independent [*cf*., arXiv:2401.07380].

- Convergence similar for all s.

Inclusive Proposal

- Intriguing to see what happens if m_b in the inclusive width is interpreted to be the MRS mass.
- Need to develop methodology to the higher-power terms:
  - lots of issues and complications left for future work (collaboration needed);
  - proposal, issues, and complications apply to other HQE applications.
- MRS mass is already being used if you take $\bar{m}_b$ from FLAG, because it is key to one of the most precise inputs [arXiv:1802.04248] to the average.

Summary

- MRS mass [arXiv:1712.04983]: a pole mass with desirable properties of a short-distance mass.
- Well-known growth actually begins at low orders [arXiv:2310.151137]:
  - factorial growth can be summed up in a consistent way, *i.e.*, neglected terms are formally smaller.
- Stability of MRS series makes it attractive for applications beyond arXiv:1802.04248, *e.g.*, as from static energy or ….
  - the heavy-quark expansion!
- Minimal renormalon subtraction is a bad name: extensions beyond one higher power, renormalons not needed, not a subtraction but a sum.

Thank you!

Backup

My Solution

- The relation between the coefficients is a matrix equation

$$f_k^{(p)} = r_k - \frac{2}{p} \sum_{j=0}^{k-1} (j+1) \beta_{k-1-j} r_j$$

$$\boldsymbol{f}^{(p)} = \left[\mathbf{1} - \frac{2}{p} \mathbf{D} \right] \cdot \boldsymbol{r} \equiv \mathbf{Q}^{(p)} \cdot \boldsymbol{r}$$

  and $\mathbf{D}$ is on the lower triangle.

- Matrix is infinite, but the lower triangular form makes a row-by-row solution straightforward.

- Notation to make the expressions compact: $\tau \equiv 2\beta_0/p$.

$$\mathbf{Q}_{\mathrm{g}}^{(p)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau^2 pb & -2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^2 & -2\tau^2 pb & -3\tau & 1 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^3 & -2\tau(\tau pb)^2 & -3\tau^2 pb & -4\tau & 1 & 0 & 0 & \cdots \\ -\tau(\tau pb)^4 & -2\tau(\tau pb)^3 & -3\tau(\tau pb)^2 & -4\tau^2 pb & -5\tau & 1 & 0 & \cdots \\ -\tau(\tau pb)^5 & -2\tau(\tau pb)^4 & -3\tau(\tau pb)^3 & -4\tau(\tau pb)^2 & -5\tau^2 pb & -6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

- As before $b = \beta_1/2\beta_0^2 \stackrel{n_f=3}{=} 32/81 \approx 0.4$.

- Scheme for α_s is chosen to simplify algebra (“geometric”):

$$\beta(\alpha_{\mathrm{g}}) = -\frac{\beta_0 \alpha_{\mathrm{g}}^2}{1 - (\beta_1/\beta_0)\alpha_{\mathrm{g}}}$$

- Inverse reveals that factorial growth begins at low orders:

$$\mathbf{Q}_{\rm g}^{(p)\,-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau^2\frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ \tau^3\frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\tau^2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3\tau & 1 & 0 & 0 & 0 & \cdots \\ \tau^4\frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\tau^3\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\tau^2\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4\tau & 1 & 0 & 0 & \cdots \\ \tau^5\frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\tau^4\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\tau^3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\tau^2\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5\tau & 1 & 0 & \cdots \\ \tau^6\frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\tau^5\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\tau^4\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\tau^3\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\tau^2\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$\boldsymbol{r} = \mathbf{Q}_{\rm g}^{(p)\,-1} \cdot \boldsymbol{f}^{(p)}$$

return

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$$\mathbf{Q}_{\mathrm{g}}^{(p)\,-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau^2\frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ \tau^3\frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\tau^2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3\tau & 1 & 0 & 0 & 0 & \cdots \\ \tau^4\frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\tau^3\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\tau^2\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4\tau & 1 & 0 & 0 & \cdots \\ \tau^5\frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\tau^4\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\tau^3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\tau^2\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5\tau & 1 & 0 & \cdots \\ \tau^6\frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\tau^5\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\tau^4\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\tau^3\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\tau^2\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$\boldsymbol{r} = \mathbf{Q}_{\mathrm{g}}^{(p)\,-1} \cdot \boldsymbol{f}^{(p)}$$

return

$$r_l = \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)} + f_l^{(p)}$$

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well-known growth

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well-known growth

Komijani's R_0 (truncated)

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$$\boldsymbol{r} = \mathbf{Q}_{\mathrm{g}}^{(p)\,-1} \cdot \boldsymbol{f}^{(p)}$$

return

$$r_l = \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1)\frac{\Gamma(1+pb)}{\Gamma(k+2+pb)}\left(\frac{p}{2\beta_0}\right)^k f_k^{(p)} + f_l^{(p)}$$

well-known growth | Komijani's R_0 (truncated) | extra

Comparing Truncations

- Standard—truncate and hope for the best:

$$\sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1}$$

- arXiv:1701.00347 + arXiv:1712.04983—add&subtract, truncate:

$$\sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{\infty} [r_l - R_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} R_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} [r_l - R_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} R_l \alpha_s^{l+1}$$

- This analysis—approximate higher orders with the dominant factorial:

$$\sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} [r_l - R_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} R_l \alpha_s^{l+1}$$

Next Approximation

- If there is another power correction with $p_2 > p_1 = p$, then f_k will grow in a similar but slower fashion.

- Apply previous procedure with p_1; then repeat with p_2:

$$\begin{aligned}
\boldsymbol{f}^{\{p_1,p_2\}} &\equiv \mathbf{Q}^{(p_2)} \cdot \mathbf{Q}^{(p_1)} \cdot \boldsymbol{r} \\
\Rightarrow \boldsymbol{r} &= \mathbf{Q}^{(p_1)^{-1}} \cdot \mathbf{Q}^{(p_2)^{-1}} \cdot \boldsymbol{f}^{\{p_1,p_2\}} \\
&= \left[\frac{p_2}{p_2 - p_1} \mathbf{Q}^{(p_1)^{-1}} + \frac{p_1}{p_1 - p_2} \mathbf{Q}^{(p_2)^{-1}} \right] \cdot \boldsymbol{f}^{\{p_1,p_2\}}
\end{aligned}$$

- Extension to any sequence of higher powers by induction.