# Sum Rules for Lifetimes

Based on a work in progress in collaboration with Prof. Alexander Lenz, Martin Lang, and Matthew Black

Zachary Wüthrich - 30.09.2024

Lattice Meets Continuum
Siegen

# Theoretical Foundations

# Overview

## What are we doing?

- Determine non-perturbative quantities related to the B-meson total decay rate

$$\Gamma(B) = \frac{1}{2M_B} \langle B|\mathcal{T}|B\rangle$$

  - Specifically, how do we calculate this hadronic matrix element?

- Sum rules employ a perturbative calculation, analyticity of the S-matrix, and quark hadron duality

- Sum rules vs Lattice QCD:

·Independent predictions ·less computationally intensive
·systematic approach ·more flexible ·worse precision

# Heavy Quark Expansion

## The Total Decay Rate

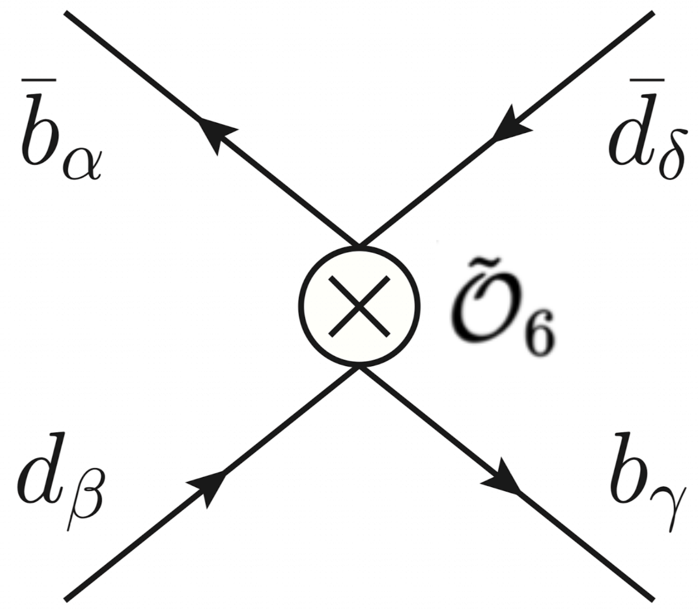
- The hadronic matrix element is a result of the **optical theorem**:

$$\Gamma(B) = \frac{1}{2M_B}\langle B|\mathcal{T}|B\rangle$$

$$\mathcal{T} \equiv \mathrm{Im}\, i \int d^4x\, \mathrm{T}\{\mathcal{H}_{eff}(x)\mathcal{H}_{eff}(0)\}$$

- We can separate the short- and long-distance dynamics with an **OPE**:

$$\Gamma(B) = \Gamma_3\langle\mathcal{O}_3\rangle + \Gamma_5\frac{\langle\mathcal{O}_5\rangle}{m_b^2} + \Gamma_6\frac{\langle\mathcal{O}_6\rangle}{m_b^3} + \ldots + 16\pi^2\left[\tilde{\Gamma}_6\frac{\langle\tilde{\mathcal{O}}_6\rangle}{m_b^3} + \tilde{\Gamma}_7\frac{\langle\tilde{\mathcal{O}}_7\rangle}{m_b^4} + \ldots\right]$$



# The Bag Parameter

## And the Vacuum Insertion Approximation

- Input a complete set of states, but assume the main contribution comes from inserting the vacuum

$$\langle \overline{B}|Q(\mu)|B\rangle = \langle \overline{B}\big|J^1_\mu\big|0\rangle \cdot \langle 0\big|J^{2,\mu}\big|B\rangle B(\mu) = A_Q f_B^2 M_B^2 B(\mu)$$

$$\langle 0\big|\bar{b}\gamma^\mu\gamma^5 q\big|B(p)\rangle = -i f_B p^\mu$$

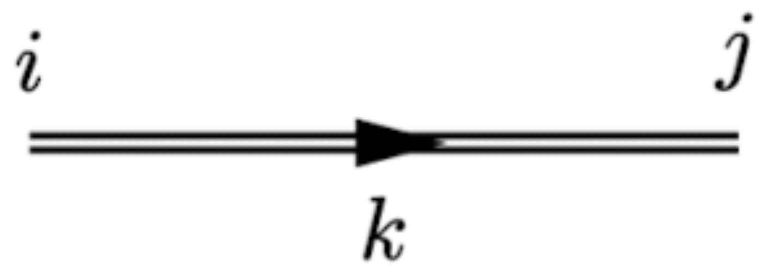
- The Bag Parameter parametrizes how good this approximation is
  - $B(\mu) = 1$ would imply the VIA is exact

# Heavy Quark Effective Theory (HQET)

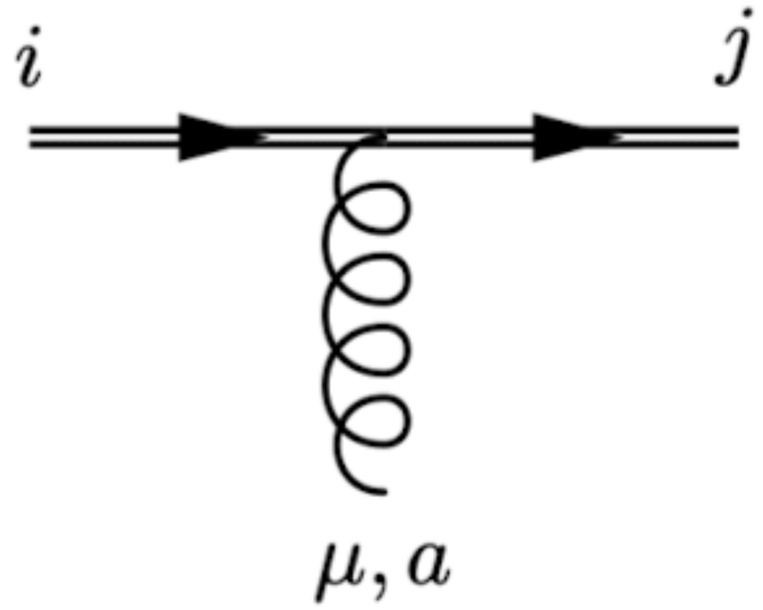
# HQET

## The Basics

- Expansion in the heavy quark mass: $\Lambda_{QCD} \ll m_Q$
- Parametrize heavy quark momentum: $p_Q^\mu = p^\mu + m_Q v^\mu$
- Parametrize heavy quark field: $Q(x) = e^{-im_Q v\cdot x} h(x) + \mathcal{O}\left(\frac{1}{m_Q}\right)$
- Basic **Feynman rules**:
  - Propagator: $$i\frac{\not{p}_Q + m_Q}{p_Q^2 - m_Q^2} = i\frac{\not{p} + m_Q(1+\not{v})}{p^2 + 2m_Q p\cdot v}, \\ = i\frac{(1+\not{v})}{2\omega} + \mathcal{O}\left(\frac{1}{m_Q}\right)$$




$$i\,\delta^{ij}\,\frac{(1+\not{v})}{2(k\cdot v)}$$




$$i\,g\,t^a_{ij}\,v^\mu$$

# QCD-HQET Matching for Lifetimes

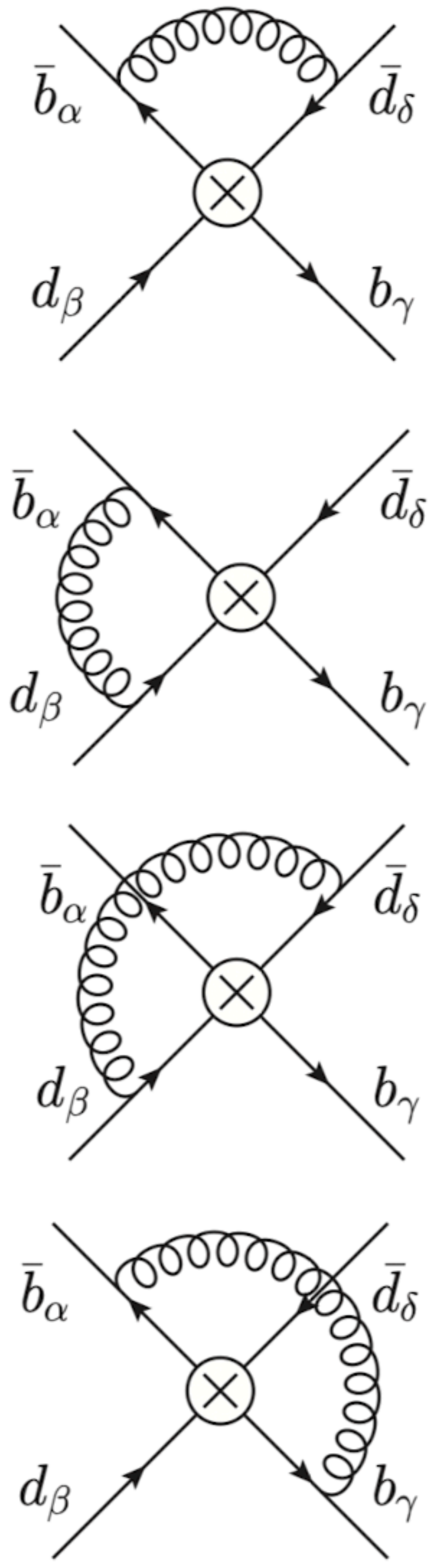
## Matching Four Quark Operators

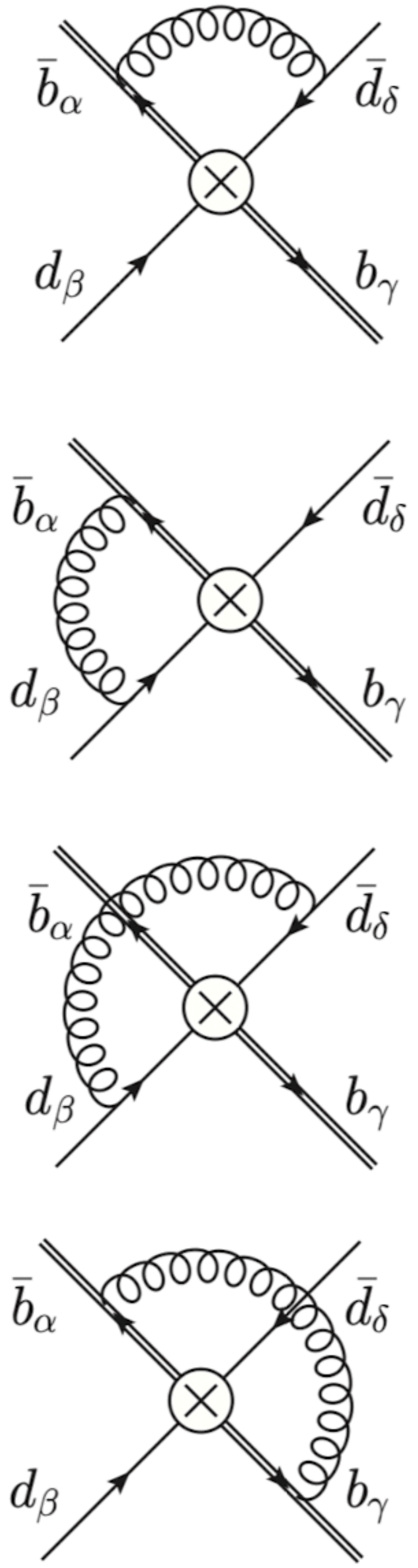
### QCD SM

$$Q_1^q = \bar{b}\gamma_\mu (1-\gamma_5)\, q\, \bar{q}\gamma^\mu (1-\gamma_5)\, b\,,$$
$$Q_2^q = \bar{b}\,(1-\gamma_5)\, q\, \bar{q}\,(1+\gamma_5)\, b\,,$$
$$T_1^q = \bar{b}\gamma_\mu (1-\gamma_5)\, T^a q\, \bar{q}\gamma^\mu (1-\gamma_5)\, T^a b\,,$$
$$T_2^q = \bar{b}\,(1-\gamma_5)\, T^a q\, \bar{q}\,(1+\gamma_5)\, T^a b$$

### QCD BSM

$$O_5^q \equiv \bar{b}\gamma_\mu (1-\gamma_5)\, q\, \bar{q}\gamma^\mu (1+\gamma_5)\, b\,,$$
$$O_6^q \equiv \bar{b}\,(1-\gamma_5)\, q\, \bar{q}\,(1-\gamma_5)\, b\,,$$
$$O_7^q \equiv \bar{b}\gamma_\mu (1-\gamma_5)\, T^a q\, \bar{q}\gamma^\mu (1+\gamma_5)\, T^a b\,,$$
$$O_8^q \equiv \bar{b}\,(1-\gamma_5)\, T^a q\, \bar{q}\,(1-\gamma_5)\, T^a b\,,$$
$$O_9^q \equiv \bar{b}\sigma_{\mu\nu} (1-\gamma_5)\, q\, \bar{q}\sigma^{\mu\nu} (1-\gamma_5)\, b\,,$$
$$O_{10}^q \equiv \bar{b}\sigma_{\mu\nu} (1-\gamma_5)\, T^a q\, \bar{q}\sigma^{\mu\nu} (1-\gamma_5)\, T^a b\,,$$
$$Q_i'^{\,q} \equiv Q_i^q\big|_{1\mp\gamma_5 \to 1\pm\gamma_5}\,, \qquad i = 1, \ldots, 10$$




### HQET

The same operators as in QCD, but $b \to h$

Tensor operators removed by HQET equations of motion

# Sum Rules

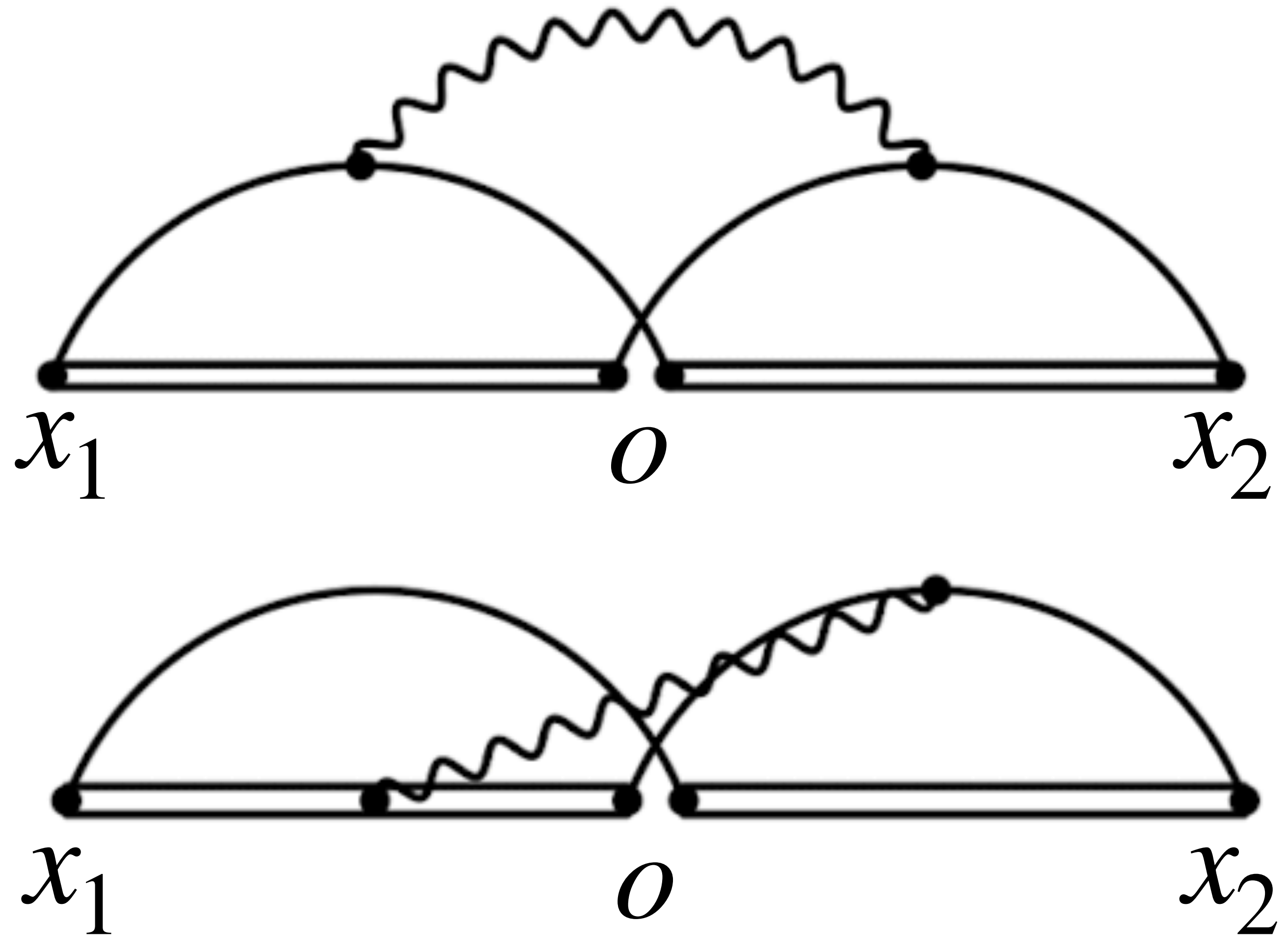
# HQET Sum Rules

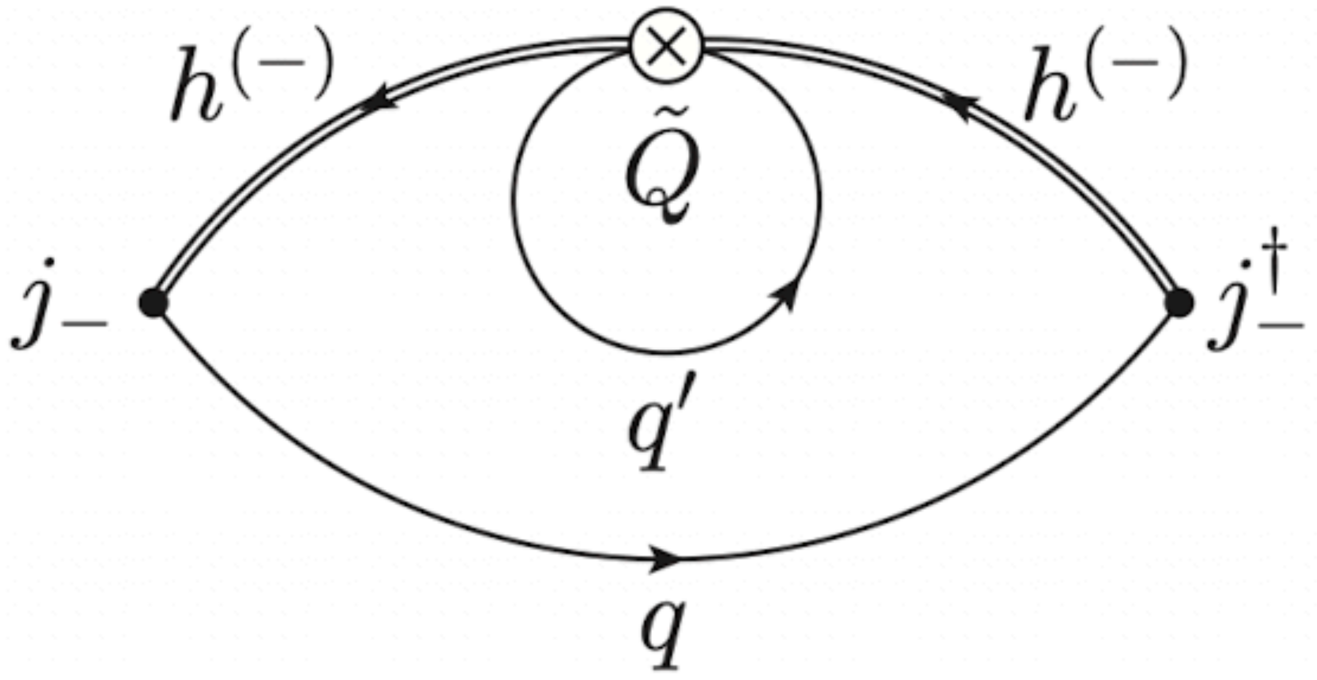
## Three-point Correlator

$$K_{\mathbf{Q}}(\omega_1, \omega_2) = \int d^d x_1 d^d x_2 e^{ip_1 \cdot x_1 - ip_2 \cdot x_2} \left\langle 0 \right| \mathrm{T}\left[\tilde{j}_+(x_2)\mathbf{Q}(0)\tilde{j}_-(x_1)\right] \left| 0 \right\rangle$$

$$\tilde{j}_- = \bar{q}\gamma^5 h^{(-)} \qquad j_+ = j_-^\dagger$$

This depends on the heavy quark's residual energy: $\omega = p \cdot v$




Nonfactorizable diagrams actually calculate

$B(\mu) - 1$

# HQET Sum Rules

## Analyticity

We relate the perturbative part of the correlator to the non-perturbative bound states via a dispersion relation:

$$\Pi(\omega) = \frac{1}{2\pi i} \oint_C ds \frac{\Pi(s)}{(s-\omega)}$$
$$= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_0^\infty ds \frac{\Pi(s+i\epsilon) - \Pi(s-i\epsilon)}{(s-\omega)} + \frac{1}{2\pi i} \int_R ds \frac{\Pi(s)}{(s-\omega)}$$
$$= \int_0^\infty ds \frac{\rho(s)}{(s-\omega)}$$

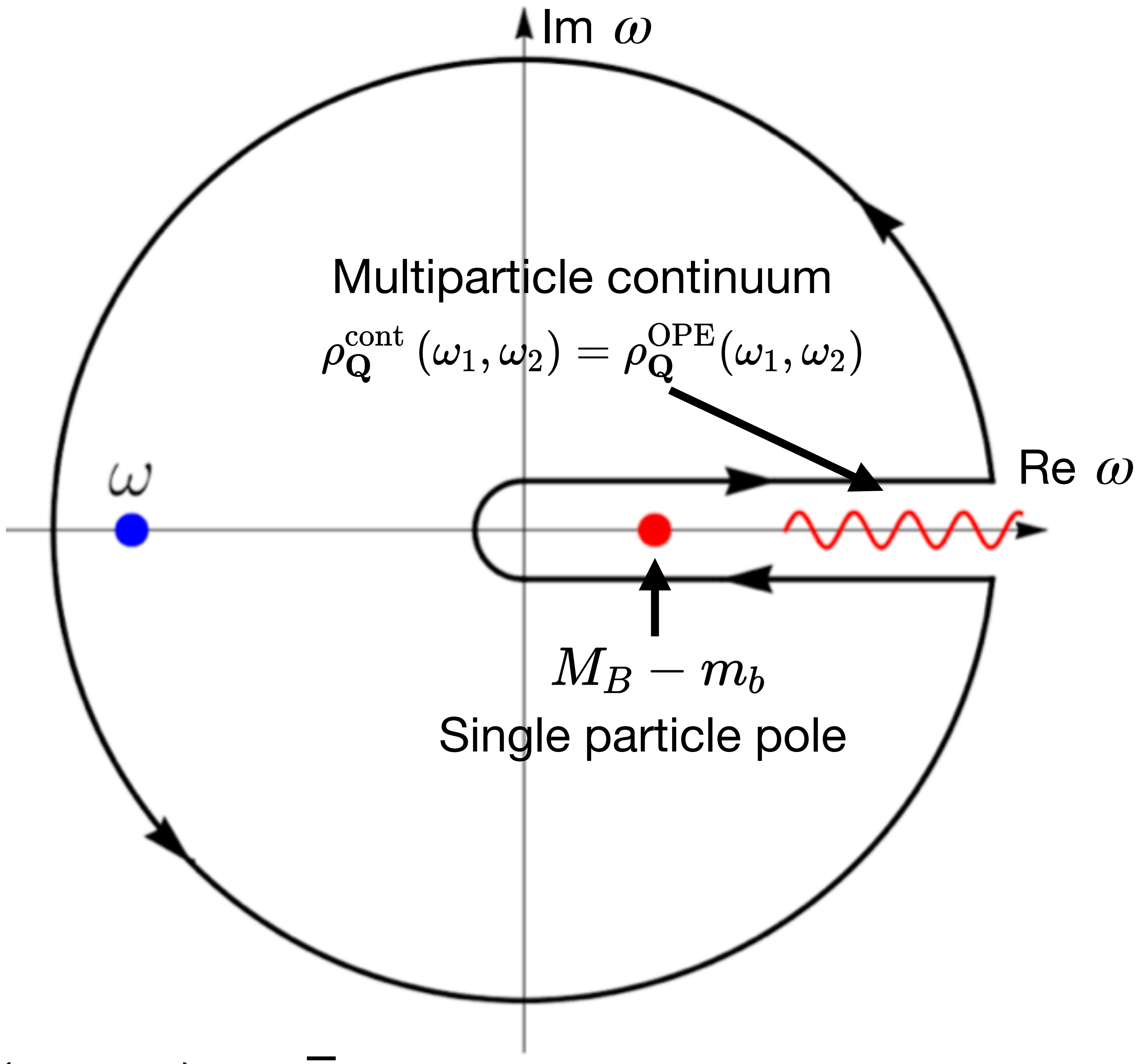
Dispersion Relation: $K_{\mathbf{Q}}(\omega_1, \omega_2) = \int_0^\infty d\eta_1 d\eta_2 \frac{\rho_{\mathbf{Q}}(\eta_1, \eta_2)}{(\eta_1 - \omega_1)(\eta_2 - \omega_2)} + [\,\text{subtraction terms}\,]$

# HQET Sum Rules

## The Hadronic Spectral Function

- A dispersion relation equates the three point correlation function to the hadronic spectral function:

$$K_{\mathbf{Q}}(\omega_1,\omega_2) = \int_0^\infty d\eta_1 d\eta_2 \frac{\rho_{\mathbf{Q}}(\eta_1,\eta_2)}{(\eta_1-\omega_1)(\eta_2-\omega_2)}$$
$$+ \left[\,\text{subtraction terms}\,\right]$$




- We have an ansatz for the hadronic spectral function:

$$\rho^{\text{had}}_{\mathbf{Q}}(\omega_1,\omega_2) = F^2(\mu)\langle\mathbf{Q}(\mu)\rangle\delta\Big(\omega_1-\bar{\Lambda}\Big)\delta\Big(\omega_2-\bar{\Lambda}\Big) + \rho^{\text{cont}}_{\mathbf{Q}}(\omega_1,\omega_2)\,,\quad \bar{\Lambda} = M_B - m_b$$

# HQET Sum Rules

## Arriving at the Sum Rule

$$K_{\mathbf{Q}}(\omega_1,\omega_2) = \int_0^{\infty} d\eta_1 d\eta_2 \frac{\rho_{\mathbf{Q}}(\eta_1,\eta_2)}{(\eta_1-\omega_1)(\eta_2-\omega_2)} + [\,\text{subtraction terms}\,]$$

$$\rho_{\mathbf{Q}}^{\text{had}}(\omega_1,\omega_2) = F^2(\mu)\langle \mathbf{Q}(\mu)\rangle \delta\left(\omega_1-\bar{\Lambda}\right)\delta\left(\omega_2-\bar{\Lambda}\right) + \rho_{\mathbf{Q}}^{\text{cont}}(\omega_1,\omega_2)$$

**Quark-Hadron duality:** $\rho_{\mathbf{Q}}^{\text{cont}}(\omega_1,\omega_2) = \rho_{\mathbf{Q}}^{\text{OPE}}(\omega_1,\omega_2)[1-\theta(\omega_c-\omega_1)\theta(\omega_c-\omega_2)]$

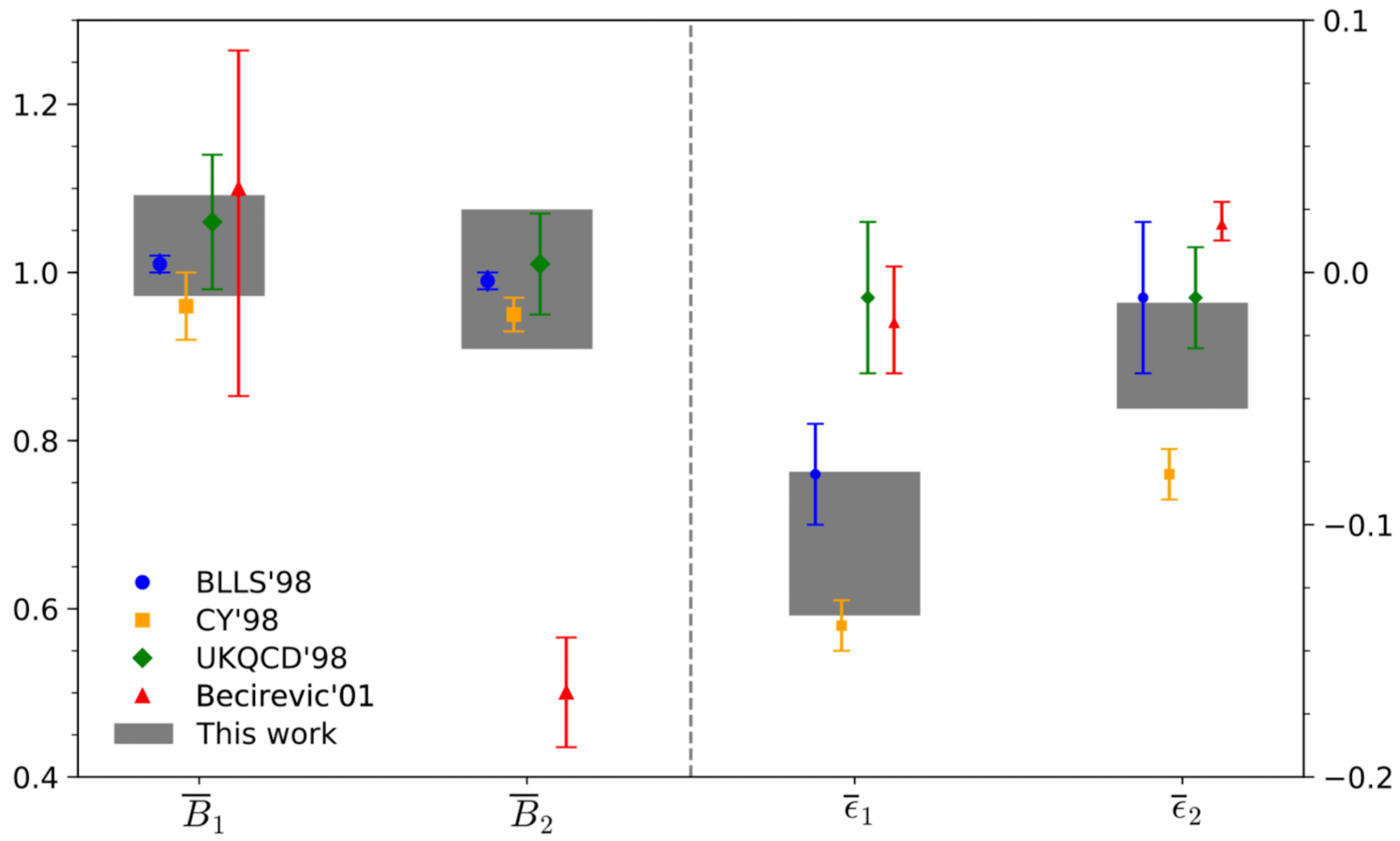
Borel Sum Rule:

$$F^2(\mu)\langle \mathbf{Q}(\mu)\rangle e^{-\frac{\bar{\Lambda}}{t_1}-\frac{\bar{\Lambda}}{t_2}} = \int_0^{\omega_c} d\omega_1 d\omega_2 e^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}} \rho_{\mathbf{Q}}^{\text{OPE}}(\omega_1,\omega_2)$$

# Results

# Previous Results in Lifetimes

## SM Operators Only



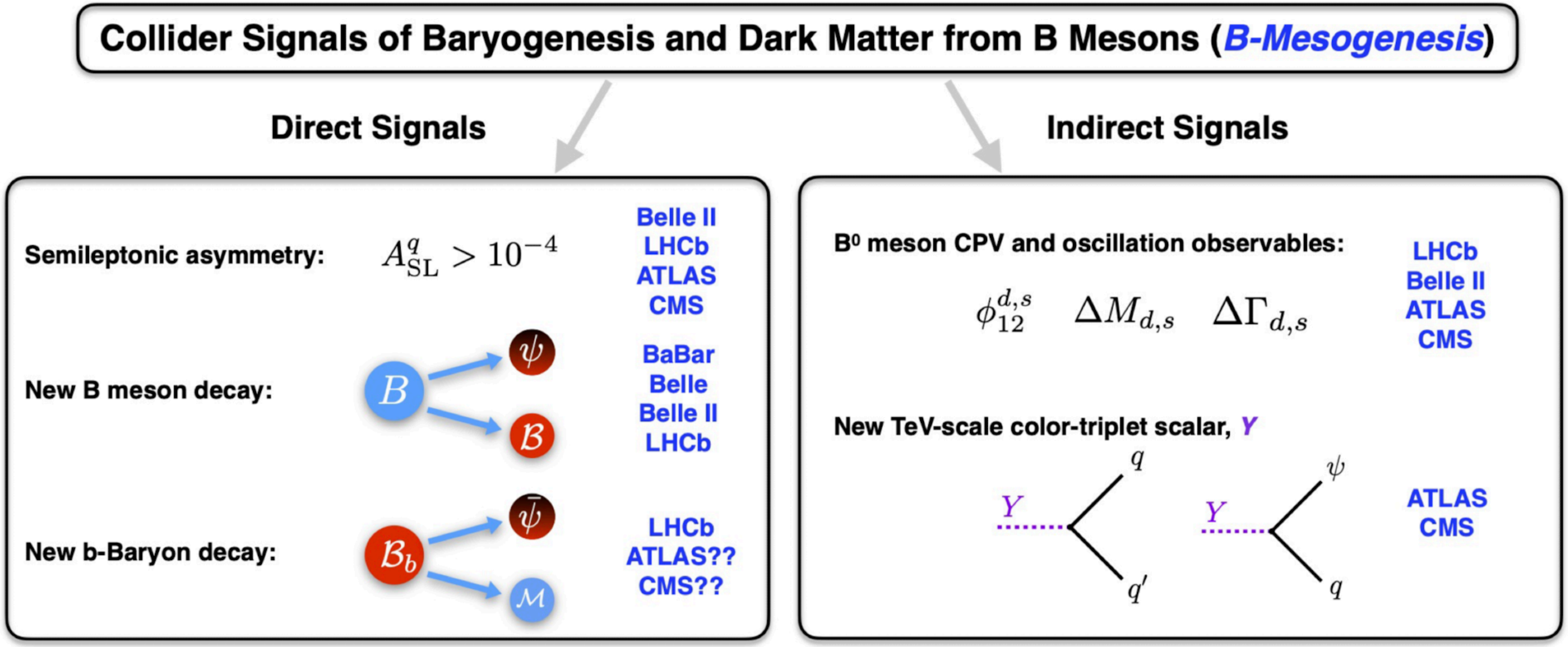

M. Kirk, A. Lenz, and T. Rauh, *Dimension-six matrix elements for meson mixing and lifetimes from sum rules*, JHEP **12** (2017) 068, arXiv:1711.02100.

### Status of the Calculation

- Three point correlation function calculated for all BSM operators
- Hadronic spectral function calculated for all BSM operators
- One loop matching and renormalization in progress
- Uncertainty analysis TBD

# Where is this needed?

## B-Mesogenesis




G. Alonso-Álvarez, G. Elor and M. Escudero, Phys. Rev. D **104**, no.3, 035028 (2021) doi:10.1103/PhysRevD.104.035028 [arXiv:2101.02706 [hep-ph]].

# Thank you!

# Backup Slides

# Previous Results

## Uncertainties

- Variation of $\bar{\Lambda} = M_B - m_b$
- Percent uncertainty for condensate contributions
- Uncertainty in NNLO $\alpha_s^2$ contributions in the spectral density
- Higher order 1/m_b corrections in the VIA
- Higher order QCD-HQET matching corrections and corrections to the RGE
    - Error estimate from varying renormalization scale

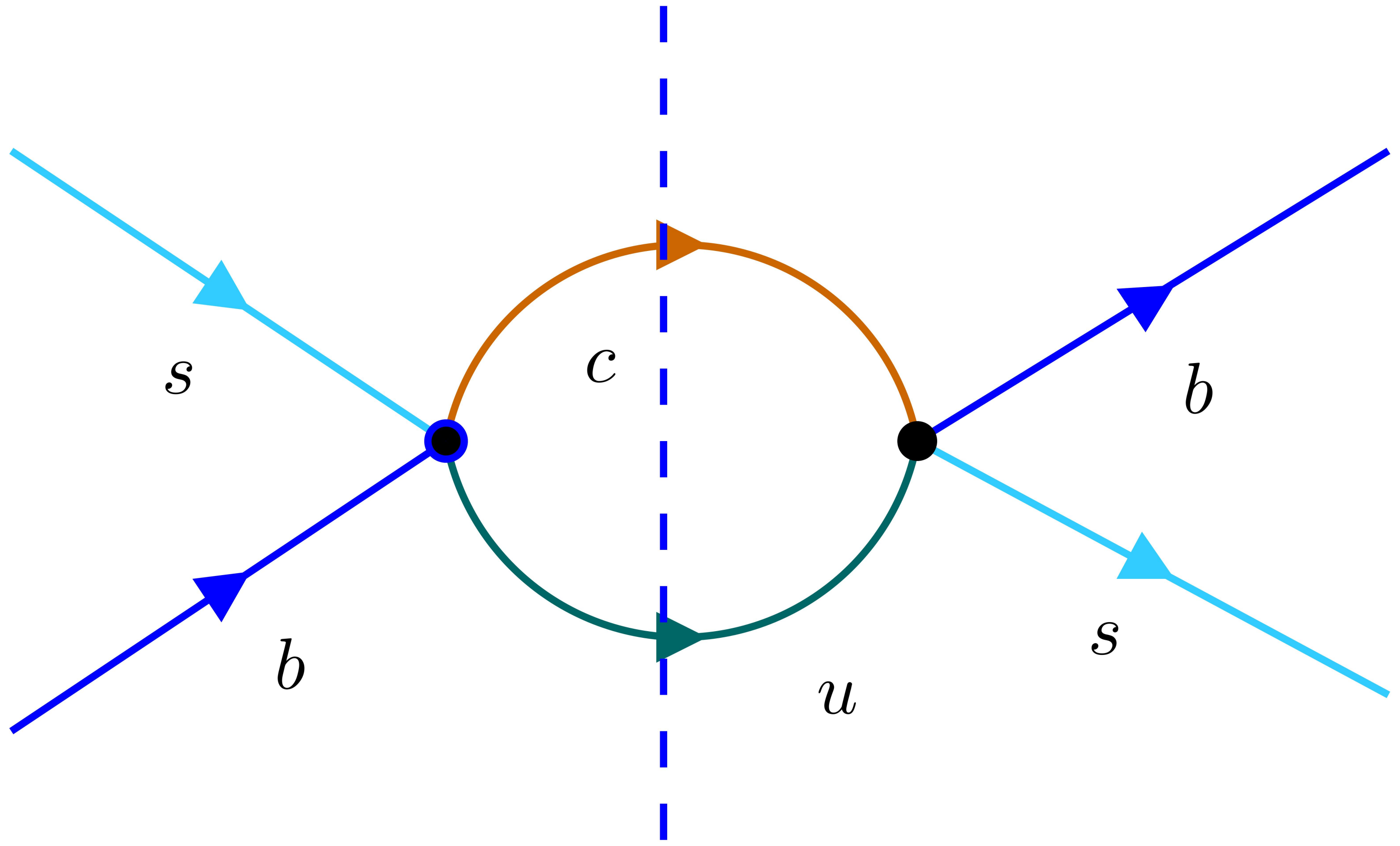
# Optical Theorem

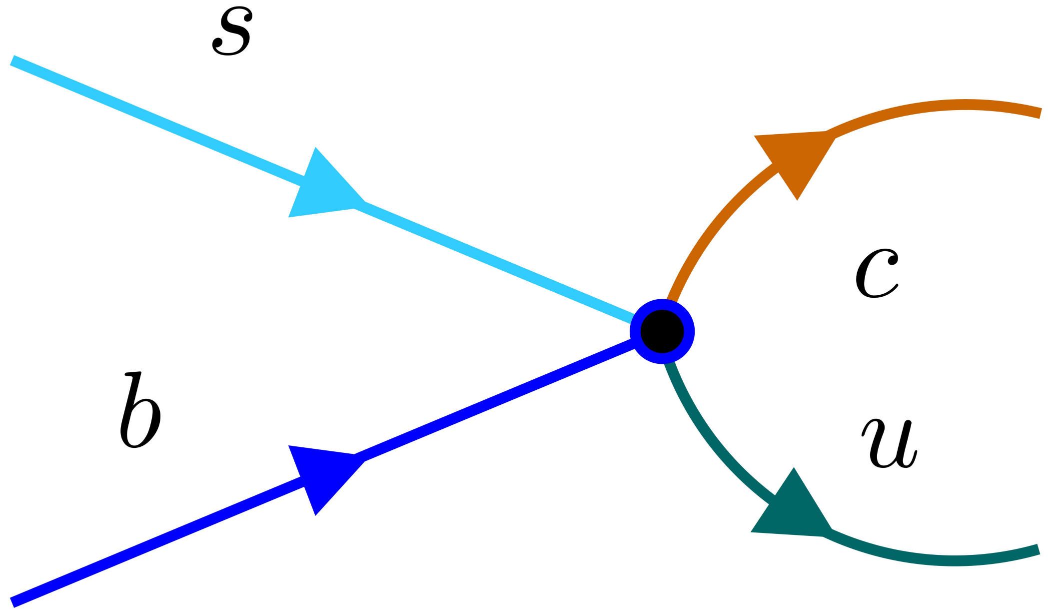
## Inclusive Calculations

$$S_{fi} \equiv \langle f|S|i\rangle = \delta_{fi} + iT_{fi}$$

$$T_{fi} = (2\pi)^4 \delta^{(4)}(p_f - p_i)\mathcal{M}_{fi}$$

$$2\,\mathrm{Im}\,\mathcal{M}_{ii} = \sum_n \int_n (2\pi)^4 \delta^{(4)}\left(\sum_{j=1}^{n} p_j - p_i\right) |\mathcal{M}_{ni}|^2$$

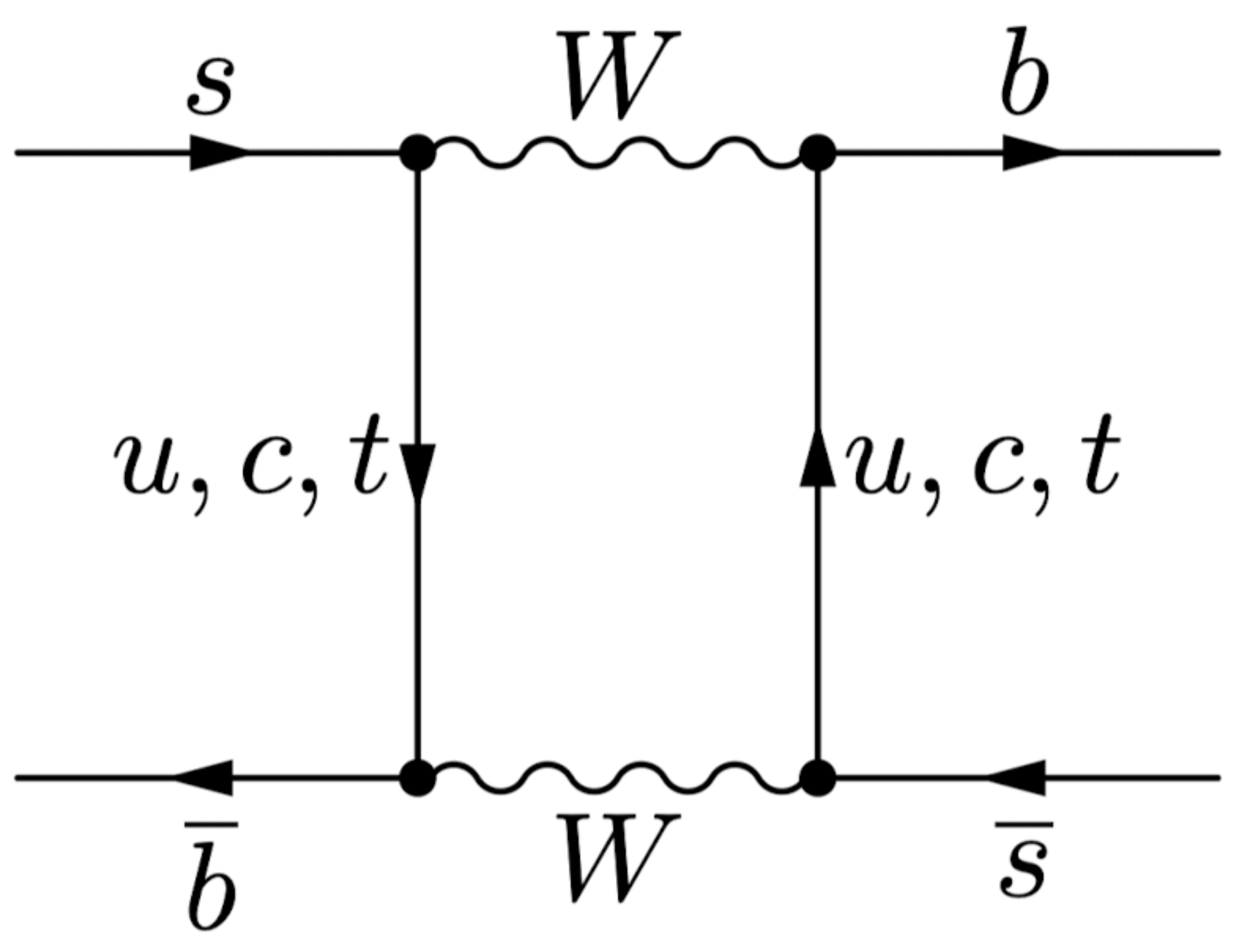
2 

= | 

+ X |$^2$

# B-Meson Mixing and Lifetimes

## Standard Model Case




- Diagram leading to B-Mixing:
  - GIM suppressed
- Leads to an effective Hamiltonian:

$$\mathcal{H}_{eff}^{\Delta B=2} = C_1 Q_1 + \text{ h.c.}, \qquad Q_1 = \bar{b}_i \gamma_\mu \frac{(1-\gamma^5)}{2} q_i \bar{b}_j \gamma^\mu \frac{(1-\gamma^5)}{2} q_j$$

- Apply Vacuum Insertion Approximation (VIA) and the correction to VIA is given by the Bag parameter $B_1(\mu)$

$$\langle \bar{B}_s | Q_1 | B_s \rangle = \left(2 + \frac{2}{N_c}\right) M_{B_s}^2 f_{B_s}^2 B_1(\mu)$$

# QCD-HQET Matching for Mixing

## Matching Operators

### QCD

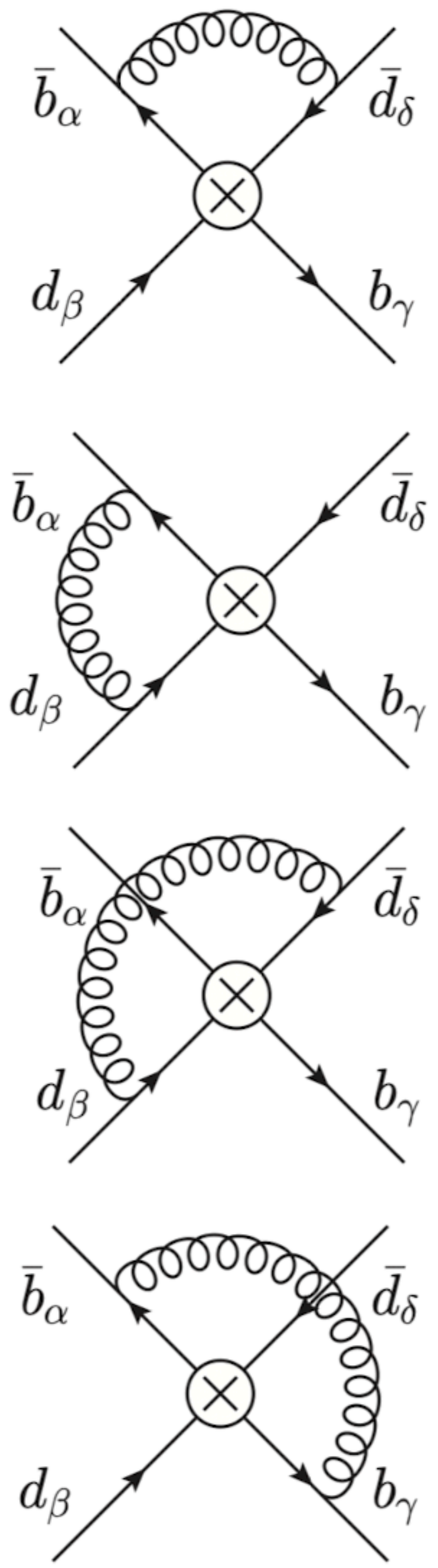
$$Q_1 = \bar{b}_i \gamma_\mu (1-\gamma^5) q_i \bar{b}_j \gamma^\mu (1-\gamma^5) q_j$$

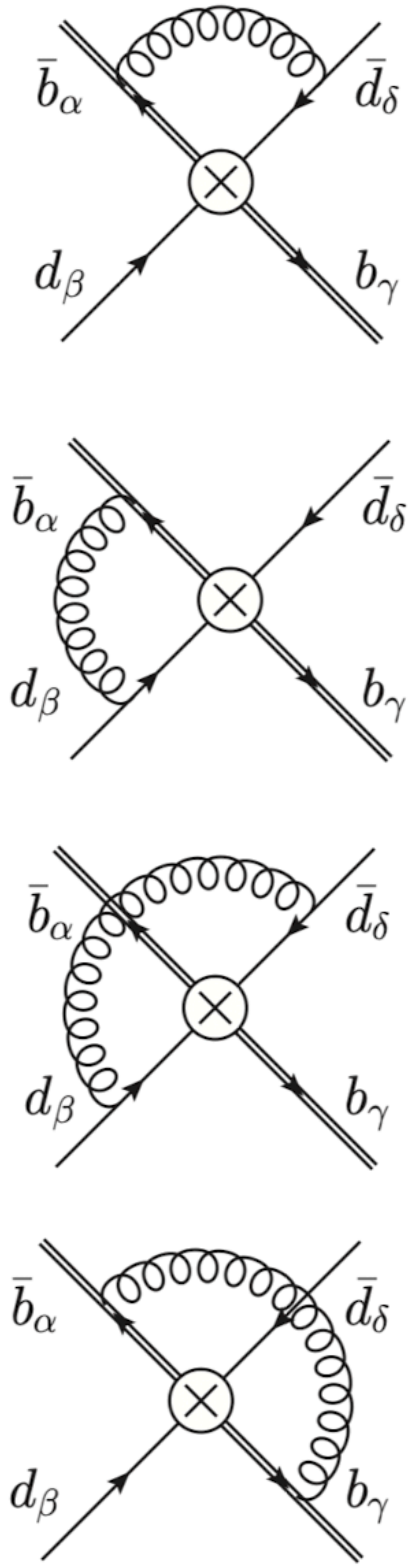
$$Q_2 = \bar{b}_i (1-\gamma^5) q_i \bar{b}_j (1-\gamma^5) q_j$$

$$Q_3 = \bar{b}_i (1-\gamma^5) q_j \bar{b}_j (1-\gamma^5) q_i$$

$$Q_4 = \bar{b}_i (1-\gamma^5) q_i \bar{b}_j (1+\gamma^5) q_j$$

$$Q_5 = \bar{b}_i (1-\gamma^5) q_j \bar{b}_j (1+\gamma^5) q_i$$




### HQET

$$\mathbf{Q}_1 = \bar{h}_i^{\{(+)}\gamma_\mu (1-\gamma^5) q_i \bar{h}_j^{(-)\}} \gamma^\mu (1-\gamma^5) q_j$$

$$\mathbf{Q}_2 = \bar{h}_i^{\{(+)} (1-\gamma^5) q_i \bar{h}_j^{(-)\}} (1-\gamma^5) q_j$$

$$\mathbf{Q}_4 = \bar{h}_i^{\{(+)} (1-\gamma^5) q_i \bar{h}_j^{(-)\}} (1+\gamma^5) q_j$$

$$\mathbf{Q}_5 = \bar{h}_i^{\{(+)} (1-\gamma^5) q_j \bar{h}_j^{(-)\}} (1+\gamma^5) q_i$$

$$\bar{h}\sigma_{\mu\nu} q\, \bar{q}\sigma^{\mu\nu} h = -2\left[\bar{h}q\,\bar{q}h - \bar{h}\gamma_\mu q\,\bar{q}\gamma^\mu h + \bar{h}\gamma_5 q\,\bar{q}\gamma_5 h + \bar{h}\gamma_\mu\gamma_5 q\,\bar{q}\gamma^\mu\gamma_5 h\right] + \mathcal{O}\left(\frac{1}{m_b}\right)$$

# B-Meson Mixing and Lifetimes

## Observable parameters

$$\begin{aligned}\Delta M_s &\equiv M_H^s - M_L^s \\ &= 2|M_{12}^s|\left(1 - \frac{|\Gamma_{12}^s|\sin^2\phi_{12}^s}{8|M_{12}^s|^2} + \ldots\right) \\ &\approx 2|M_{12}^s|,\end{aligned}$$

$$\phi_{12}^s \equiv \left(-\frac{M_{12}^s}{\Gamma_{12}^s}\right).$$

$$\begin{aligned}\Delta\Gamma_s &\equiv \Gamma_H^s - \Gamma_L^s \\ &= 2|\Gamma_{12}^s|\cos\phi_{12}^s\left(1 - \frac{|\Gamma_{12}^s|\sin^2\phi_{12}^s}{8|M_{12}^s|^2} + \ldots\right) \\ &\approx 2|\Gamma_{12}^s|\cos\phi_{12}^s,\end{aligned}$$

# HQET Sum Rules

## Borel Transform

$$\Pi(t) \equiv \mathcal{B}_t \Pi(\omega) = \lim_{\substack{-\omega, n \to \infty \\ -\omega/n \to t}} \frac{(-\omega)^{n+1}}{n!} \left[\frac{d}{d\omega}\right]^n \Pi(\omega)$$

$$\mathcal{B}_t \left[\omega^i\right] = 0$$

Removes subtraction terms

$$\mathcal{B}_t \left[\frac{1}{(s-\omega)^i}\right] = \lim_{\substack{-\omega, n \to \infty \\ -\omega/n \to t}} \frac{(-\omega)^{n+1}}{n!} \left[\frac{d}{d\omega}\right]^n (s-\omega)^{-i}$$

$$= \lim_{n \to \infty} \frac{1}{(i-1)!\, t^{(i-1)}} \frac{(i+n-1)!}{(n-1)!\, n^i} \left(1 + \frac{s}{nt}\right)^{-(i+n)}$$

$$= \frac{e^{\frac{-s}{t}}}{(i-1)!\, t^{(i-1)}}$$

Exponential suppresses the continuum tail, improving the QHD assumption

# HQET

## Hadronic Matrix Elements

$$|B(p)\rangle = \sqrt{2M_B}|\mathbf{B}(v)\rangle + \mathcal{O}(1/m_b),$$

$$\left\langle \mathbf{B}\left(v'\right) \mid \mathbf{B}(v)\right\rangle = \frac{v^0}{M_B^3}(2\pi)^3\delta^{(3)}\left(\mathbf{v}' - \mathbf{v}\right).$$

$$f_B = \sqrt{\frac{2}{M_B}}C(\mu)F(\mu) + \mathcal{O}(1/m_b)$$

$$C(\mu) = 1 - 2C_F\frac{\alpha_s(\mu)}{4\pi} + \mathcal{O}\left(\alpha_s^2\right)$$

# QCD-HQET Matching For Mixing

## Matching Bag Parameters

**QCD**

$$\langle Q(\mu)\rangle = A_Q f_B^2 M_B^2 B_Q(\mu)$$

$$\left\langle 0\middle|\bar{b}\gamma^\mu\gamma^5 q\middle|B(p)\right\rangle = -i f_B p^\mu$$

**HQET**

$$\langle \mathbf{Q}(\mu)\rangle = A_{\mathbf{Q}} F^2(\mu) B_{\mathbf{Q}}(\mu)$$

$$\left\langle 0\middle|\bar{h}^{(-)}\gamma^\mu\gamma^5 q\middle|\mathbf{B}(v)\right\rangle = -iF(\mu)v^\mu$$

$$\langle Q_i\rangle(\mu) = \sum C_{Q_i\mathbf{Q_j}}(\mu)\langle \mathbf{Q_j}\rangle(\mu) + \mathcal{O}\left(\frac{1}{m_b}\right)$$

$$B_{Q_i}(\mu) = \sum_j \frac{A_{\mathbf{Q}_j}}{A_{Q_i}} \frac{C_{Q_i\mathbf{Q}_j}(\mu)}{C^2(\mu)} B_{\mathbf{Q}_j}(\mu) + \mathcal{O}(1/m_b)$$

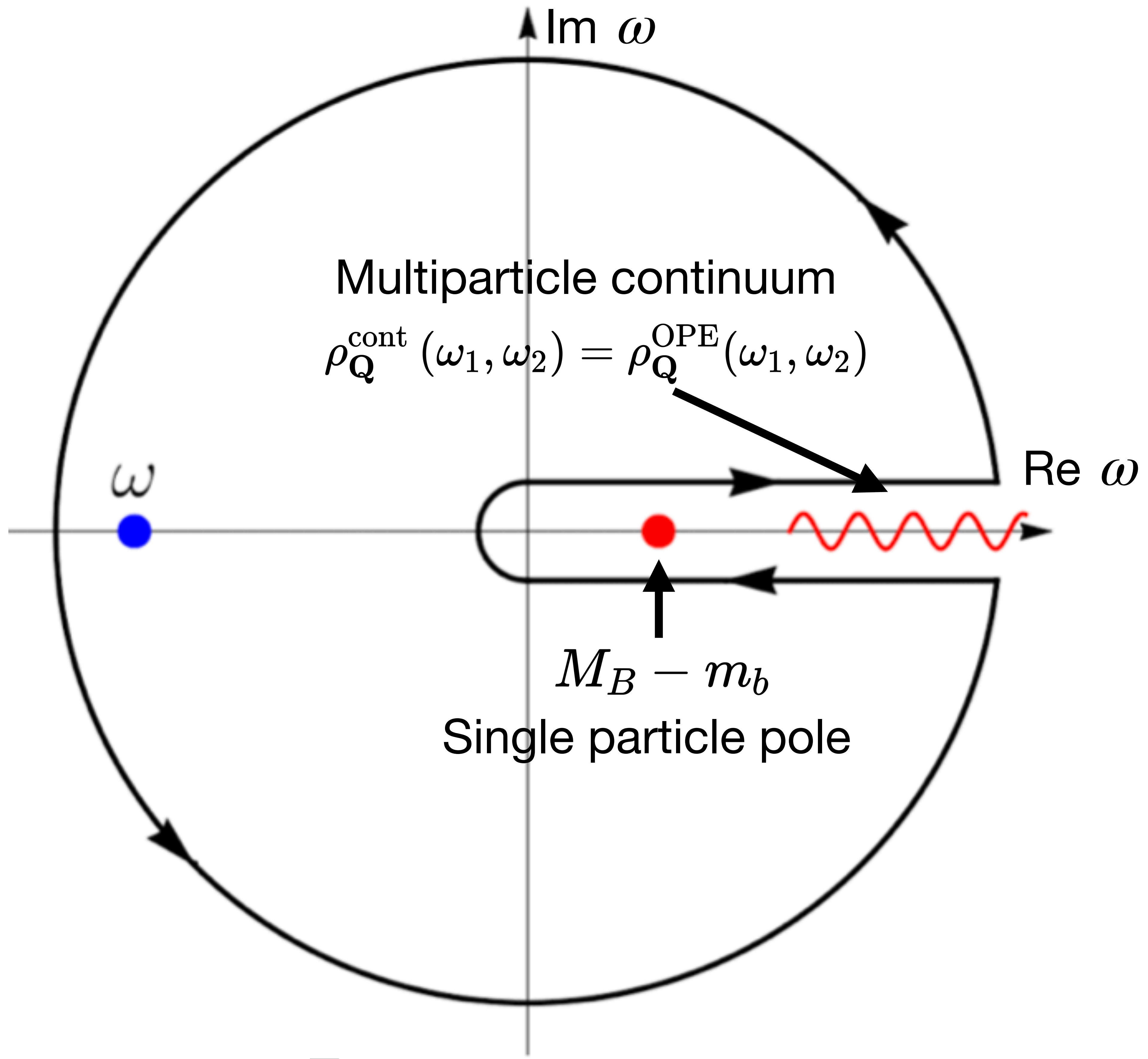
# HQET Sum Rules

## Analyticity

What is calculable?

If the heavy quark has a large negative residual energy, the light quarks must have very large momenta.

QCD is perturbative at high energies, so the **large negative residual energy limit** is calculable in perturbation theory.




$$\rho_{\mathbf{Q}}^{\text{had}}(\omega_1,\omega_2) = F^2(\mu)\langle\mathbf{Q}(\mu)\rangle\delta\left(\omega_1 - \bar{\Lambda}\right)\delta\left(\omega_2 - \bar{\Lambda}\right) + \rho_{\mathbf{Q}}^{\text{cont}}(\omega_1,\omega_2), \quad \bar{\Lambda} = M_B - m_b$$

# HQET Sum Rules

## Nonfactorizable Contribution

$$K^{\text{pert}}_{\tilde{Q}}(\omega_1, \omega_2) = K^{(0)}_{\tilde{Q}}(\omega_1, \omega_2) + \frac{\alpha_s}{4\pi} K^{(1)}_{\tilde{Q}}(\omega_1, \omega_2) + \ldots$$

$$\rho^{\text{pert}}_{\tilde{Q}_i}(\omega_1, \omega_2) = A_{\tilde{Q}_i} \rho_\Pi(\omega_1) \rho_\Pi(\omega_2) + \Delta\rho_{\tilde{Q}_i}$$

$$\Delta B_{\tilde{Q}_i} = \frac{1}{A_{\tilde{Q}_i} F(\mu)^4} \int_0^{\omega_c} d\omega_1 d\omega_2 e^{\frac{\bar{\Lambda}-\omega_1}{t_1} + \frac{\bar{\Lambda}-\omega_2}{t_2}} \Delta\rho_{\tilde{Q}_i}(\omega_1, \omega_2)$$

$$= \frac{1}{A_{\tilde{Q}_i}} \frac{\int_0^{\omega_c} d\omega_1 d\omega_2 e^{-\frac{\omega_1}{t_1} - \frac{\omega_2}{t_2}} \Delta\rho_{\tilde{Q}_i}(\omega_1, \omega_2)}{\left(\int_0^{\omega_c} d\omega_1 e^{-\frac{\omega_1}{t_1}} \rho_\Pi(\omega_1)\right)\left(\int_0^{\omega_c} d\omega_2 e^{-\frac{\omega_2}{t_2}} \rho_\Pi(\omega_2)\right)}$$

# HQET Sum Rules

## Utilizing a Weight Function

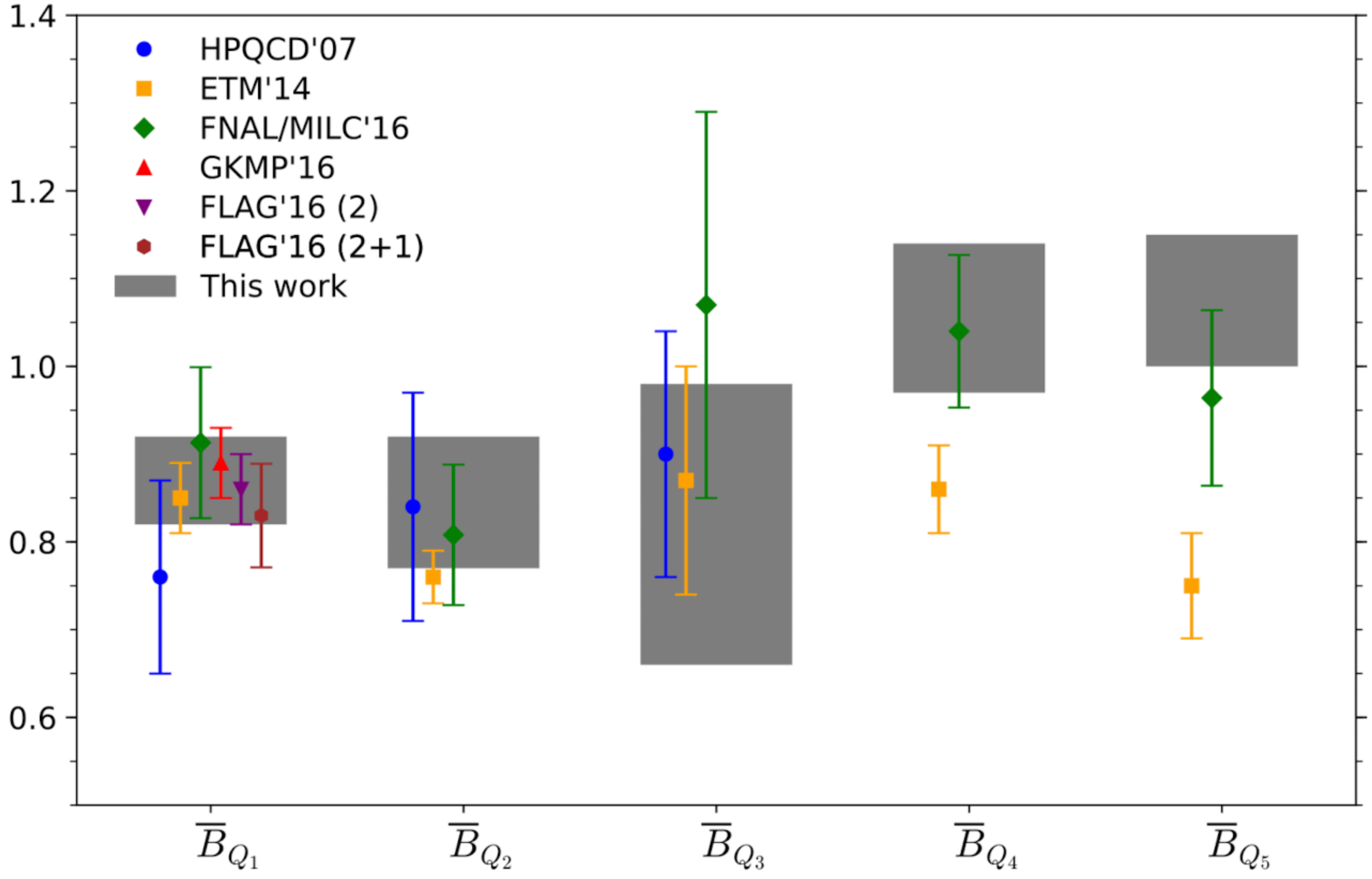
$$F^4(\mu)e^{-\frac{\bar\Lambda}{t_1}-\frac{\bar\Lambda}{t_2}}w(\bar\Lambda,\bar\Lambda) = \int_0^{\omega_c} d\omega_1 d\omega_2 e^{-\frac{\omega_1}{t_1}-\frac{\omega_2}{t_2}} w(\omega_1,\omega_2)\rho_\Pi(\omega_1)\rho_\Pi(\omega_2) + \dots$$

$$w_{\tilde{Q}_i}(\omega_1,\omega_2) = \frac{\Delta\rho^{\mathrm{pert}}_{\tilde{Q}_i}(\omega_1,\omega_2)}{\rho^{\mathrm{pert}}_\Pi(\omega_1)\rho^{\mathrm{pert}}_\Pi(\omega_2)} = \frac{C_F}{N_c}\frac{\alpha_s}{4\pi} r_{\tilde{Q}_i}(x, L_\omega)$$

$$\Delta B^{\mathrm{pert}}_{\tilde{Q}_i}(\mu_\rho) = \frac{C_F}{N_c A_{\tilde{Q}_i}}\frac{\alpha_s(\mu_\rho)}{4\pi} r_{\tilde{Q}_i}\left(1, \log\frac{\mu_\rho^2}{4\bar\Lambda^2}\right)$$

# Previous Results in Mixing

## Full BSM Basis

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