Perturbative Computations in Quantum Field Theory

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P3H

Particle Physics Phenomenology after the Higgs Discovery

CPPS retreat Meinerzhagen, February 14th – 16th, 2024

Equations of motion in classical mechanics

- Example: Bead on rotating ring
- Lagrangian function

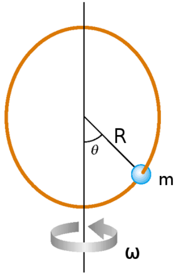
$$L = \frac{m}{2}\left[R^2\,\dot\theta^2 + \omega^2 R^2 \sin^2\theta\right] + m\,g\,R\,\cos\theta$$

- Equation of motion
  (Euler-Lagrange equation):

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0$$

- Yields:

$$\ddot\theta = \omega^2 \sin\theta\cos\theta - \frac{g}{R}\,\sin\theta$$



Equations of motion in field theory

- Example: QED
- Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi + e\,\bar{\psi}A\!\!\!/\psi$$

  - ψ: Fermion field (electron, positron)
  - A^μ: Photon field
  - $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$: Field strength tensor
  - $e = \sqrt{4\pi\alpha}$: coupling. $\alpha \sim \frac{1}{137}$

- Euler-Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

- Yields:

$$(i\partial\!\!\!/ - m)\psi = -e\,A\!\!\!/\psi$$

$$\partial_\mu F^{\mu\nu} = -e\,\bar{\psi}\gamma^\nu\psi$$

- Coupled, nonlinear equations
  - Cannot be solved in closed form

Perturbative expansion

- Idea: Series expansion in small coupling

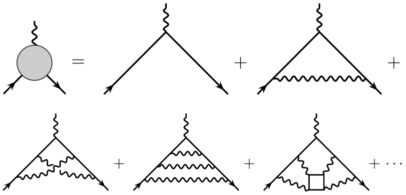
$$\alpha \sim \frac{1}{137} \qquad \alpha_s \sim 0.1 \;-\; 0.2$$

$$\mathcal{A} = \mathcal{O}(1) + \mathcal{O}(\alpha) + \mathcal{O}(\alpha^2) + \ldots$$

- At each order, more Feynman diagrams with more loops
- For each loop, have one unconstrained four-momentum ℓ^μ, to be integrated over with measure

$$\int \frac{d^4\ell}{(2\pi)^4}$$

- $a_e = (g_e - 2)/2$ of electron



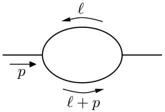
$a_e = 0$	(leading order)
$a_e = \dfrac{\alpha}{2\pi} \approx 0.0011614$	(to $\mathcal{O}(\alpha)$)
$a_e = 0.001159652181643(764)$	(to $\mathcal{O}(\alpha^5)$)
$a_e = 0.00115965218059(13)$	(experiment)

IBP reduction

- A multi-loop amplitude contains easily millions of terms (integrals)
- Not all integrals independents
- Integration-by-parts (IBP) equations [Tkachov'81;Chetyrkin,Tkachov'81]

$$\int \frac{d^d \ell_1}{i\pi^{d/2}} \cdots \int \frac{d^d \ell_L}{i\pi^{d/2}} \, \frac{\partial}{\partial \ell_i^\mu} \, \frac{v_j^\mu}{D_1^{a_1} \cdots D_n^{a_n}} = 0$$

- For the one-loop bubble



$(p^2 \neq 0)$

$\underline{v = \ell:}$

$$0 = (d - a_2 - 2a_1) F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ - a_2 F(a_1 - 1, a_2 + 1)$$

$\underline{v = p:}$

$$0 = (a_1 - a_2) F(a_1, a_2) + a_2 p^2 F(a_1, a_2 + 1) - a_1 p^2 F(a_1 + 1, a_2) \\ + a_2 F(a_1 - 1, a_2 + 1) - a_1 F(a_1 + 1, a_2 - 1)$$

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2}}\,, \qquad a_i \in \mathbb{Z}$$

$$\begin{aligned} D_1 &= -\ell^2 \\ D_2 &= -(\ell + p)^2 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \ell^2 &= -D_1 \\ \ell \cdot p &= \frac{1}{2}(D_1 - D_2 - p^2) \end{aligned}$$

IBP reduction: Laporta's algorithm

- Laporta's algorithm [Laporta'01]
  - Specialize set of IBP equations to a range of integer indices
  - Generate system of linear equations

include symmetries

directly discard scaleless integrals

$$
\begin{aligned}
0 &= F(2,1) - F(1,2) \\
0 &= (d-3)F(1,1) - p^2 F(1,2) \\
0 &= (d-4)F(1,2) - 2p^2 F(1,3) \\
0 &= 2p^2 F(1,3) - p^2 F(2,2) - F(1,2) - F(2,1) \\
0 &= (d-5)F(2,1) - p^2 F(2,2) - F(1,2) \\
&\vdots
\end{aligned}
$$

  - Solve system by Gaussian elimination.

    Express each integral as a linear combination of independent master integrals.

IBP reduction: Laporta's algorithm

- Calcuation of higher loop-orders entails several challenges
  - number of Feynman diagrams
  - size of intermediate expressions
  - complexity of loop integrals (# loops, legs, scales)
- Many sophisticated public and private tools exist
  - AIR [Anastasiou,Lazopoulos'04]
  - Reduze [Studerus'09; Manteuffel,Studerus'12]
  - FIRE [Smirnov et al.'08+]
  - LiteRed [Lee'12+]
  - Kira [Maierhöfer,Usovitsch,Uwer'17; Klappert,Lange,Maierhöfer,Usovitsch'20]
  - Crusher . . . [Marquard,Seidel]

- Mostly based on Laporta's algorithm [Laporta'01]
  - Solves IBP equations for numerical values of indices with Gaussian elimination.
- Several refinements exist, e.g.
  - Parallelization
  - Methods from finite fields
    [v. Manteuffel,Schabinger'14; Smirnov,Chukharev'19]
    [Peraro'16'19; Klappert,Klein,Lange'19'20]
- Drawbacks
  - Compute many more integrals than required
  - Large storage required for results of $10^{\sim 4-6}$ integrals

Algebraic-geometry approach

- New ideas from
  - algebraic geometry [Larsen,Zhang'14; Böhm et al.'18'19]
  - syzygy equations [Kosower et al.'10'18; Schabinger et al.'11'20; Ita'15; Böhm et al.'17]

$$A \cdot B = 0$$

  - Linear algebra
- New approach [Barakat,TH et al. 2020+]
  - Leave propagator powers symbolic
  - Derive so-called normal-form IBP relations

- Normal-form IBP equations

$$R_i \;=\; a_i\, D_i^- - \mathrm{NF}_G\left(a_i D_i^-\right) \;\in\; I_{\mathrm{IBP}}$$

  - e.g. for one-loop bubble

$$\mathrm{NF}_G\left(a_1 D_1^-\right) = \frac{(d-a_1-a_2-1)(d-2a_1-2a_2)}{p^2(d-2a_1-2)}$$

$$\mathrm{NF}_G\left(a_2 D_2^-\right) = \frac{(d-a_1-a_2-1)(d-2a_1-2a_2)}{p^2(d-2a_2-2)}$$

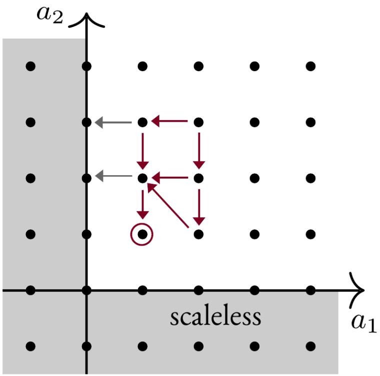
  - Allow for a straightforward reduction
  - All you have to store
  - Easy to implement in Mathematica or FORM
  - Well-suited for parallelization
  - Allows for fast reduction, also of high propagator powers

Laporta vs. Normal-form IBPs

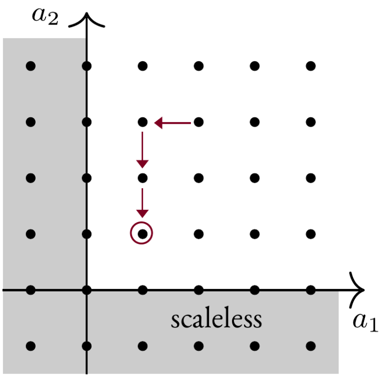
- Compare reduction with Laporta vs. Normal-form IBPs for the one-loop massless bubble

Reduction of $F(3,2)$:

using standard IBPs



using $\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}_G(\widehat{a}_i \widehat{D}_i^-)$



Computing needs

- Jobs run, depending on complexity, on Laptop, desktop, OMNI, cluster in Karlsruhe
- Need modern CAS (Mathematica, C++, Fermat, ...)
- RAM, up to $\mathcal{O}(10^{2-3}\mathrm{GB})$
- storage space up to $\mathcal{O}(1\mathrm{TB})$
- Fast I/O traffic, read/write on local disk (scratch)
- Parallelizable ($\mathcal{O}(20)$ cores)
- Job runtimes of up to $\mathcal{O}(\mathrm{weeks})$

- Training for students in CAS, programming needed

- Local resources improvable in machines with a lot of RAM, Mathematica licenses, job length

- Not so much in need of GPUs, network