β function, Λ parameter and more

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in collaboration with
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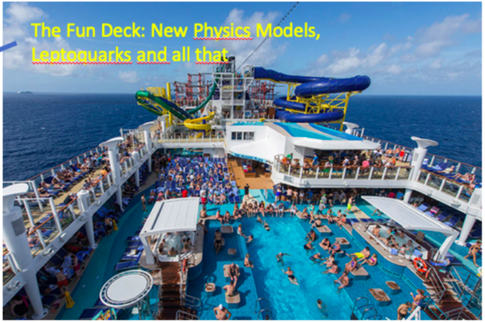
Quirks in Quark Flavour Physics
Zadar, Croatia · June 21, 2024



Legendary slide by Thomas Mannel

- No anomalies
- No CKM matrix elements
- Only two (massless) flavors
- Only QCD
- Small quirks to be improved
- Originates from studying many flavor BSM systems
- Powerful prospects

Renormalization Group β function

$$\beta(g^2) = \mu^2 \frac{dg^2}{d\mu^2}$$

- Encodes dependence of coupling g^2 on the energy scale μ^2
- β has no explicit dependence on μ^2, only implicit through $g^2(\mu)$
- Known perturbatively up to 5-loop order in the $\overline{\text{MS}}$ scheme (1- and 2-loop are universal)
  [Baikov, Chetyrkin, Kühn PRL118(2017)082002] [Ryttov and Shrock PRD94(2016)105015]
- Known perturbatively at 3-loop order in the gradient flow scheme [Harlander, Neumann JHEP06(2016)161]
- Perturbative predictions reliable at weak coupling,
  nonperturbative methods needed for strong coupling

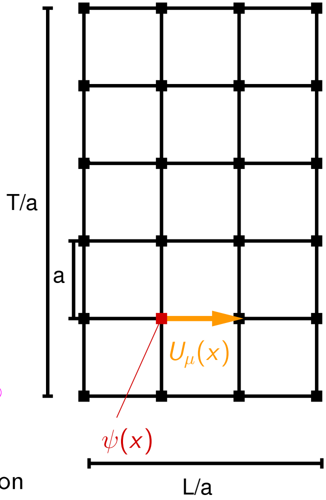
Lattice calculations

- Wick-rotate to Euclidean time $t \to i\tau$
- Discretize space-time and set up a hypercube of finite extent $(L/a)^3 \times T/a$ and spacing a
- Use path integral formalism

$$\langle \mathcal{O} \rangle_E = \frac{1}{Z} \int \mathcal{D}[\psi, \overline{\psi}]\, \mathcal{D}[U]\, \mathcal{O}[\psi, \overline{\psi}, U]\, e^{-S_E[\psi, \overline{\psi}, U]}$$

$\Rightarrow$ Large but finite dimensional path integral

- Finite lattice spacing $a \to$ UV regulator
- Finite volume of length $L \to$ IR regulator
  $\to$ Study physics in a finite box of volume $(aL)^3$ plus limit $L \to \infty$
- Different discretizations for gauge and fermion actions possible
  $\to$ Wilson, Symanzik gauge; Wilson, staggered, domain-wall fermion
  $\to$ Discretization effects disappear after taking $a \to 0$ continuum limit



Gradient flow

[Narayanan and Neuberger JHEP 0603 (2006) 064] [Lüscher CMP 293 (2010) 899][JHEP 1008 (2010) 071]

- Add flow time coordinate t with dimension [-2] and define gauge field $B_\mu(x, t)$

$$\frac{\partial}{\partial t} B_\mu(t) = \mathcal{D}_\nu(t) G_{\nu\mu}(t) \qquad \text{with} \qquad B_\mu(t=0) = A_\mu$$

  → Ordinary differential equation (ODE) ⇝ solve numerically using Runge-Kutta

  → Covariant derivative defined in terms of the flow field $B_\mu(t)$ and the Yang-Mills action $S_{YM}(B)$

- Gradient flow is a smoothing/averaging transformation

  → Can be related to RG flow ⇝ perfect to define RG β function, ...

- Energy density $\langle E(t)\rangle = -\frac{1}{2} Tr\langle G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\rangle$

  → Renormalized coupling, scale setting ($\sqrt{8t_0}$, w_0)

- Different gradient flows: Wilson flow, Symanzik flow, Zeuthen flow, ...

Step-Scaling β function

- Discretized β function determined using numerical lattice field theory calculations [Lüscher et al. NPB359(1991)221]
  - → Choose symmetric L^4 setup where the size L of the lattice is the only scale
  - → Determine β function by calculating scale change $L \to s \cdot L$
  - → Conventionally differs by a minus sign compared to the continuum definition
- Use gradient flow to define a renormalized coupling

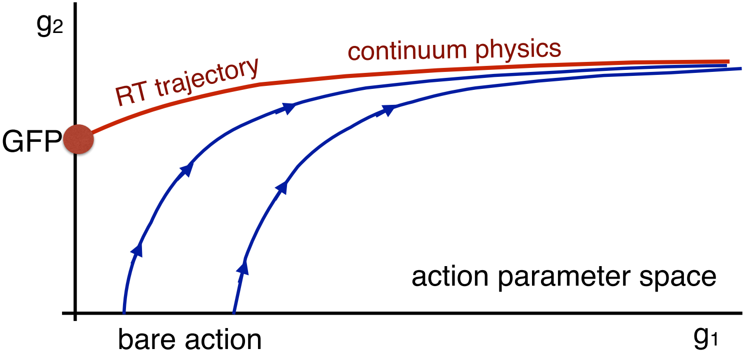
$$g_c^2(L) = \frac{128\pi^2}{3(N_c^2 - 1)} \frac{1}{C(c, L)} t^2 \langle E(t) \rangle$$

  - → Relate flow time t to scale L: $\sqrt{8t} = c \cdot L$ [Fodor et al. JHEP11(2012)007][JHEP09(2014)018]
  - → Calculate scale difference

$$\beta_s^c(g_c^2; L) = \frac{g_c^2(sL) - g_c^2(L)}{\log(s^2)}$$

  - → Extrapolate $L/a \to \infty$ to remove discretization effects and take combined continuum and infinite volume limit

Gradient flow and real-space renormalization Group (RG) flow



Gradient flow and real-space renormalization Group (RG) flow

- RG flow: change of (bare) parameters and coarse graining (blocking)

- Gradient flow is a continuous transformation
  → Define real-space RG blocked quantities by incorporating coarse graining as part of calculating expectation values [Carosso, Hasenfratz, Neil PRL 121 (2018) 201601]

- Relate GF time t/a^2 to RG scale change $b \propto \sqrt{t/a^2}$
  → Quantities at flow time t/a^2 describe physical quantities at energy scale $\mu \propto 1/\sqrt{t}$
  → Local operator with non-vanishing expectation value can be used to define running coupling
  ⇝ Simplest choice: $t^2\langle E(t)\rangle$ [Lüscher JHEP 1008 (2010) 071]

- Continuous RG β function

$$\beta_{GF}(g^2_{GF}) = \mu^2 \frac{dg^2_{GF}}{d\mu^2} = -t\frac{dg^2_{GF}}{dt}$$

Continuous RG β function

[Fodor et al. EPJ Web Conf. 175 (2018) 08027]
[Hasenfratz, OW PRD 101 (2020) 034514] [Hasenfratz, OW PoS LATTICE2019 (2019) 094]
[Wong et al. PoS LATTICE2022 (2023) 043] [Hasenfratz, Peterson, Van Sickle, OW PRD108 (2023) 014502]

New in next FLAG edition!

- Extract $g^2_{GF}(t; \beta_b, L/a)$ its derivative $\beta_{GF}(t; \beta_b, L/a)$ for a range of GF times on each ensembles
  $\rightarrow$ Different bare coupling β_b on different volumes $(L/a)^4$ or $(L/a)^3 \times T/a$
- Perform infinite volume extrapolation at fixed bare coupling β_b and GF time t
  $\rightarrow$ Obtain $g^2_{GF}(t; \beta_b)$ and $\beta_{GF}(t; \beta_b)$
- Interpolate discrete infinite volume values to get continuous values at fixed flow time
  $\rightarrow$ $g^2_{GF}(t)$ and $\beta_{GF}(t; g^2_{GF})$
- Take continuum limit $(a^2/t \rightarrow 0)$ for fixed g^2_{GF} and obtain $\beta_{GF}(g^2_{GF})$

Continuous RG β function

- Extract $g^2_{GF}(t;\beta_b,L/a)$ its derivative $\beta_{GF}(t;\beta_b,L/a)$ for a range of GF times on each ensembles
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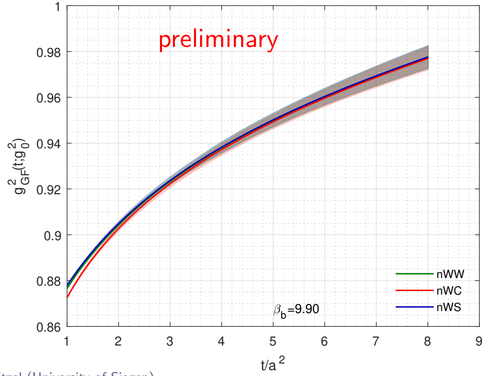

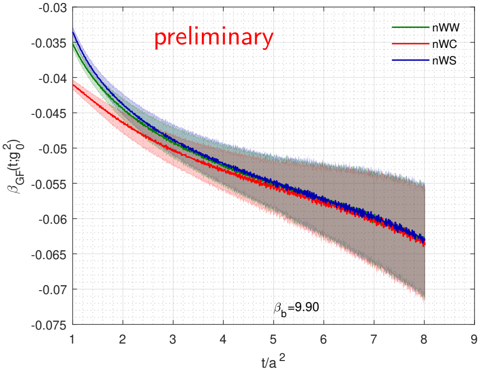

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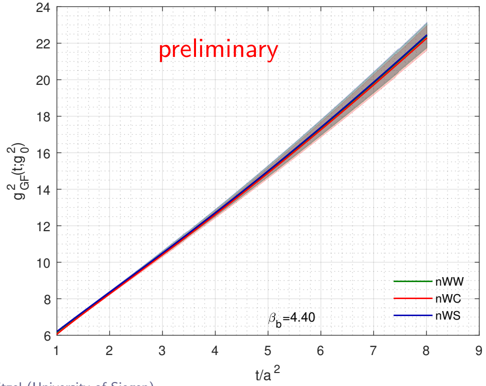
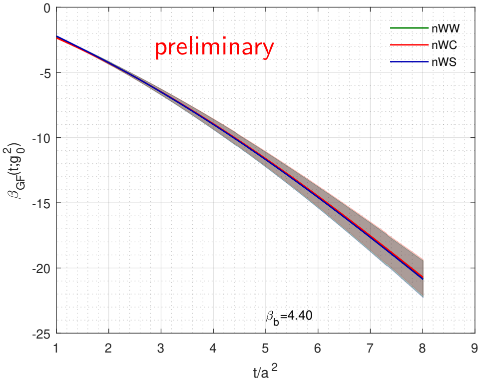
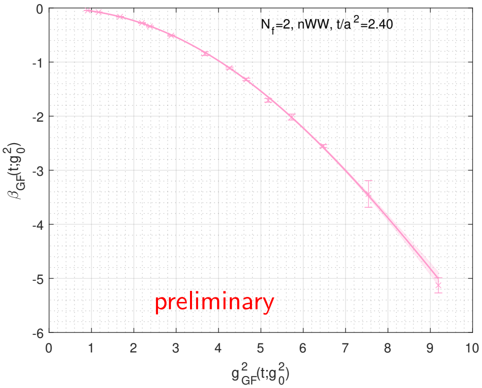
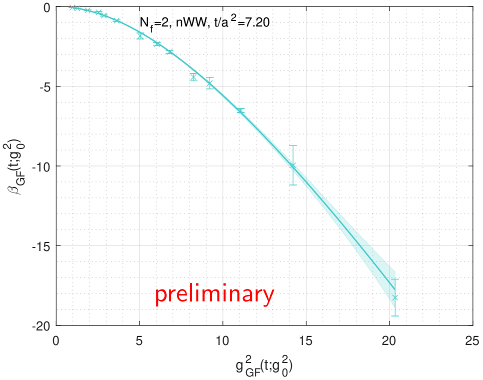




Continuous RG β function

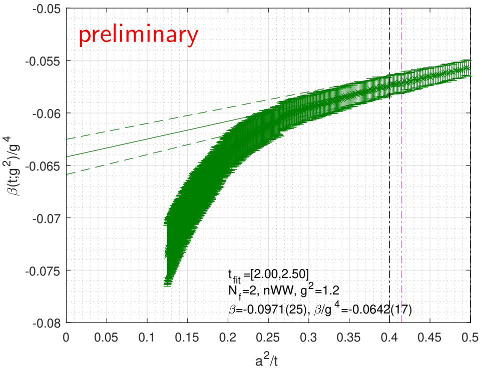
- Interpolate discrete infinite volume values to get continuous values at fixed flow time
  $\rightarrow g^2_{GF}(t)$ and $\beta_{GF}(t; g^2_{GF})$

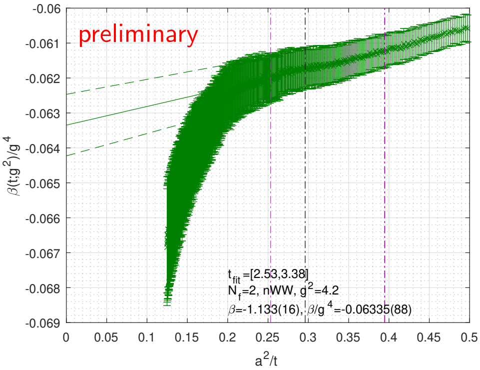




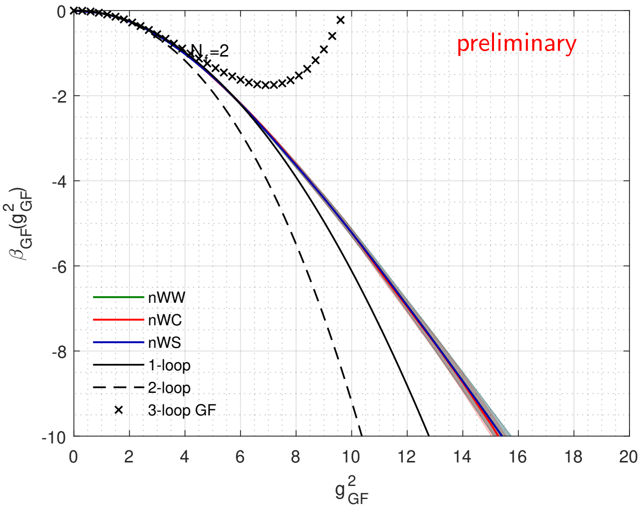
Continuous RG β function

- Take continuum limit ($a^2/t \to 0$) for fixed g^2_{GF} and obtain $\beta_{GF}(g^2_{GF})$



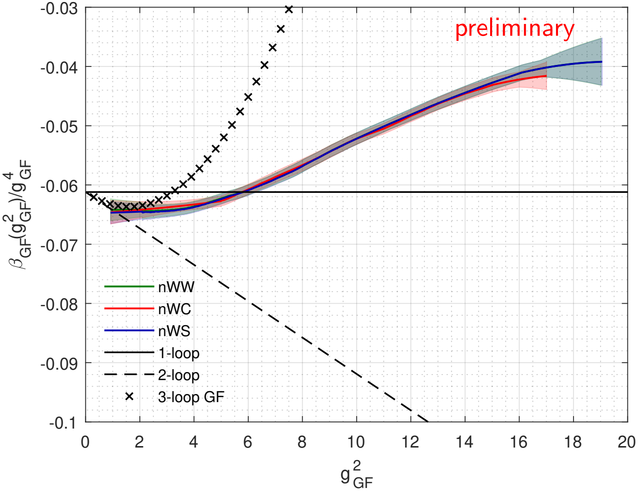


Continuum limit of continuous RG β function for $N_f = 2$



- Weak coupling well described by PT 3-loop GF result
  [Harlander, Neumann JHEP06(2016)161]
- Qualitatively more similar to 1-loop result at strong coupling
  → Very different to 3-loop GF
- Apparently linear β_{GF} function at strong coupling
  → Nonperturbative phenomenon

Continuum limit of continuous RG β function for $N_f = 2$



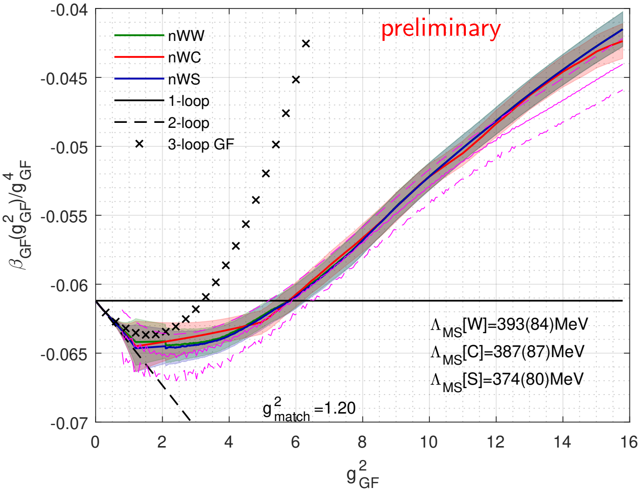
- Weak coupling well described by PT 3-loop GF result
  [Harlander, Neumann JHEP06(2016)161]
- Qualitatively more similar to 1-loop result at strong coupling
- Simulations at very weak coupling are challenging (critical slowing down)

Λ parameter

- Integrate inverse β function to obtain Λ_{GF}
  - $\to$ g_m^2 GF renormalized coupling at energy scale $\mu = 1/\sqrt{8t_0}$
  - $\to$ Only need t_0 lattice scale
  - $\to$ b_0, b_1 universal 1-loop coefficients

$$\Lambda_{GF} = \mu\left(b_0 g_m^2\right)^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 g_m^2}\right) \exp\left[-\int_0^{g_m^2} \mathrm{d}g^2 \left(\frac{1}{\beta(g^2)} + \frac{1}{b_0 g^4} - \frac{b_1}{b_0^2 g^2}\right)\right]$$

Match to 3-loop GF β function at small g^2_{GF}



- Presently large uncertainty in Λ due to matching at weak coupling
- $g_m^2 \approx 15.8$
- In principal could relate Λ to α_s
  - $\rightarrow$ $N_f = 2$ requires running "through" strange quark threshold
  - $\rightarrow$ Better repeat with $N_f = 3$ or 4

Gradient Flow renormalization [Hasenfratz, Monahan, Rizik, Shindler, OW PoS Lattice2021 155]

$$\mathcal{O}_R^{\overline{\text{MS}}}(\mu_{\text{UV}}) = c^{\overline{\text{MS}}\leftarrow\text{GF}}(\mu_{\text{UV}}, \mu_{\text{IR}}) \quad Z_{\mathcal{O}}^{\text{GF}}(a; \mu_{\text{IR}}) \quad \mathcal{O}(a)$$

PT matching to $\overline{\text{MS}}$

connect $\mu_{\text{IR}} \to \mu_{\text{UV}}$

define matching factor $Z_{\mathcal{O}}^{\text{GF}}(a; t)$

- Need fermionic gradient flow

$$\partial_t \chi(t, x) = \mathcal{D}^2(t)\chi(t, x) \qquad \text{with} \qquad \chi(0, x) = q(x)$$

Gradient flow renormalization scheme [Hasenfratz, Monahan, Rizik, Shindler, OW PoS Lattice2021 155]

$$\bar{Z}_{\mathcal{O}}^{\text{GF}}(a;t_0)\bar{R}_{\mathcal{O}}(a;x_4,t_0) = \bar{R}_{\mathcal{O}}^{\text{tree}}(a;x_4,t_0) \quad \text{for } x_4/\sqrt{8t_0} \to \infty$$

- Obtain anomalous dimension $\gamma_{\mathcal{O}}(a;t)$ with $R_{\mathcal{O}} = G_{\mathcal{O}}/G_V$

$$\gamma_{\mathcal{O}}(a;t) = \mu \frac{d \log \bar{Z}_{\mathcal{O}}^{\text{GF}}(a;\mu)}{d\mu} = 2t \frac{d \log R_{\mathcal{O}}(a;x_4,t)}{dt} \qquad \text{with} \quad \mu = 1/\sqrt{8t}$$

- Typical 2-pt correlation function parametrized by

$$G_{\mathcal{O}}(t) = A_1(t)e^{-m_1 x_4} + A_2(t)e^{-m_2 x_4} + \dots$$

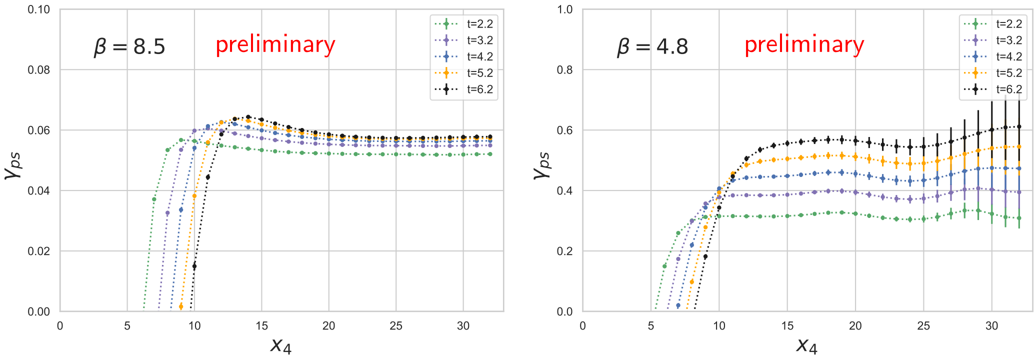
$$\Rightarrow \quad 2t\frac{d \log G_{\mathcal{O}}(t)}{dt} = \frac{d \log A_1(t)}{dt} + \mathcal{O}(e^{-(m_2-m_1)x_4})$$

- Expect $\gamma_{\mathcal{O}}(t)$ independent of x_4 (for $x_4 \ll \sqrt{8t}$)
  → Corresponds to the lightest state because others die out

Example: extract anomalous dimension for the pseudoscalar

[Hasenfratz, Monahan, Rizik, Shindler, OW PoS Lattice2021 155]

- Signal at weak and strong coupling



Use gradient flow to run and match at μ_{UV}

[Hasenfratz, Monahan, Rizik, Shindler, OW PoS Lattice2021 155]

- Connect $\mu_{\text{IR}} \to \mu_{\text{UV}}$ by integrating RG equations

$$\lim_{a\to 0} \frac{Z_{\mathcal{O}}^{\text{GF}}(a,\mu_{\text{UV}})}{Z_{\mathcal{O}}^{\text{GF}}(a,\mu_{\text{IR}})} = \exp\left\{\int_{\bar{g}_{\text{IR}}}^{\bar{g}_{\text{UV}}} dg' \frac{\gamma_{\mathcal{O}}^{\text{GF}}(g')}{\beta^{\text{GF}}(g')}\right\}$$

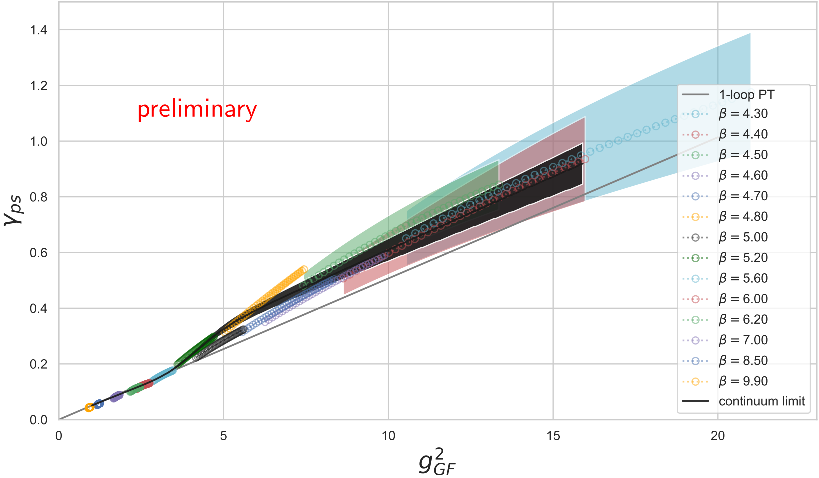
- Requires continuum limit of β^{GF} and $\gamma_{\mathcal{O}}^{\text{GF}}$ calculated in the GF scheme

$$\lim_{a\to 0}\left[\beta^{\text{GF}}(a; g^2_{\text{GF}}) = -t\frac{dg^2_{\text{GF}}(a;t)}{dt}\right]$$

$$\lim_{a\to 0}\left[\gamma_{\mathcal{O}}^{\text{GF}}(a; g^2_{\text{GF}}) = -2t\frac{d\log Z_{\mathcal{O}}^{\text{GF}}(a;t)}{dt}\right]$$

Running anomalous dimension for the pseudoscalar

[Hasenfratz, Monahan, Rizik, Shindler, OW PoS Lattice2021 155]



Summary

- Gradient flow is a powerful tool
- Continuous β function can be calculated in the deconfined as well as the confined regime
  → Provides new method to nonperturbatively determine the Λ parameter ($\rightsquigarrow \alpha_s$)
  → Only t_0 scale in GeV needed to convert to physical units
- Fermionic gradient flow allows for new nonperturbative renormalization method
  → Using gradient flow to run and match at higher energies

Quirky details: gradient flown 2-pt correlation functions

- Evaluate gradient flown quark propagator on the lattice: $\chi_{\mathrm{GF}}(t) = Z_\chi^{1/2}\chi(t)$
  $\rightarrow$ Picks-up unknown wave-function renormalization $Z_\chi^{1/2}$

- Calculate 2-pt function to determine $G_{\mathcal{O}}(a; x_4, t)$ with operator $\mathcal{O}_\Gamma(t) = \bar{\chi}(t)\Gamma\chi(t)$

$$G_{\mathcal{O}}(a; x_4, t) = \int d^3\vec{x}\, d^3\vec{x}' \langle \mathcal{O}(\vec{x}, x_4, t)\mathcal{O}(\vec{x}', 0, t = 0)\rangle$$

  $\rightarrow$ t gradient flow time, x_4 Euclidean time, $\vec{x}$ spatial components

- If $\mathcal{O}$ is a scaling operator, RG transformation with scale change $b \propto \sqrt{8t/a^2}$ predicts
  $G_{\mathcal{O}}(g_i, x_4) = b^{-\Delta_{\mathcal{O}}} G_{\mathcal{O}}(g_i^{(b)}, x_4/b)$ with $x_4 \gg b$ and $g_i^{(b)}$ RG flown couplings
  [Carosso et al. PRL 121 (2018) 201601]

Quirky details: gradient flown ratios of correlation functions

- Calculate log derivative of $G_{\mathcal{O}}$ w.r.t. GF time t

$$t\frac{\mathrm{d}}{\mathrm{d}t}\log\, G_{\mathcal{O}}(a;x_4,t) = \Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}} + \eta$$

$\rightarrow d_{\mathcal{O}}$ canonical dimension of the operator

$\rightarrow \gamma_{\mathcal{O}}$ anomalous dimension of the operator

$\rightarrow \eta$ field anomalous dimension

- Define ratios using vector currents ($\gamma_V = 0$) to cancel η (wave-function renormalization Z_χ)

$$R_{\mathcal{O}}(a;x_4,t) = \frac{G_{\mathcal{O}}(a;x_4,t)}{G_V(a;x_4,t)}$$

- Use double-ratios to eliminate the unflown probe

$$\bar{R}_{\mathcal{O}}(a;x_4,t) = \frac{R_{\mathcal{O}}(a;x_4,t=0)}{R_{\mathcal{O}}(a;x_4,t)}$$

$\rightarrow$ i.e. $\bar{R}_{\mathcal{O}}$ renormalizes with $\bar{Z}_{\mathcal{O}} = Z_{\mathcal{O}}/Z_V$