Future Theory

Maxwell T. Hansen

July 21st, 2023

THE UNIVERSITY of EDINBURGH

The future of the theory of charm…

Lenz, Alexander, Prof. Dr.
Re: Plenary talk at CHARM 2023
To: Max Hansen, Cc: Witzel, Oliver, Dr. rer. nat., Malami, Eleftheria, Nico Gubernari & 1 r

Dear Max,

we would like to invite you to give a plenary talk
about future theory aspects of charm physics - what can lattice do in 10, 20 years...
at CHARM 2023 in Siegen, Germany.

- My thoughts…
  - I definitely want to go to this conference
  - It's fun to think about what lattice can do in 10 - 20 years
  - I am probably not the best person to summarise the *future of theory in general*

↓

accept invitation and hope for the best!

Landscape of non-lattice charm theory

Charm Physics: From Standard Model to New Physics	*Svjetlana Fajfer*	
US-C 116, Hörsaalzentrum Unteres Schloss	11:00 - 11:45	
Charmed hadron lifetimes and the status of D-Dbar mixing	*Prof. Blazenka Melic*	
US-C 116, Hörsaalzentrum Unteres Schloss	16:30 - 17:00	
Spectroscopy	*Antonio Polosa*	
US-C 116, Hörsaalzentrum Unteres Schloss	09:00 - 09:45	
Tau Lepton Physics	*Antonio Pich*	
US-C 116, Hörsaalzentrum Unteres Schloss	09:00 - 09:45	
Status of Intrinsic Charm	*Ramona Vogt*	
US-C 116, Hörsaalzentrum Unteres Schloss	11:00 - 11:45	
Charmonia in Media	*Krista Smith*	
US-C 116, Hörsaalzentrum Unteres Schloss	11:45 - 12:30	
Semi-leptonic decays of charmed hadrons	*Keri Vos*	
US-C 116, Hörsaalzentrum Unteres Schloss	09:00 - 09:45	
Hadronic charm decays and CP Violation	*Fu-Sheng Yu*	
US-C 116, Hörsaalzentrum Unteres Schloss	09:00 - 09:45	

Future of non-lattice charm theory

- Improve precision/reliability for pre- and post-dictions of
  - hadronic spectra, properties (Svjetlana)
  - decays (Svjetlana, Fu-Sheng)
- Constrain *new physics* parameters using charm experiment + theory (Svjetlana)

- Revisit methods and extend available orders in $1/m_Q$ and α_s expansions (Blazenka, Keri)
- Use lattice to determine inputs for such expansions (Blazenka, Keri)
- Progress methods for $D^0 \leftrightarrow \overline{D}^0$ mixing, multi-hadron decays (Blazenka)
- Improve techniques and *theoretical defintions* for nature of states, e.g. *X*, *Y*, *Z* (Polosa)

- Take advantage of improved experimental data inputs, especially for
  - τ physics + charm (Pich)
  - intrinsic charm (Ramona)
  - charmonia in media (Krista)
  - heavy quark expansions (Keri)
  - multi-hadron decays (Fu-Sheng)

Future of non-lattice charm theory

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Will endeavour to connect to this at the end of the talk ✓

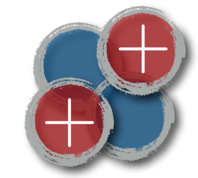
The following will be lattice motivated… but keep an eye out for non-lattice theory

- Take advantage of improved experimental data inputs, especially for
  - τ physics + charm (Pich)
  - intrinsic charm (Ramona)
  - charmonia in media (Krista)
  - heavy quark expansions (Keri)
  - multi-hadron decays (Fu-Sheng)

Lattice QCD: *Recipe for strong force predictions*

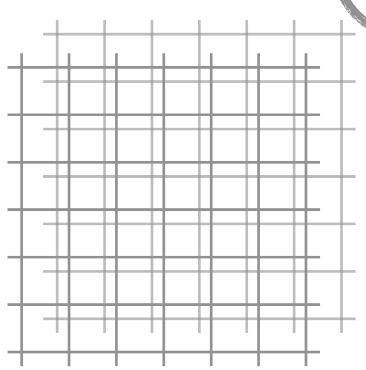
1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice QCD)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



Wide range of precision pre-/post-dictions

$m_q^{\overline{\rm MS}}(\mu)$ m_p $\alpha_{\overline{\rm MS}}(\mu)$ m_n f_π g_A f_K $(g-2)_\mu^{\rm HVP}$ f_D f_B



FLAG 2019 Flavour Lattice Averaging Group

$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1182(8)$ lattice average

$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ Lattice QCD as a reliable tool ✓

More challenging observables? 😮

Observables for lattice QCD

- Essential to classify by types of states and inserted operators

  - 1. Single-hadron form factors

$$\langle \text{hadron} | \mathcal{O}(0) | \text{hadron} \rangle$$

$$D \to \pi \ell \nu \qquad \Lambda_c \to \Lambda \ell \nu$$

  - 2. Multi-hadron scattering and decays

$$\langle \text{multi-hadron state} | \text{multi-hadron state} \rangle$$
$$\langle \text{multi-hadron state} | \mathcal{O}(0) | \text{hadron} \rangle$$

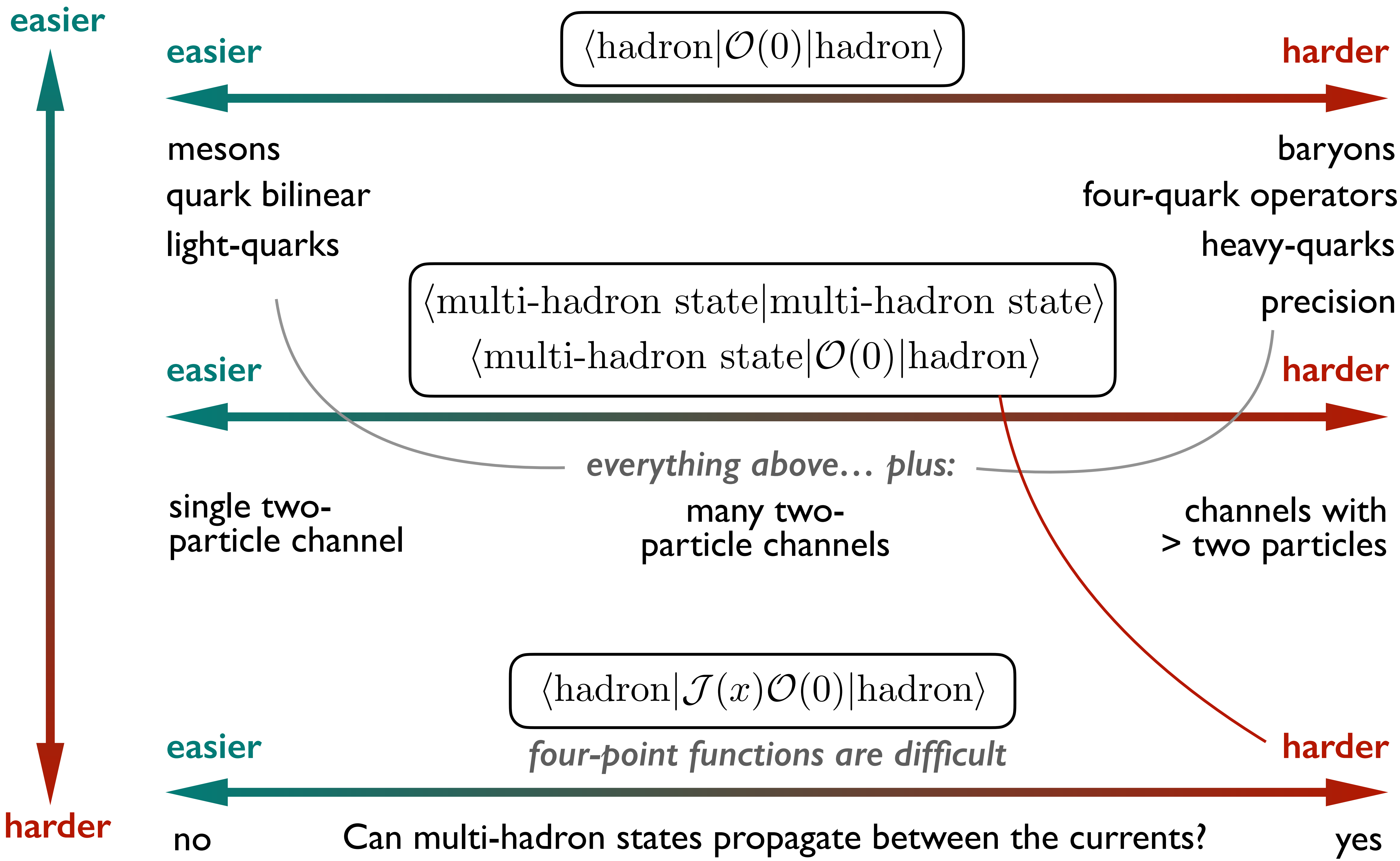
$$D\pi \to D\pi \quad \to D\eta \qquad D^*K \to D^*K \qquad D \to \pi\pi \quad \to K\pi\pi$$

  - 3. Intermediate multi-hadron states

$$\langle \text{hadron} | \mathcal{J}(x) \mathcal{O}(0) | \text{hadron} \rangle$$

$$D^0 \quad \leftrightarrow \quad \pi\pi, K\overline{K} \quad \leftrightarrow \quad \overline{D}^0$$

(Incomplete) landscape of lattice observables



Single-hadron quantities

$\langle \text{hadron} \vert \text{hadron} \rangle$	$\langle \text{hadron} \vert \mathcal{O}(0) \vert \text{vacuum} \rangle$	$\langle \text{hadron} \vert \mathcal{O}(0) \vert \text{hadron} \rangle$
masses	***decay constants***	***form factors***

- Partly covered in excellent talks at this conference by…

Sara Collins — *Meson and baryon spectroscopy with charm quarks from lattice QCD*

Will Jay — *Lattice Results for Semileptonic Decays of Charmed Hadrons*

Felix Erben — *D-meson mixing from lattice QCD*

Juan Andreas Urrea Nino — *Toward the physical charmonium spectrum with improved distillation*

Brian Colquhoun — *Precise determination of the decay rates of* $\eta_c \to \gamma\gamma$, $J/\psi \to \gamma\eta_c$, $J/\psi \to \eta_c e^+ e^-$, *from lattice QCD*

Tomas Korzek — *Iso-scalar states from LQCD*

Roman Höllwieser — *Charmonium and glueballs including light hadrons*

Single-hadron quantities

$\langle \text{hadron} \vert \text{hadron} \rangle$	$\langle \text{hadron} \vert \mathcal{O}(0) \vert \text{vacuum} \rangle$	$\langle \text{hadron} \vert \mathcal{O}(0) \vert \text{hadron} \rangle$
masses	***decay constants***	***form factors***

- Precision is always very challenging
  - Managing large data sets
  - Blinding data
  - Difficult fits and uncertainty determination/interpretation
- Lowering statistical uncertainty → new systematics become important
  - Enhanced logarithmic lattice spacing dependence
  - More detailed knowledge of pion-mass/volume dependence
- Incorporating QED

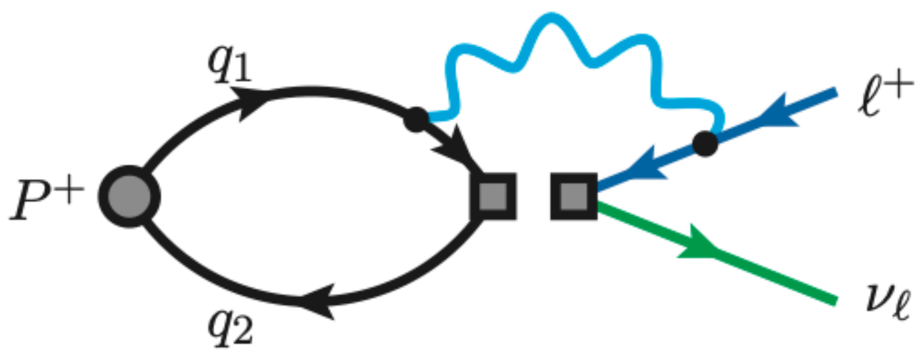
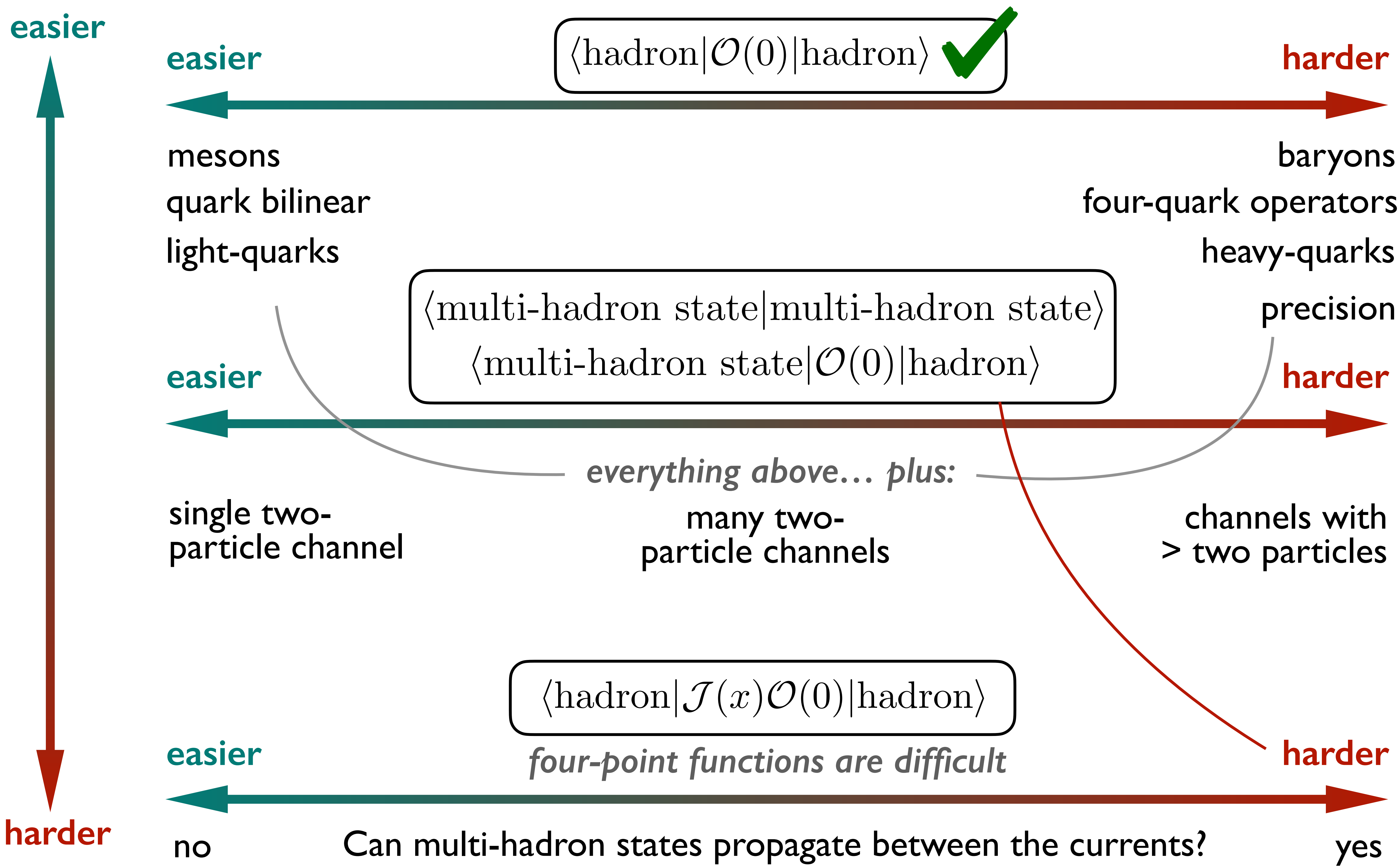


Figure by M. Di Carlo, see *Boyle et al., 2211.12865*

(Incomplete) landscape of lattice observables



Resonances

- If multi-hadron states play a role… *resonances* could be relevant
- Meson decays

CP violation in strange $K \to \pi\pi$

What is the role of the $\sigma/f_0(500)$?

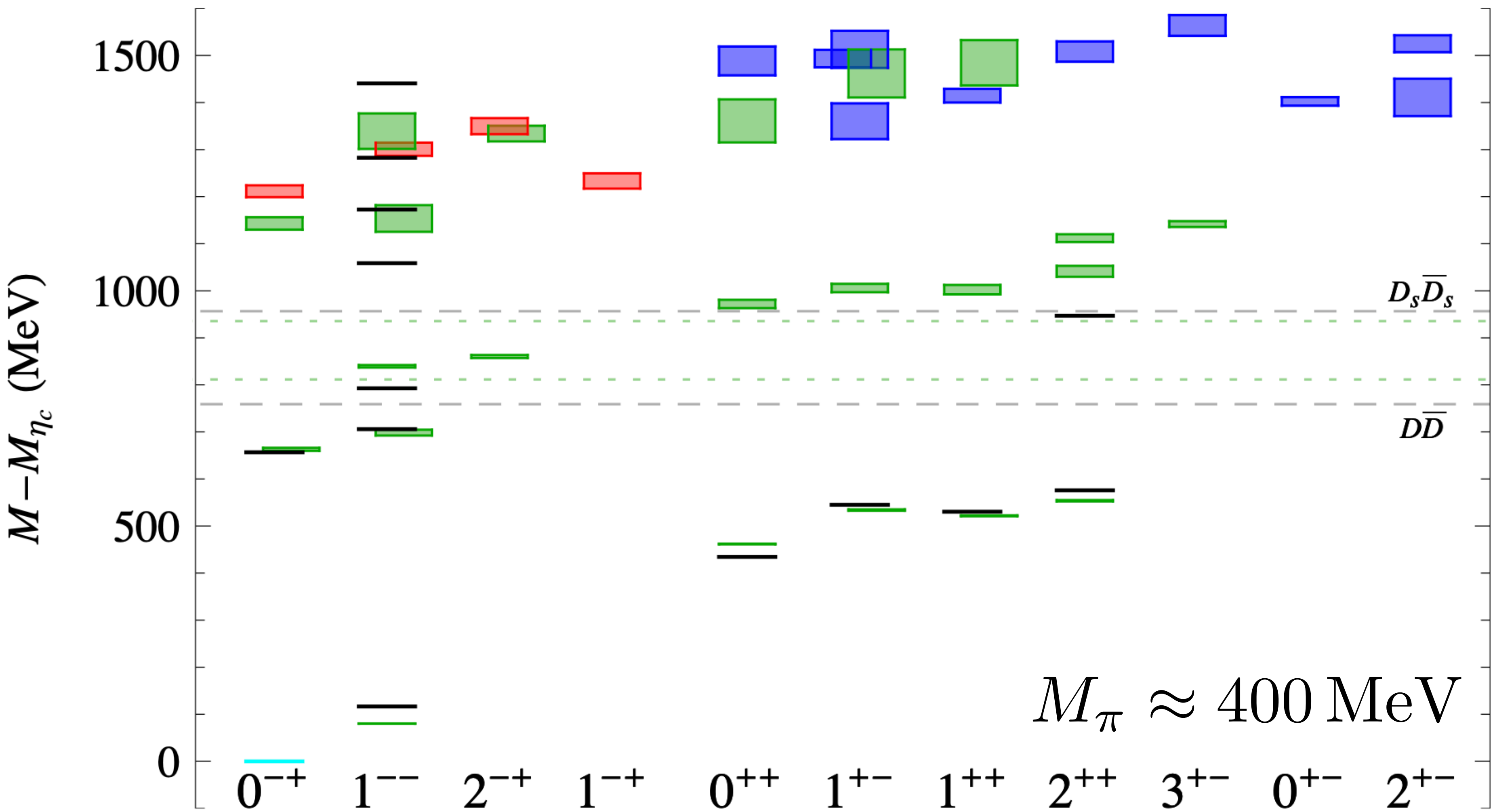
CP violation in charm $D \to \pi\pi, K\overline{K}$

$f_0(1710)$ could enhance ΔA_{CP}
• Soni (2017) •

Resonant D and B decays $B \to K^* \ell\ell \to K\pi\, \ell\ell$

- Any reliable approach should consider whether such effects are relevant
- Can be both a challenge and an opportunity (also for lattice QCD)

Charmonium resonances



heavier than physical M_π

Height of rectangle = statistical uncertainty

States created using local operators

Not quite right to say "*States cannot decay*" (more later)

Remarkable progress… but not the complete picture!

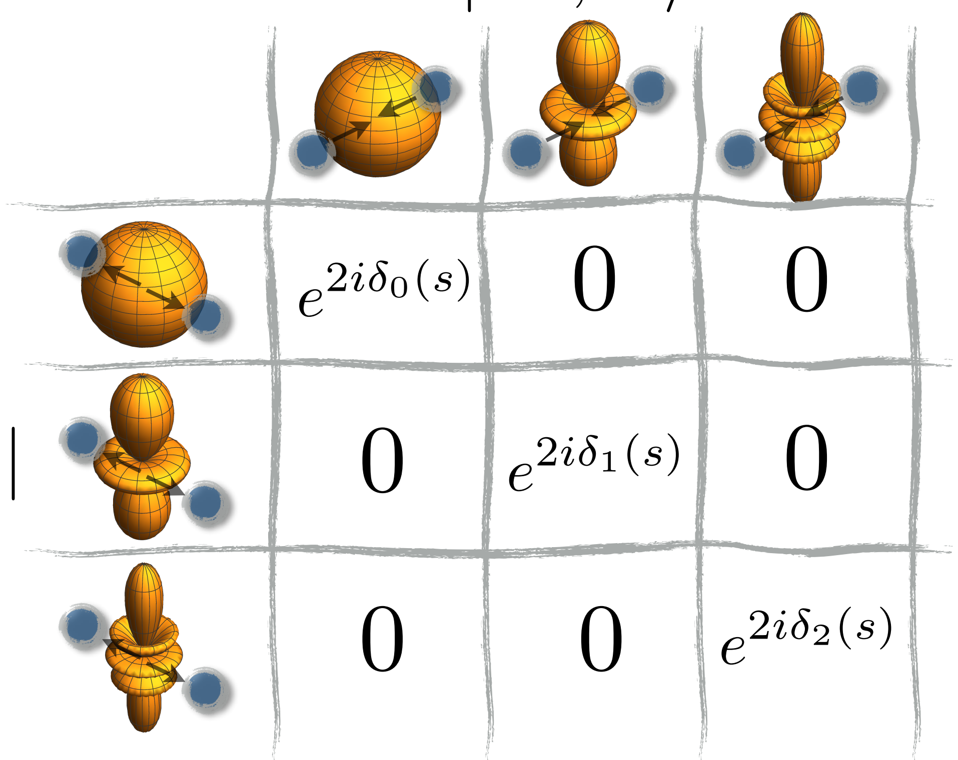
- Liu et al. (Hadron Spectrum Collaboration), *Excited and exotic charmonium spectroscopy from lattice QCD* JHEP, 2012 •

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$$|\pi\pi, \text{in}\rangle$$

$$S(s) \equiv \langle \pi\pi, \text{out}|$$



depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

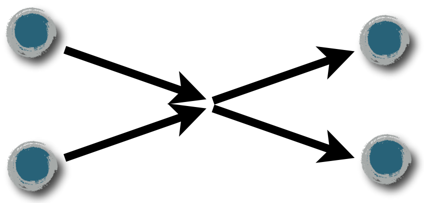
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

- ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

- Poles on the second Riemann sheet give resonances

Full QCD demands this description… lattice QCD cannot escape it

Unitarity and Analyticity



- For $s < (2M_\pi)^2$, the optical theorem tells us…

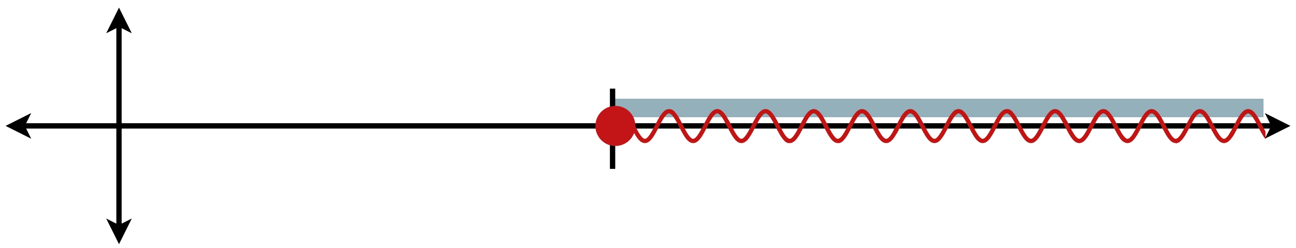
$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \operatorname{Im}\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance) phase-space cut (long distance)

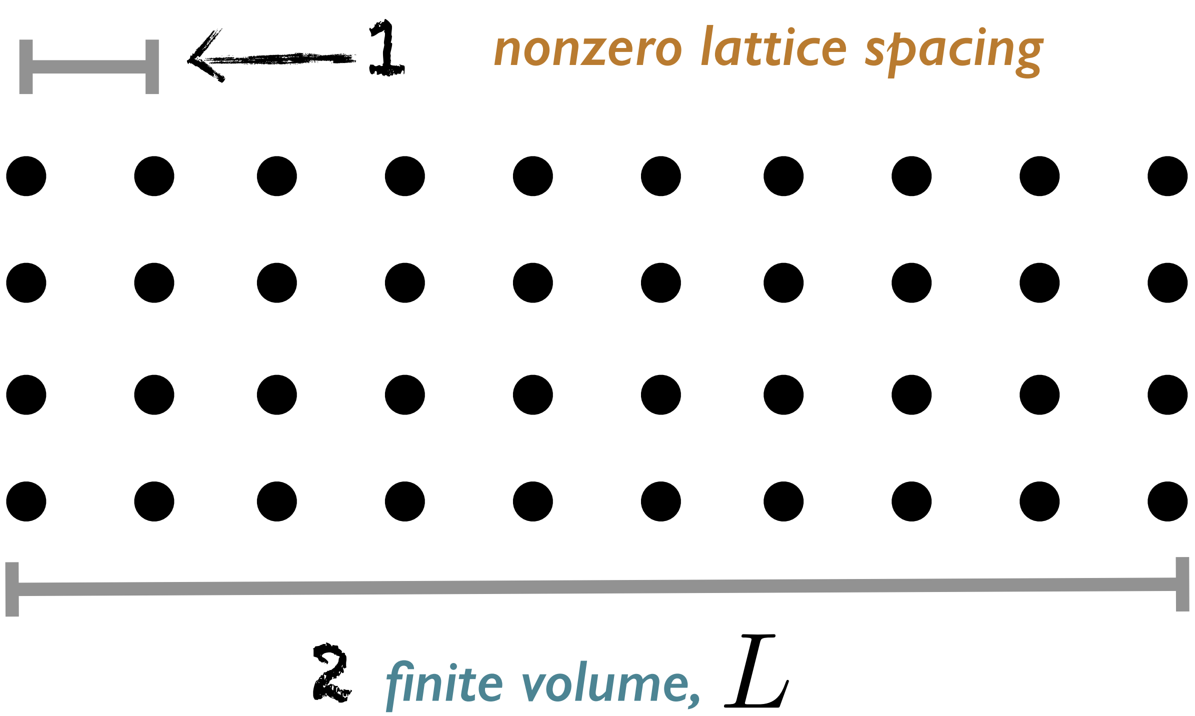


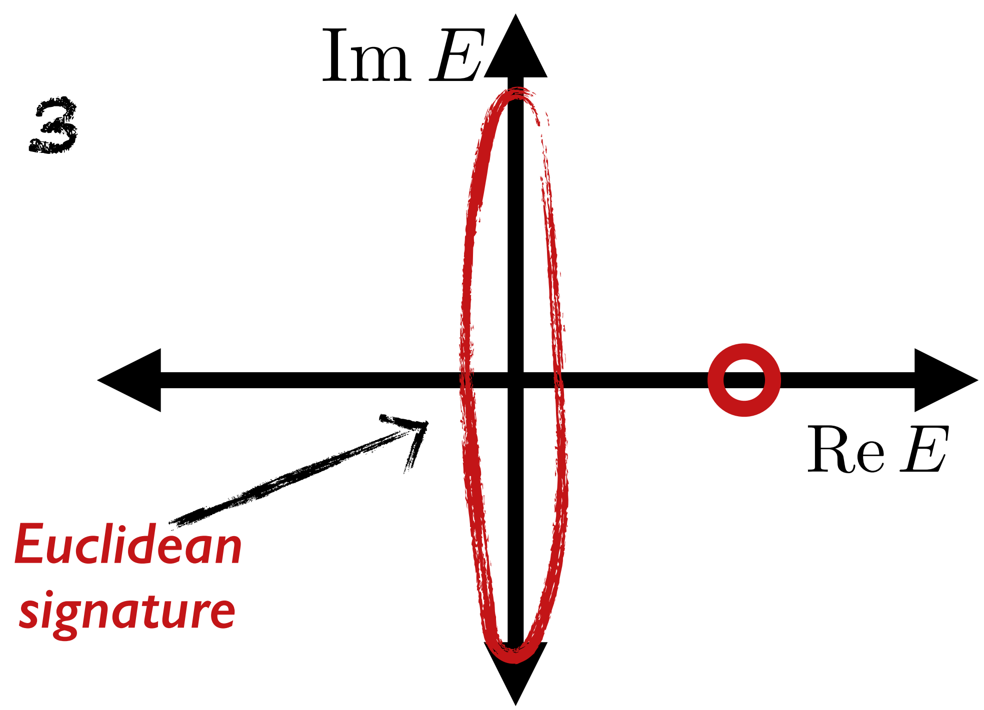
Amplitude has a branch cut ✓ K-matrix is useful for parametrizing ✓

Lattice QCD

$$\text{observable?} = \int d^N\!\phi \, e^{-S} \left[\begin{array}{c}\textbf{interpolator} \\ \textbf{for observable}\end{array} \right]$$

To proceed we have to make *three modifications*



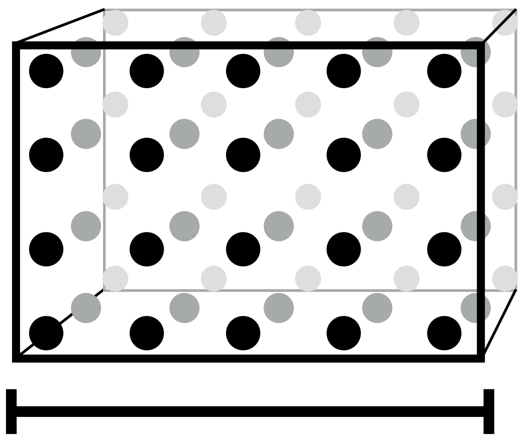


Also… $M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$

(but physical masses → increasingly common)

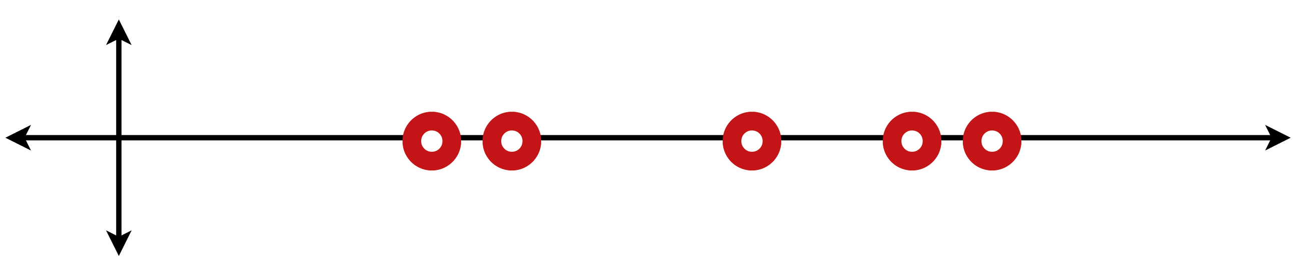
Difficulties for multi-hadron observables

- The ***Euclidean signature***…
  - ***Prevents*** usual on-shell approach (want $p_4^2 = -E(p)^2$, but have only $p_4^2 > 0$)



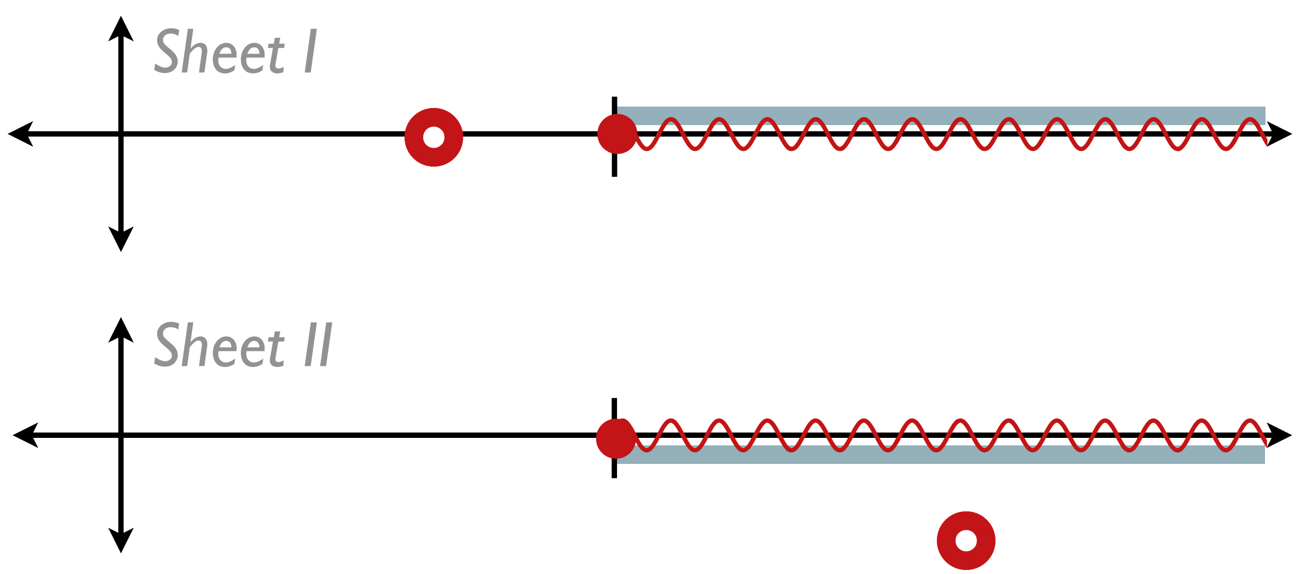
- The ***finite volume***…
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and extra sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



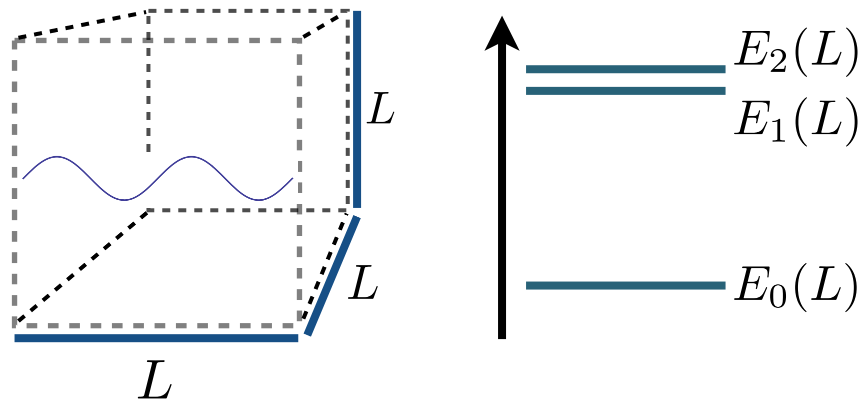
Note: cannot count finite-volume energies to count resonance poles!

Infinite-volume analytic structure



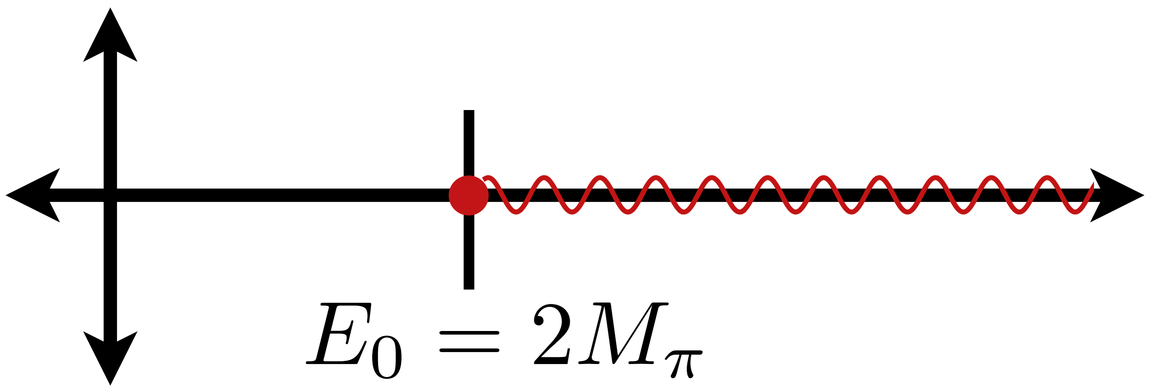
The finite-volume as a tool

- ❑ Finite-volume set-up



- ❑ cubic, spatial volume (extent *L*)
- ❑ periodic
- ❑ *L* is large enough to neglect $e^{-M_\pi L}$

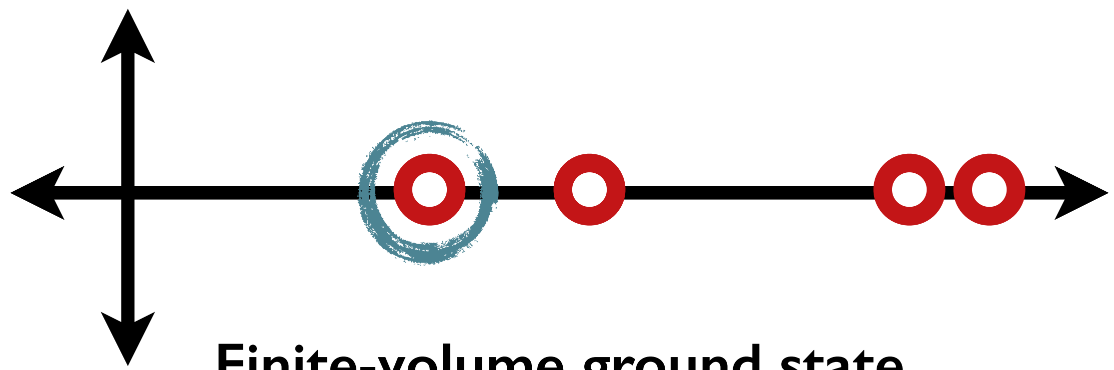
- ❑ Scattering leaves an *imprint* on finite-volume quantities



$$E_0 = 2M_\pi$$

Infinite-volume threshold

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$



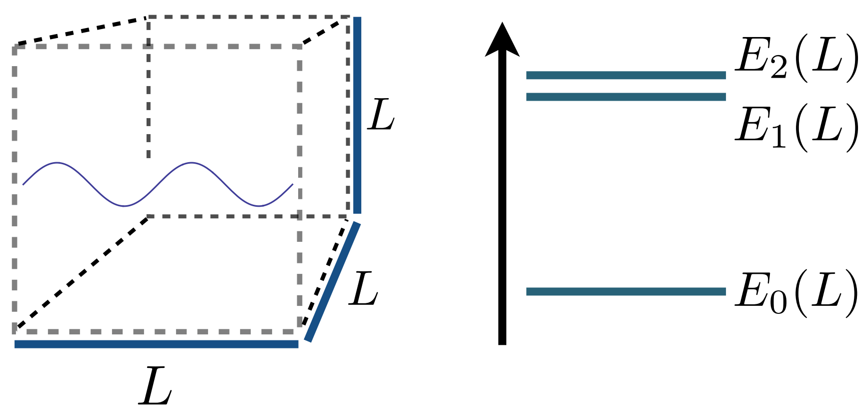
Finite-volume ground state

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

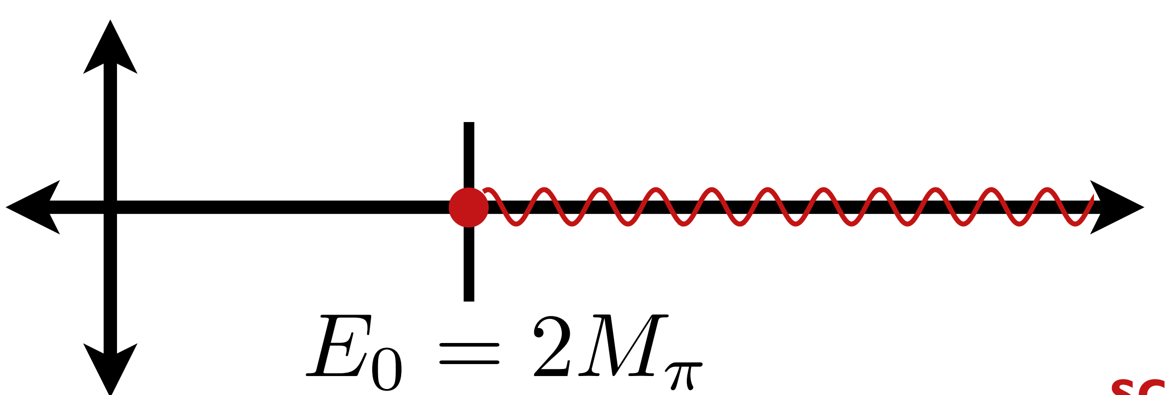
The finite-volume as a tool

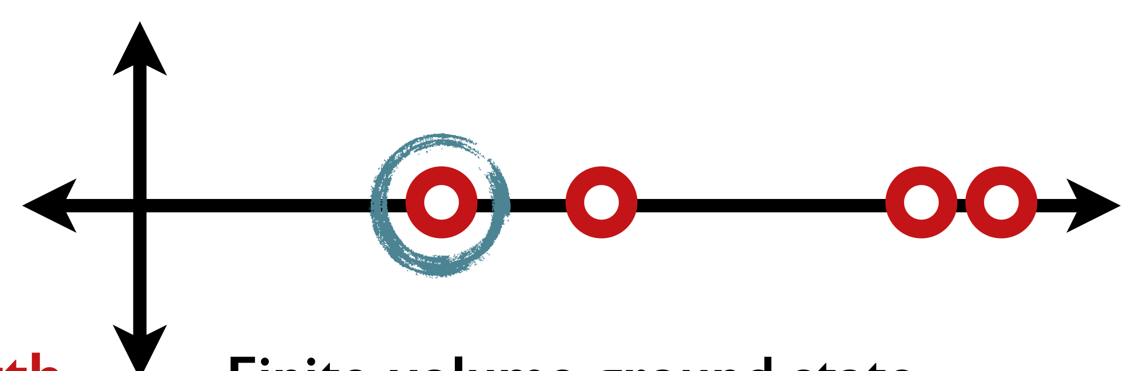
- Finite-volume set-up



- **cubic**, spatial volume (extent *L*)
- **periodic**
- *L* is large enough to neglect $e^{-M_\pi L}$

- Scattering leaves an *imprint* on finite-volume quantities





scattering length

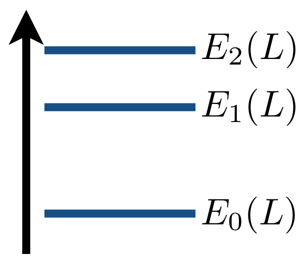
$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

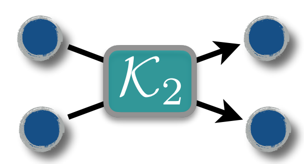
• Huang, Yang (1958) •

General relation

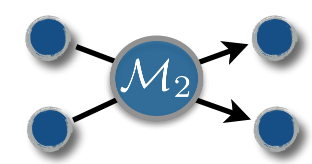
$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0 \qquad F(P,L) \equiv \text{Matrix of known geometric functions}$$



finite volume ⟷



unitarity ⟷



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

• Lüscher (1989) • *many others* •

Scattering processes and resonances from lattice QCD

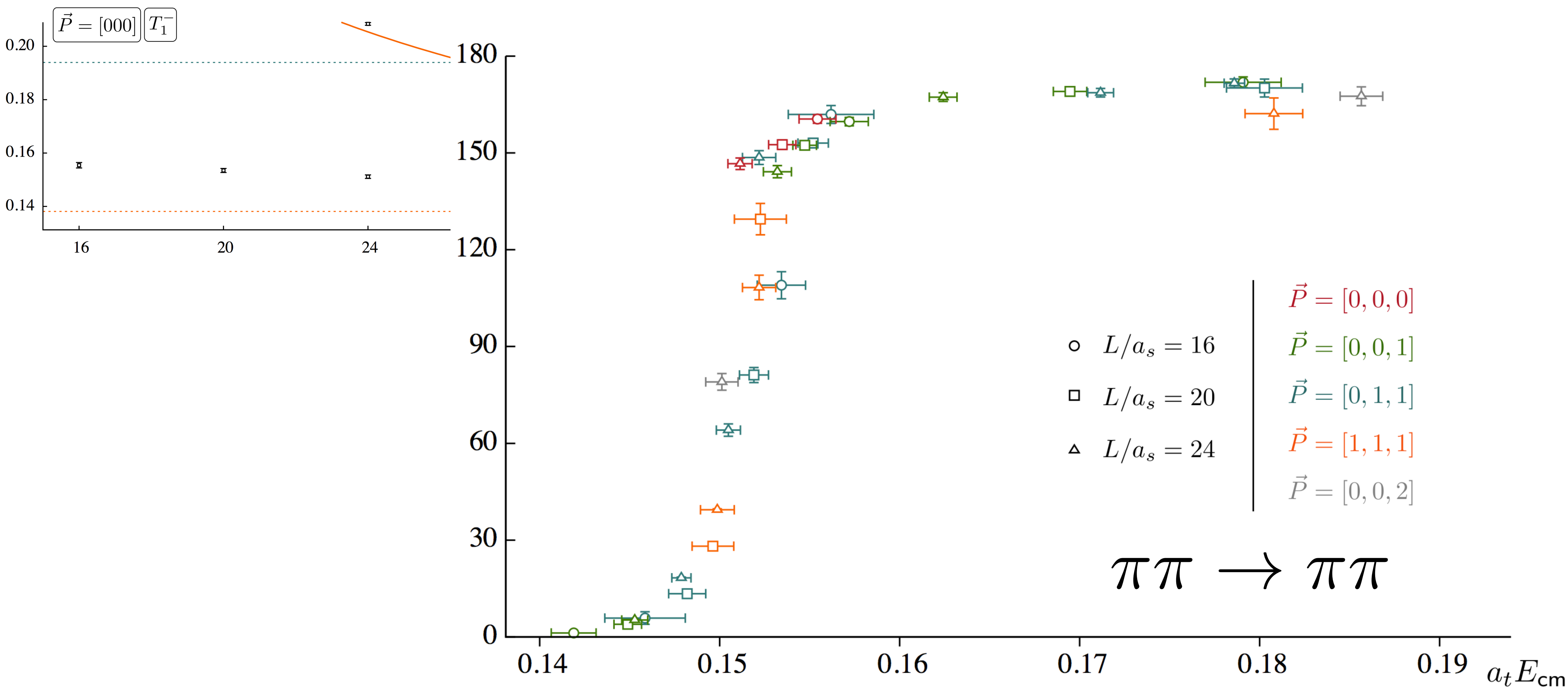
Raúl A. Briceño,[1,*] Jozef J. Dudek,[1,2,†] and Ross D. Young[3,‡]

REVIEWS OF MODERN PHYSICS

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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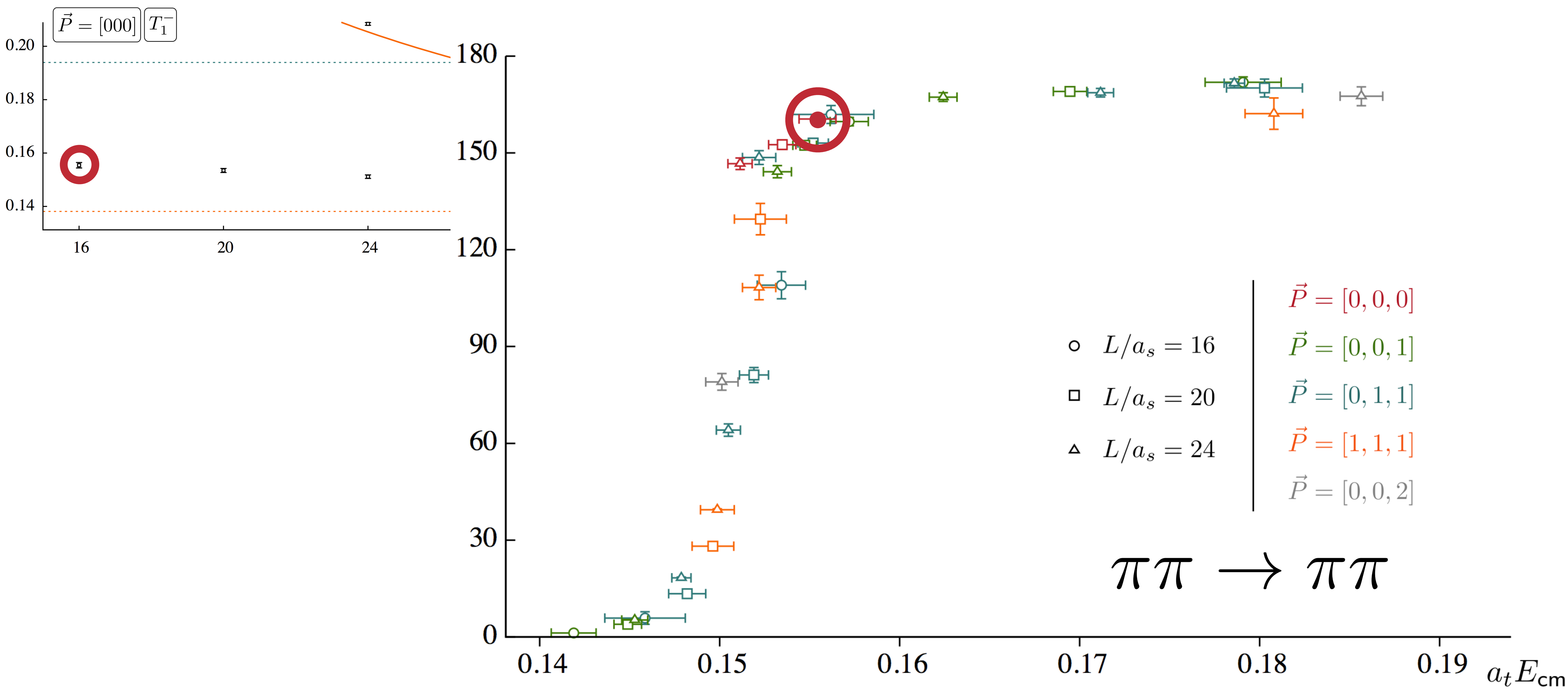

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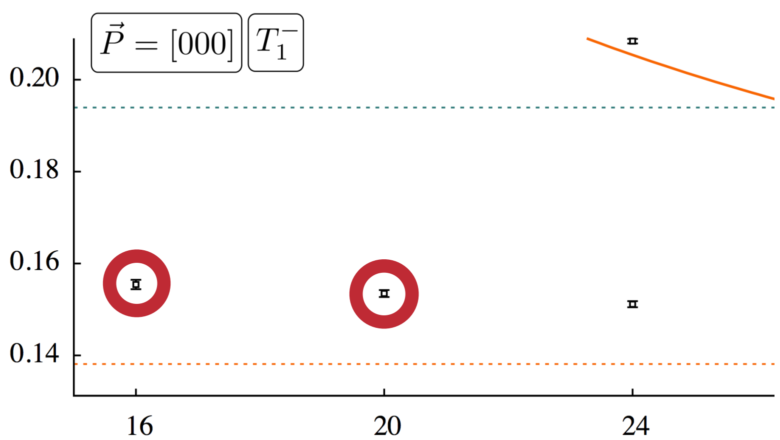

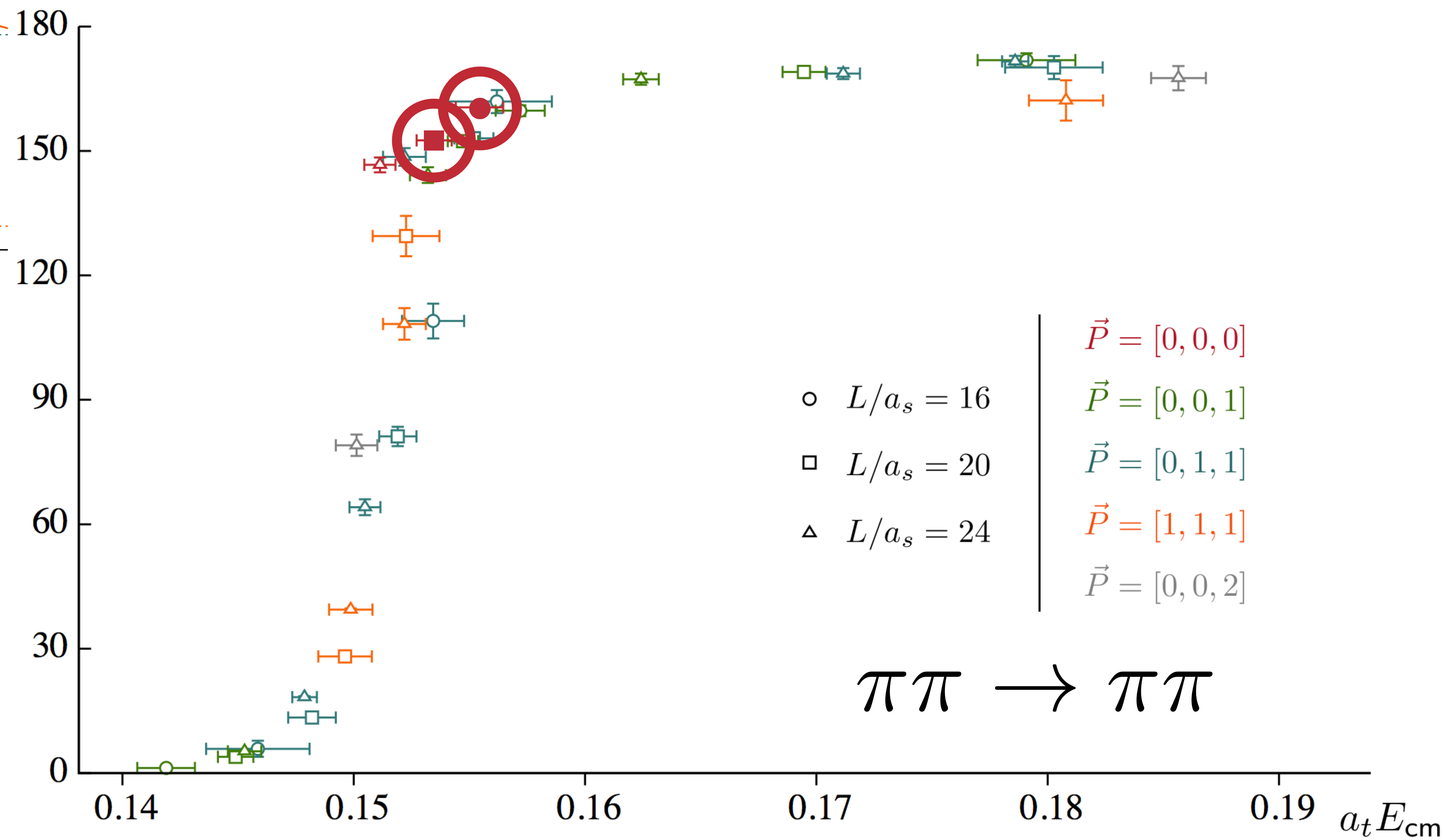

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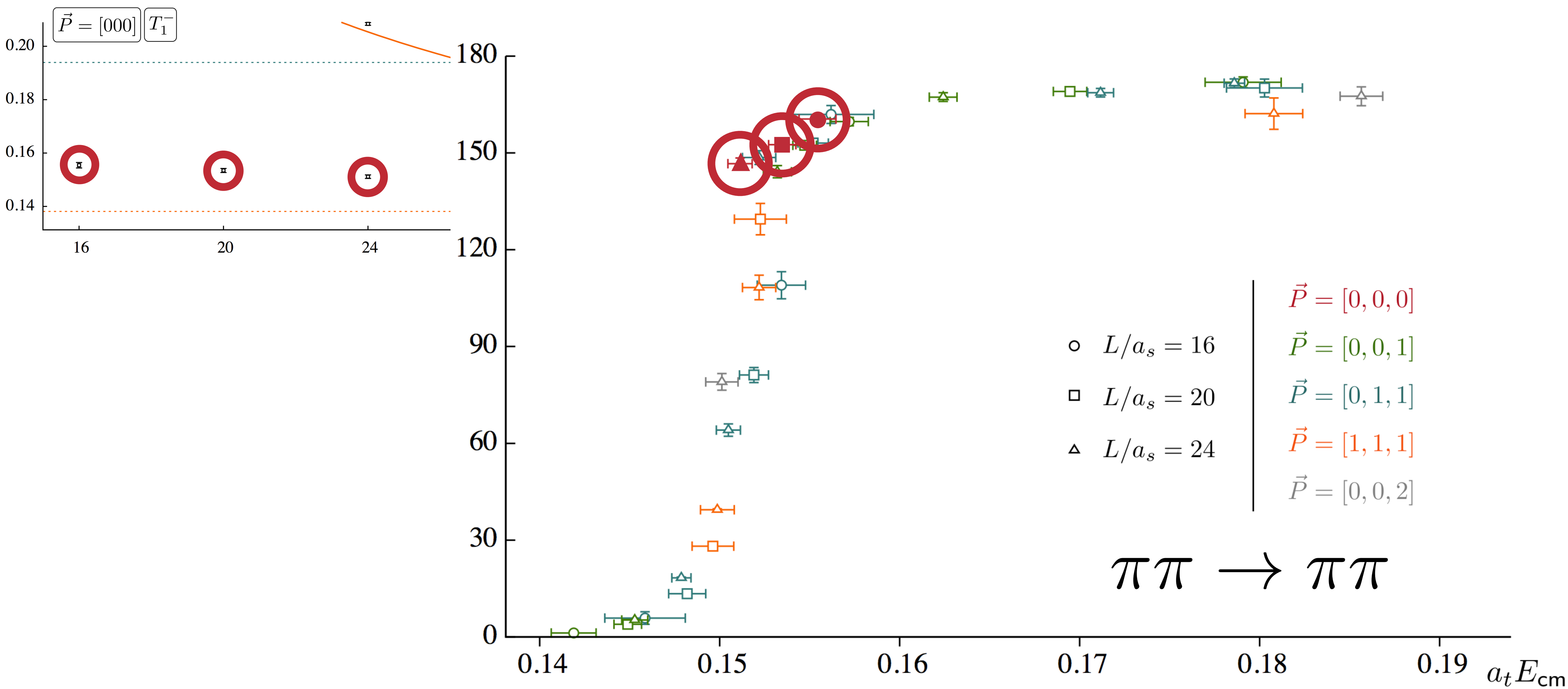

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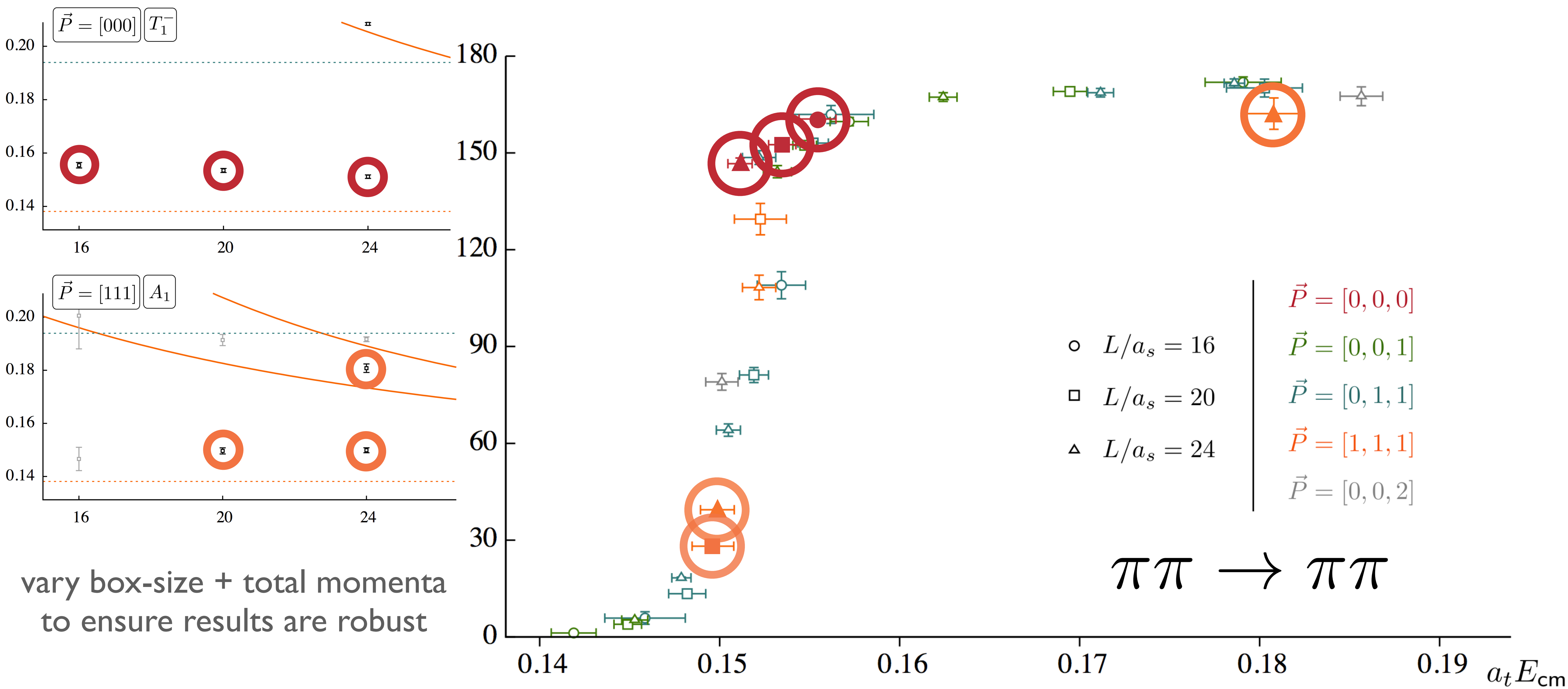
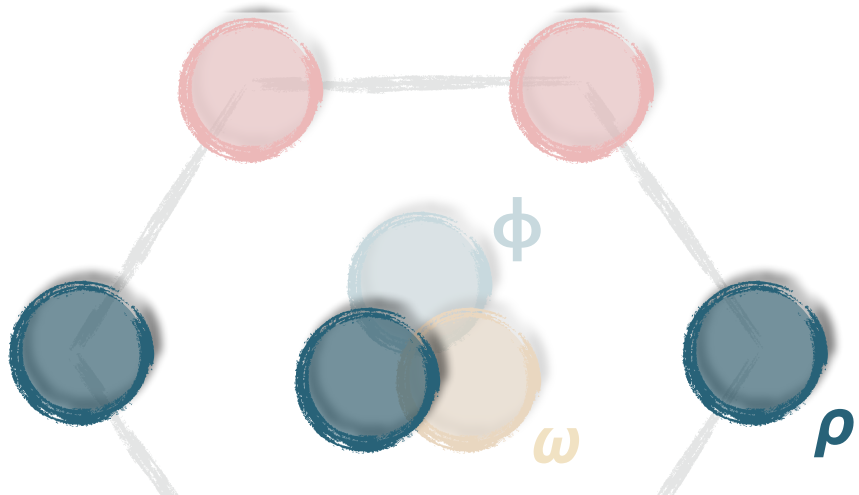


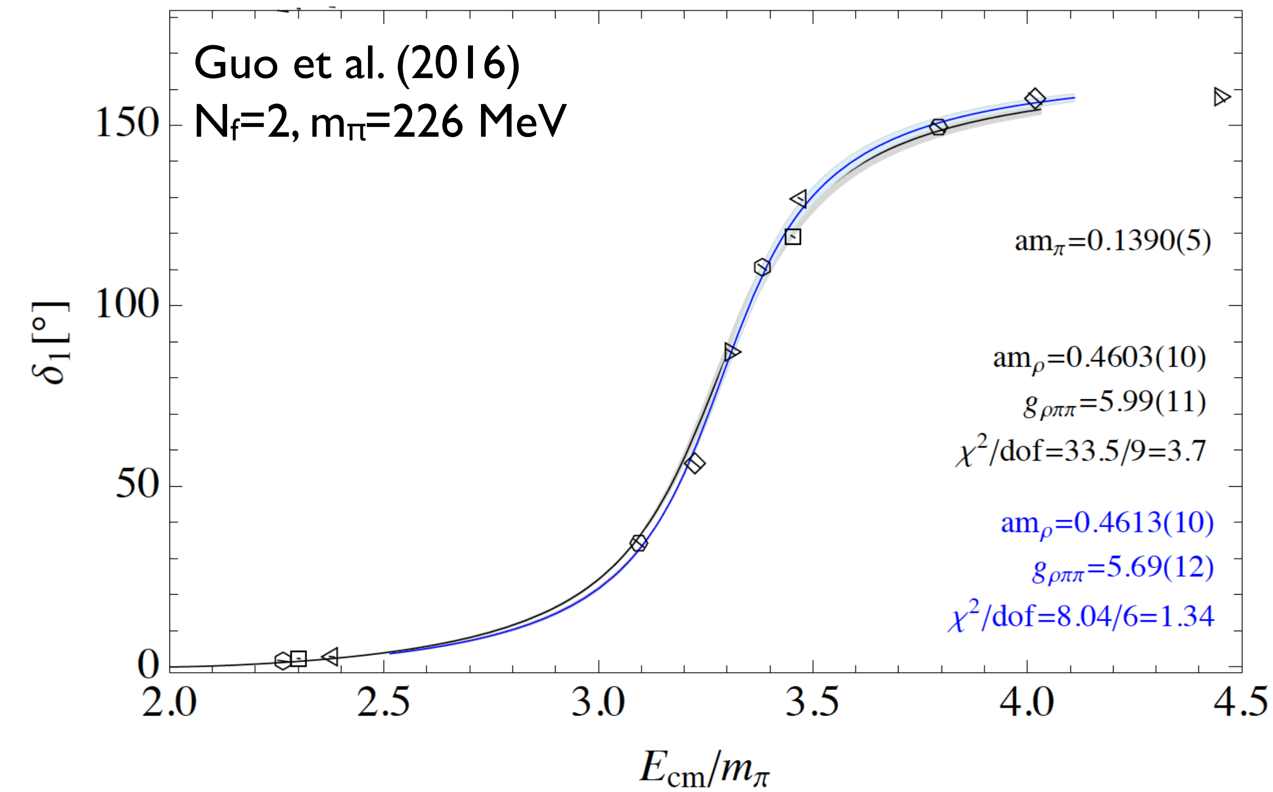
vary box-size + total momenta
to ensure results are robust

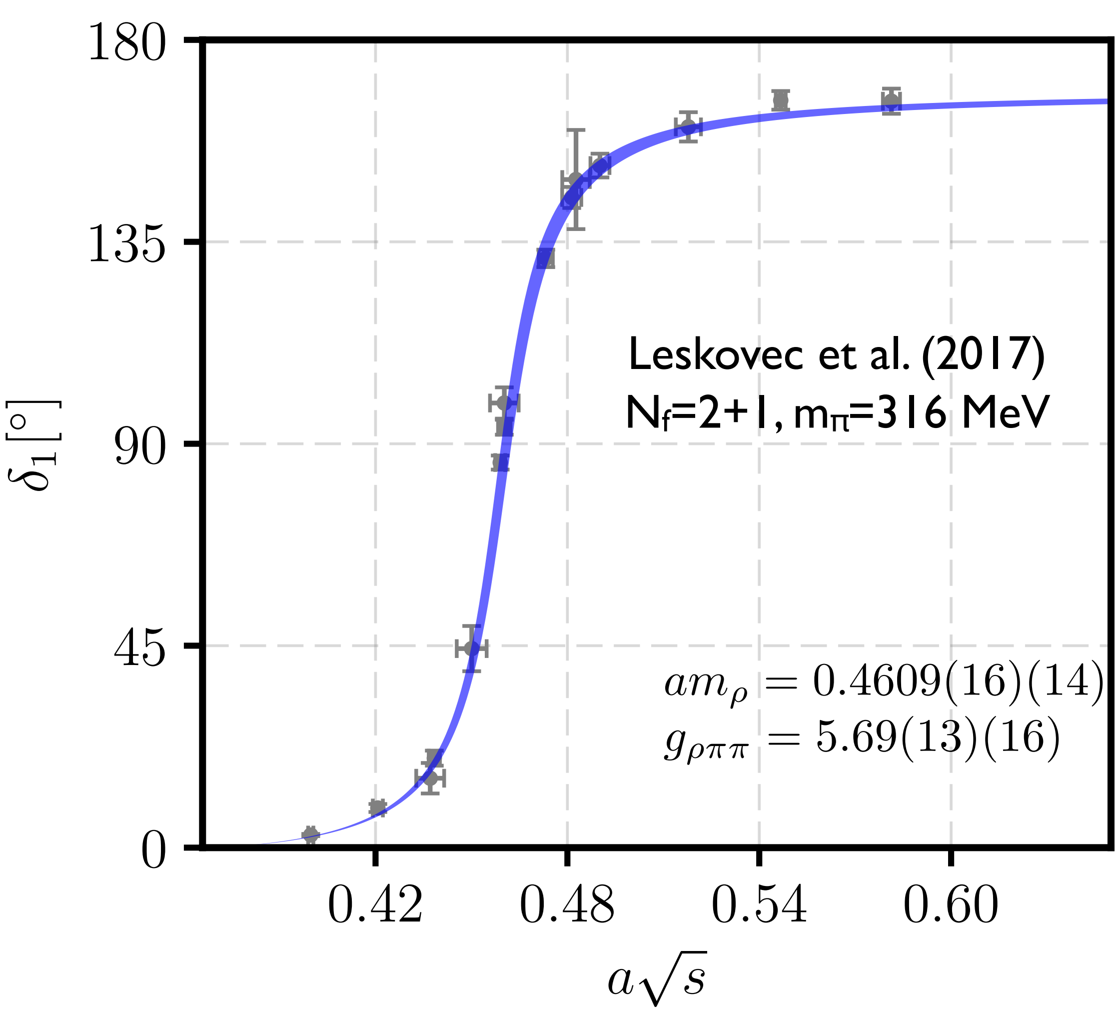
• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

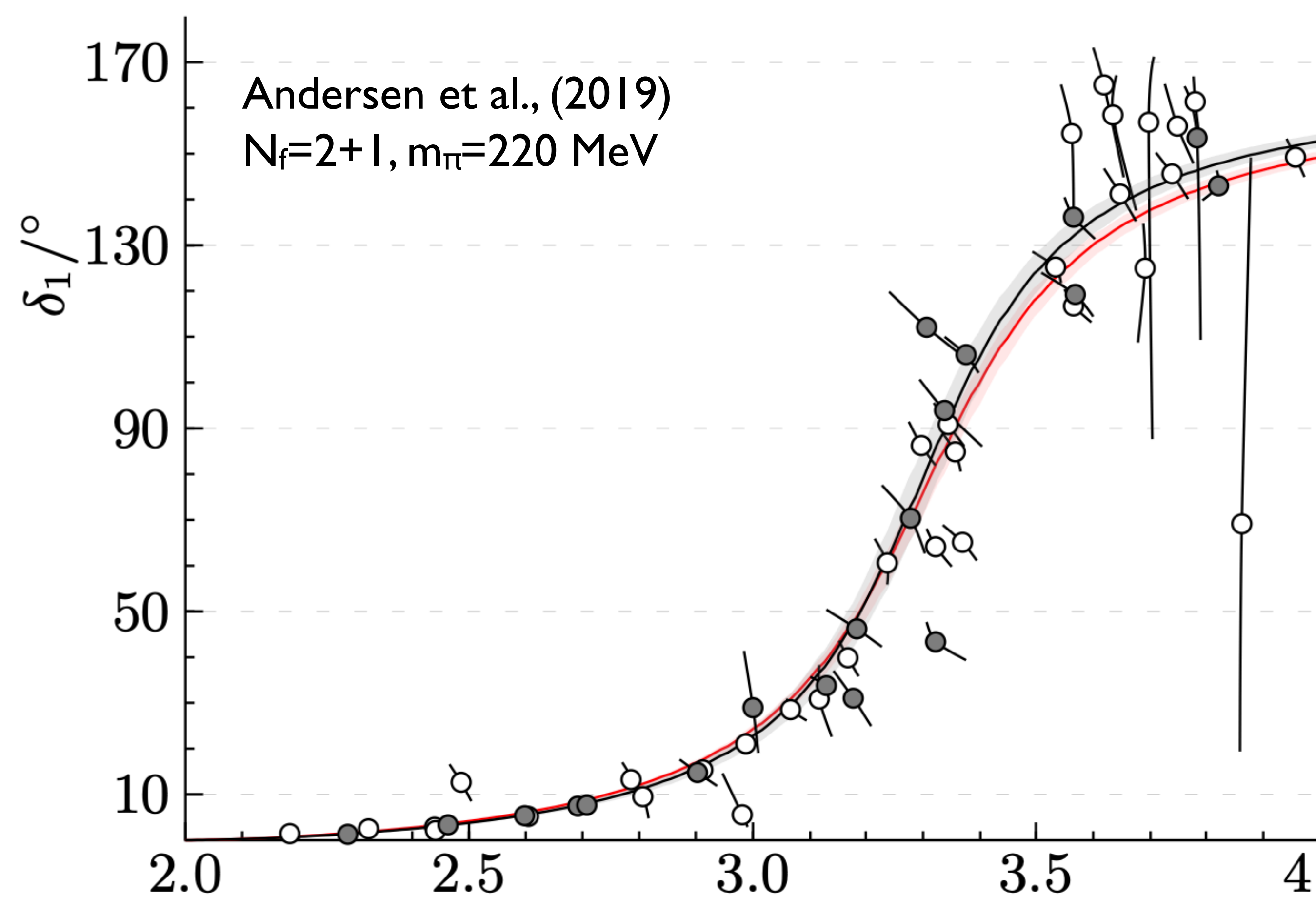
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$



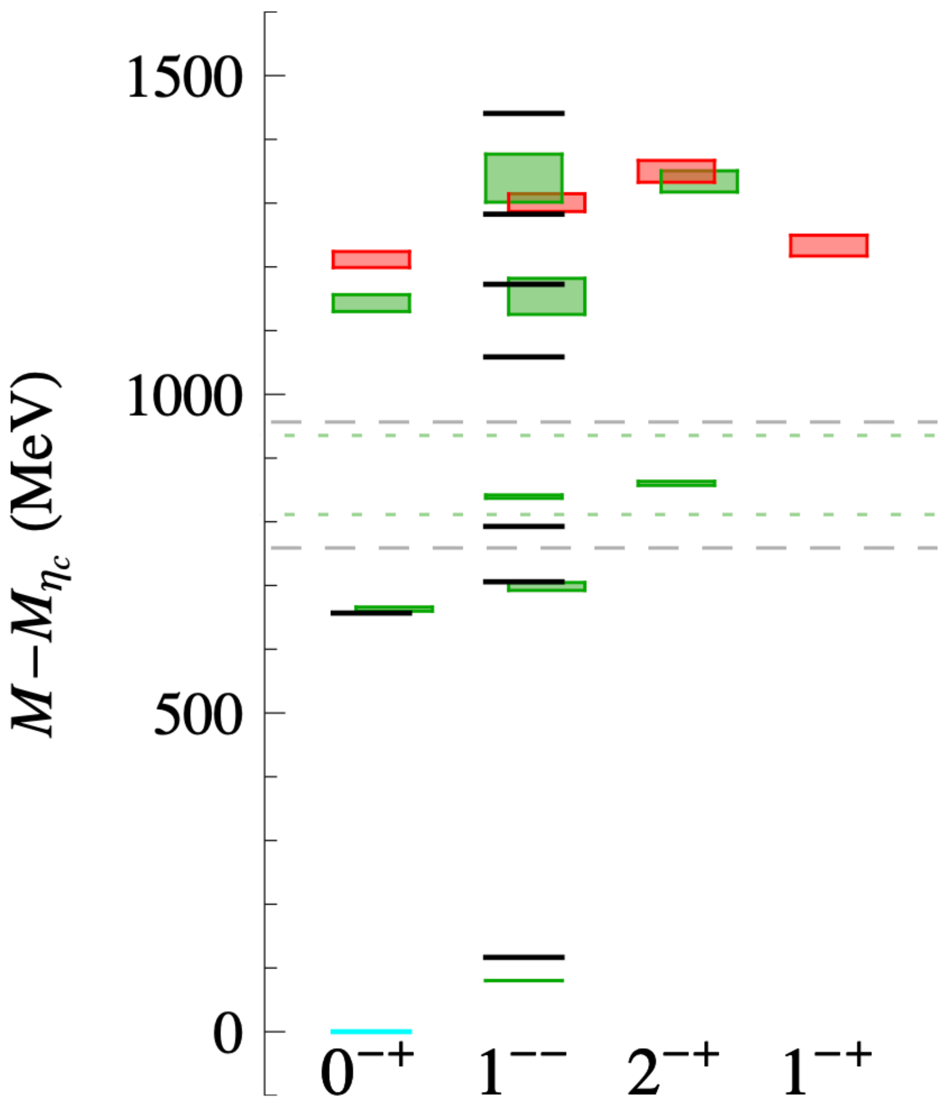






Two types of spectroscopy

Explore the spectrum of compact QCD excited states
(via quark-model inspired local operators)



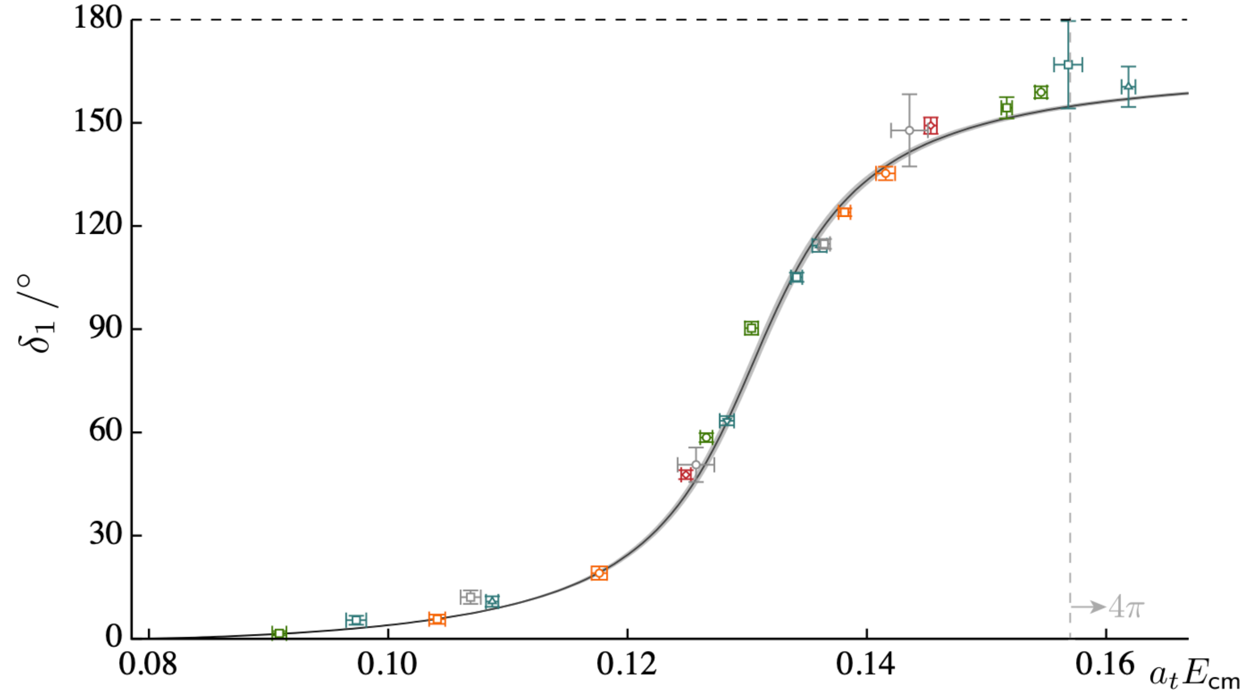
Dudek, Edwards (2012)

Extract the full finite-volume energy spectrum

local operators

+ many multi-hadron operators

$$\pi\pi \to \rho \to \pi\pi$$



Wilson, Briceño, Dudek, Edwards, Thomas (2015)

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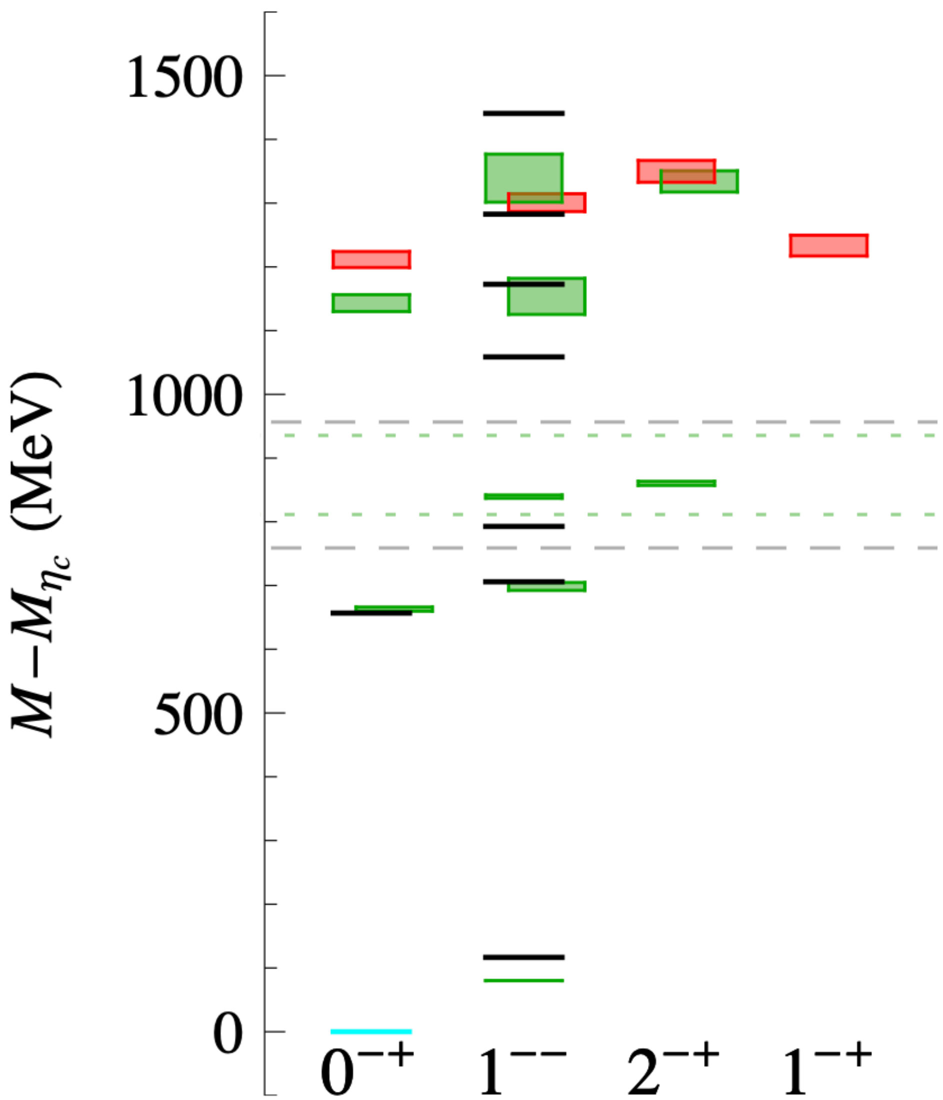
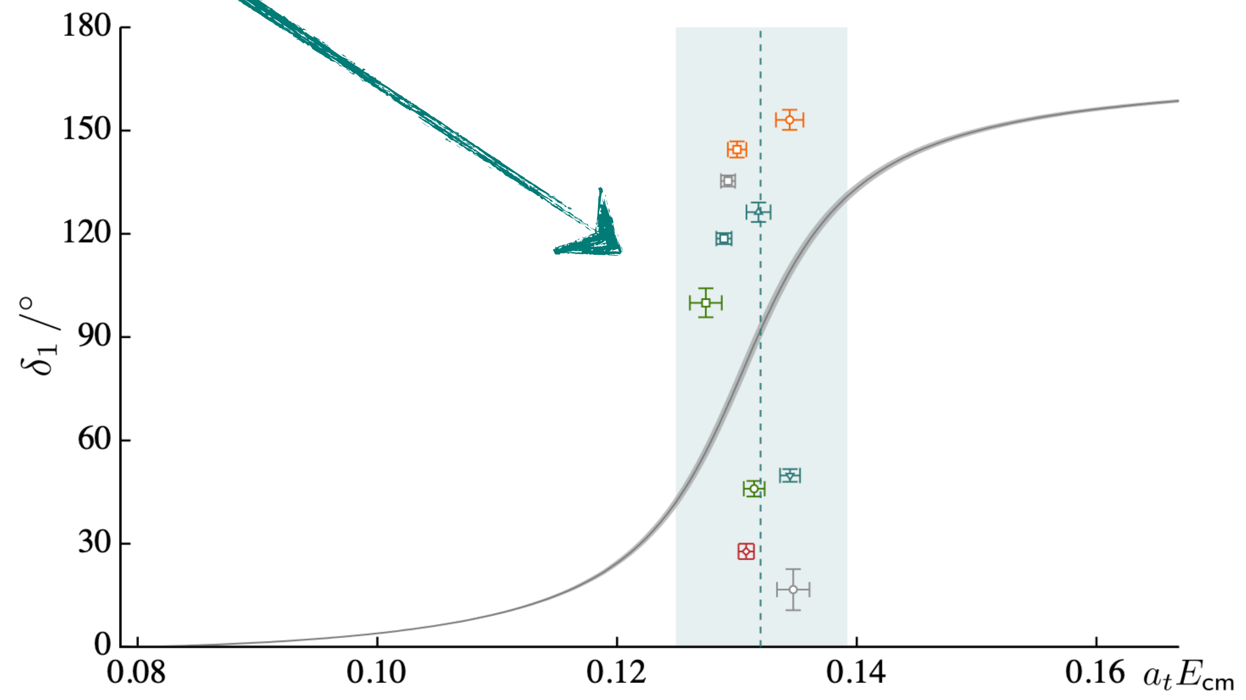


Dudek, Edwards (2012)

local operator spectrum =
not suitable for phase shift extraction

$$\pi\pi \to \rho \to \pi\pi$$



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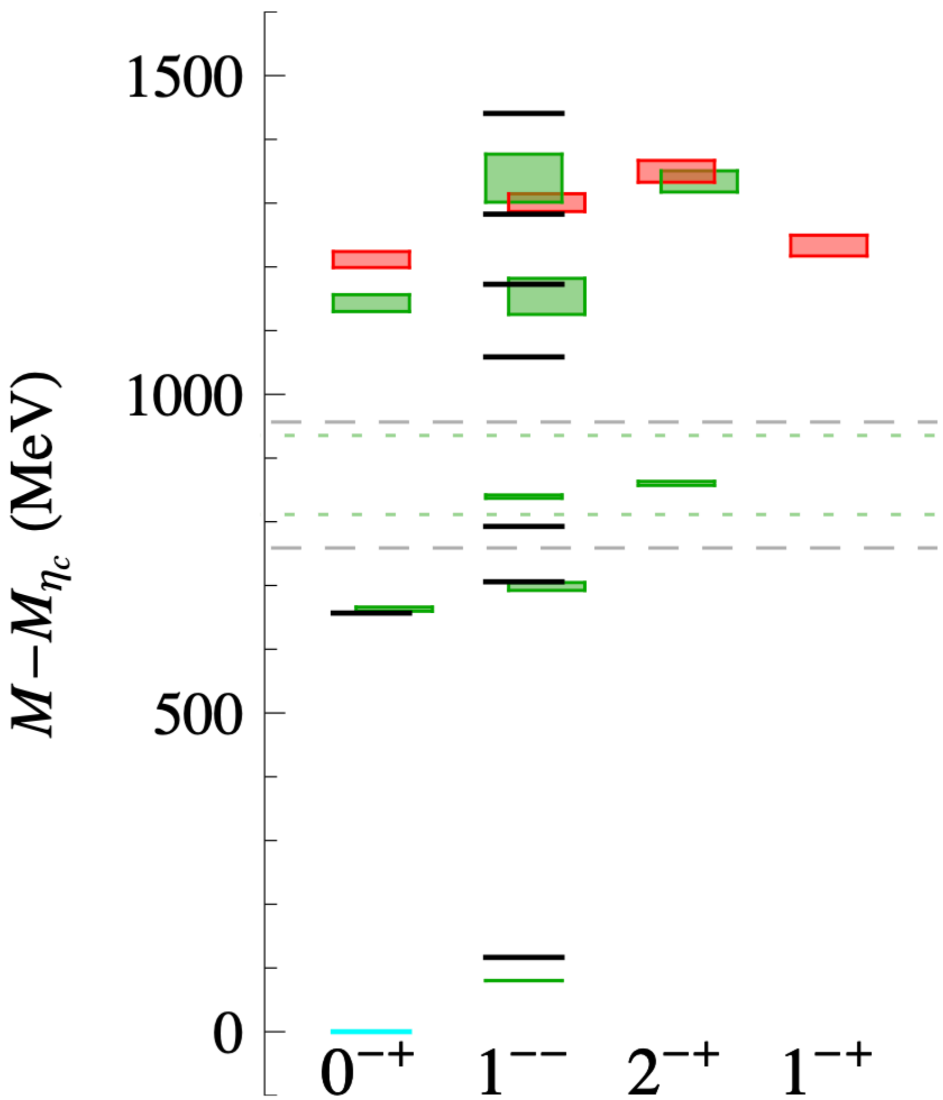
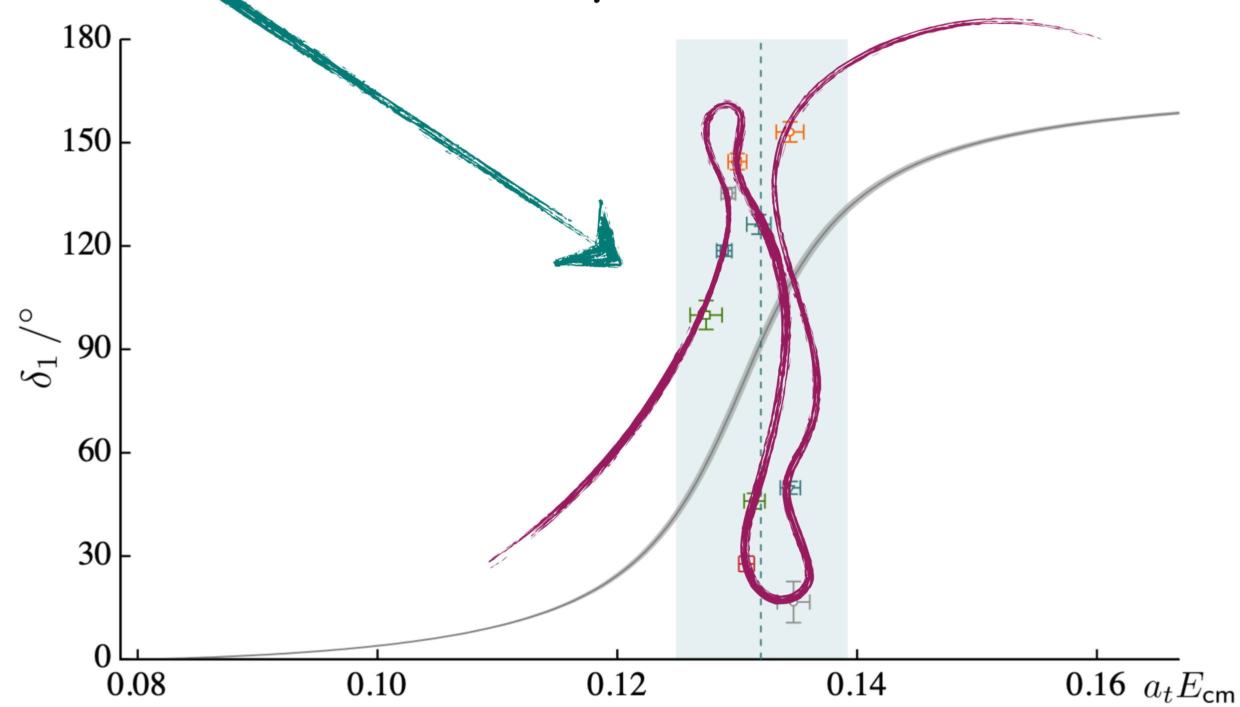
Explore the spectrum of compact QCD excited states
(via quark-model inspired local operators)



Dudek, Edwards (2012)

local operator spectrum =
not suitable for phase shift extraction

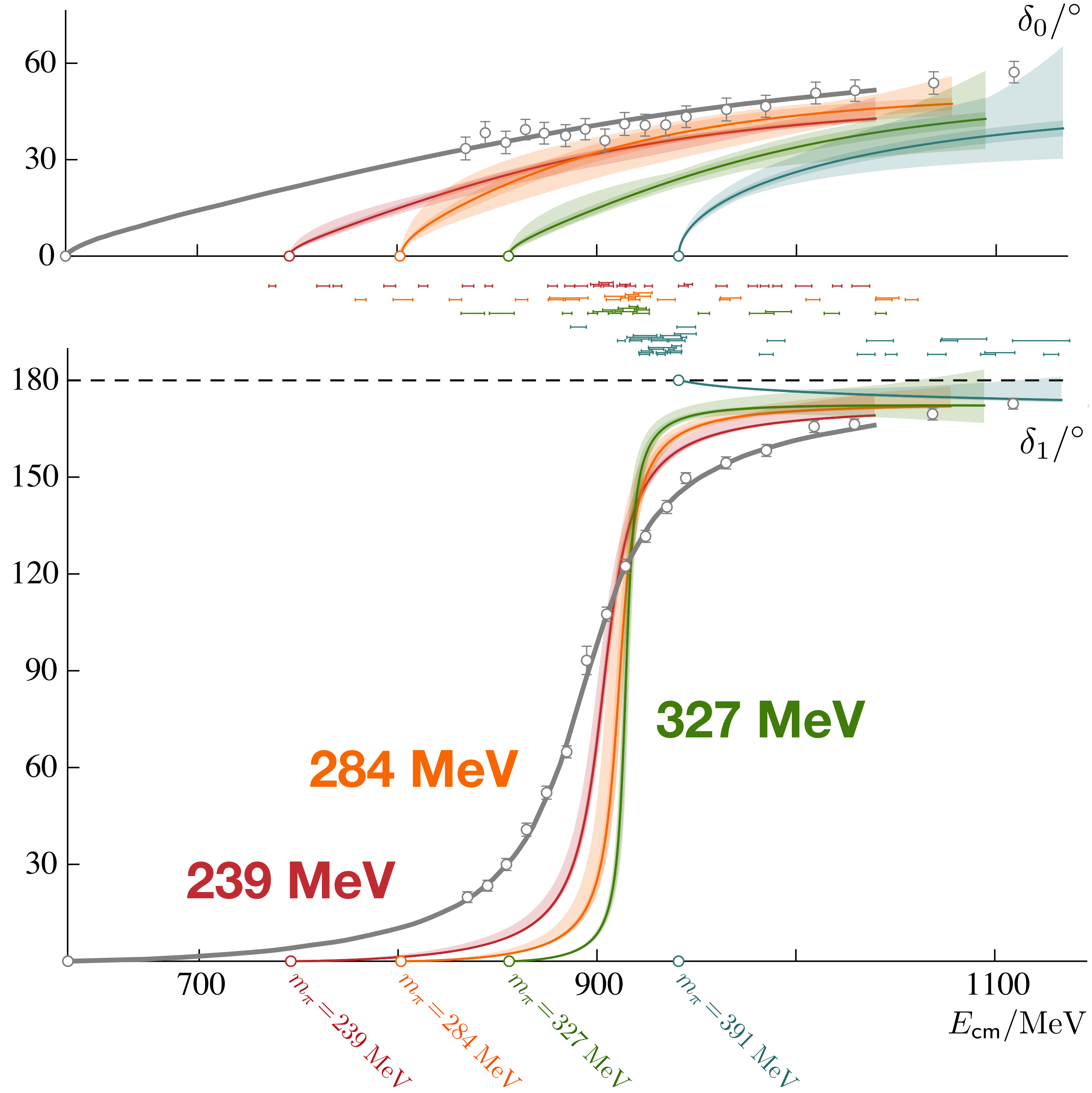
$$\pi\pi \to \rho \to \pi\pi$$

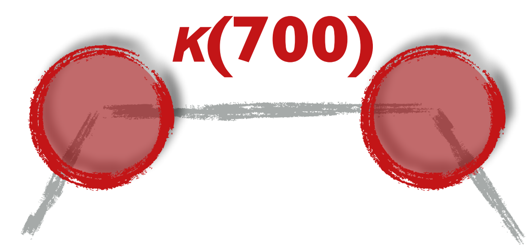


Note: cannot count finite-volume energies to count resonance poles!

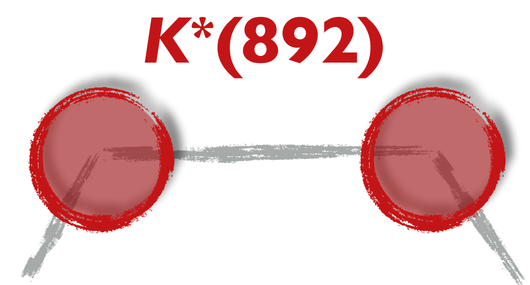
Wilson, Briceño, Dudek, Edwards, Thomas (2015)

$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$

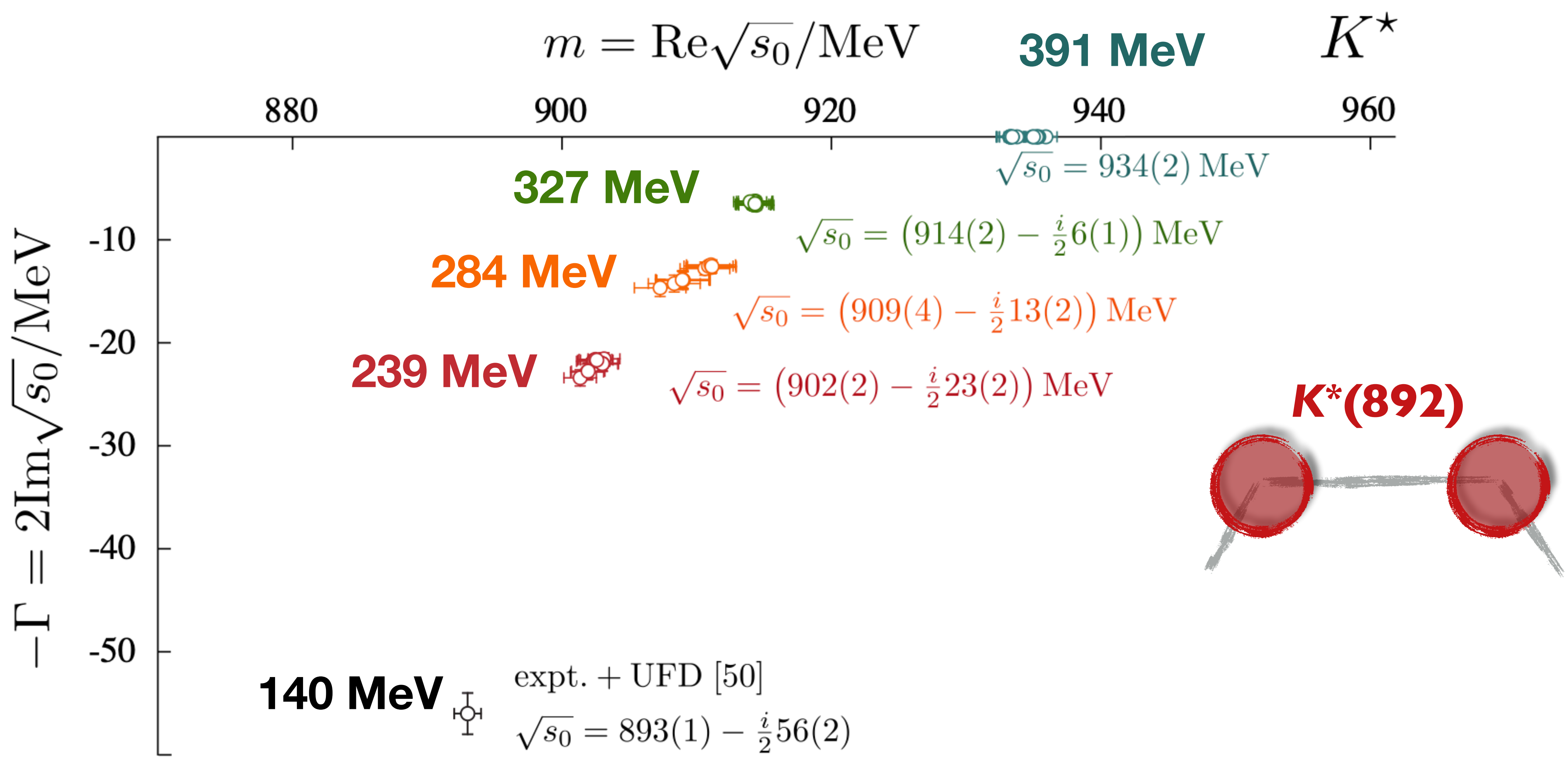


$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* • See also Nelson Lachini, *Lattice2022* •

$\kappa, K^* \to K\pi$

$I(J^P) = 1/2(1^-)$



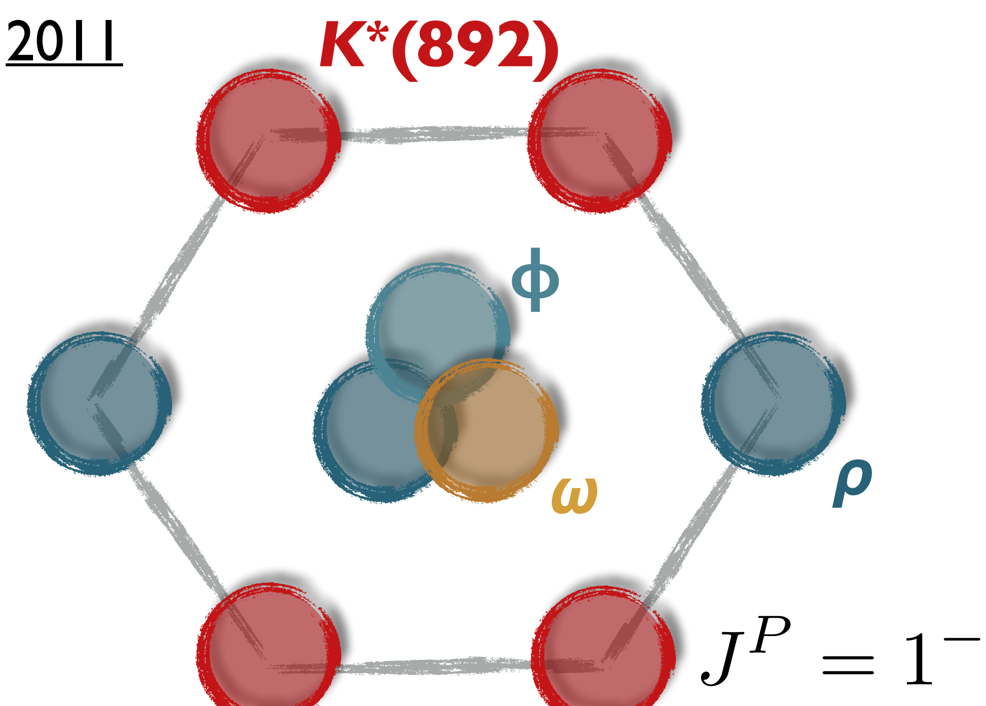
- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* • See also Nelson Lachini, *Lattice2022* •

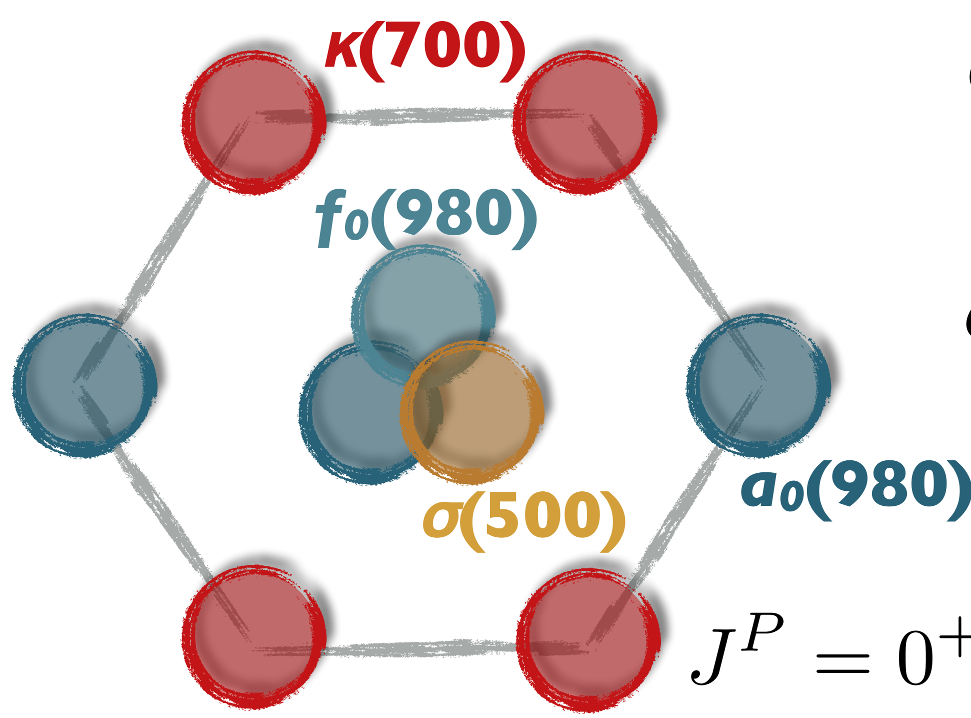
$\rho \to \pi\pi$

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, **2016**
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017
- Andersen et al. 2018
- Fischer et al. 2020
- Erben et al. 2020

$\sigma \to \pi\pi$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017
- Guo et al. 2018





$\kappa \to K\pi$

$K^* \to K\pi$

- Lang et al. 2012
- Prelovsek et al. 2013
- **Wilson et al. 2015**
- RQCD 2015
- **Brett et al. 2018**
- Wilson et al. 2019
- Rendon et al. 2020

$b_1 \to \pi\omega, \pi\phi$

- **Woss et al. 2019**

$a_0(980) \to \pi\eta, K\overline{K}$

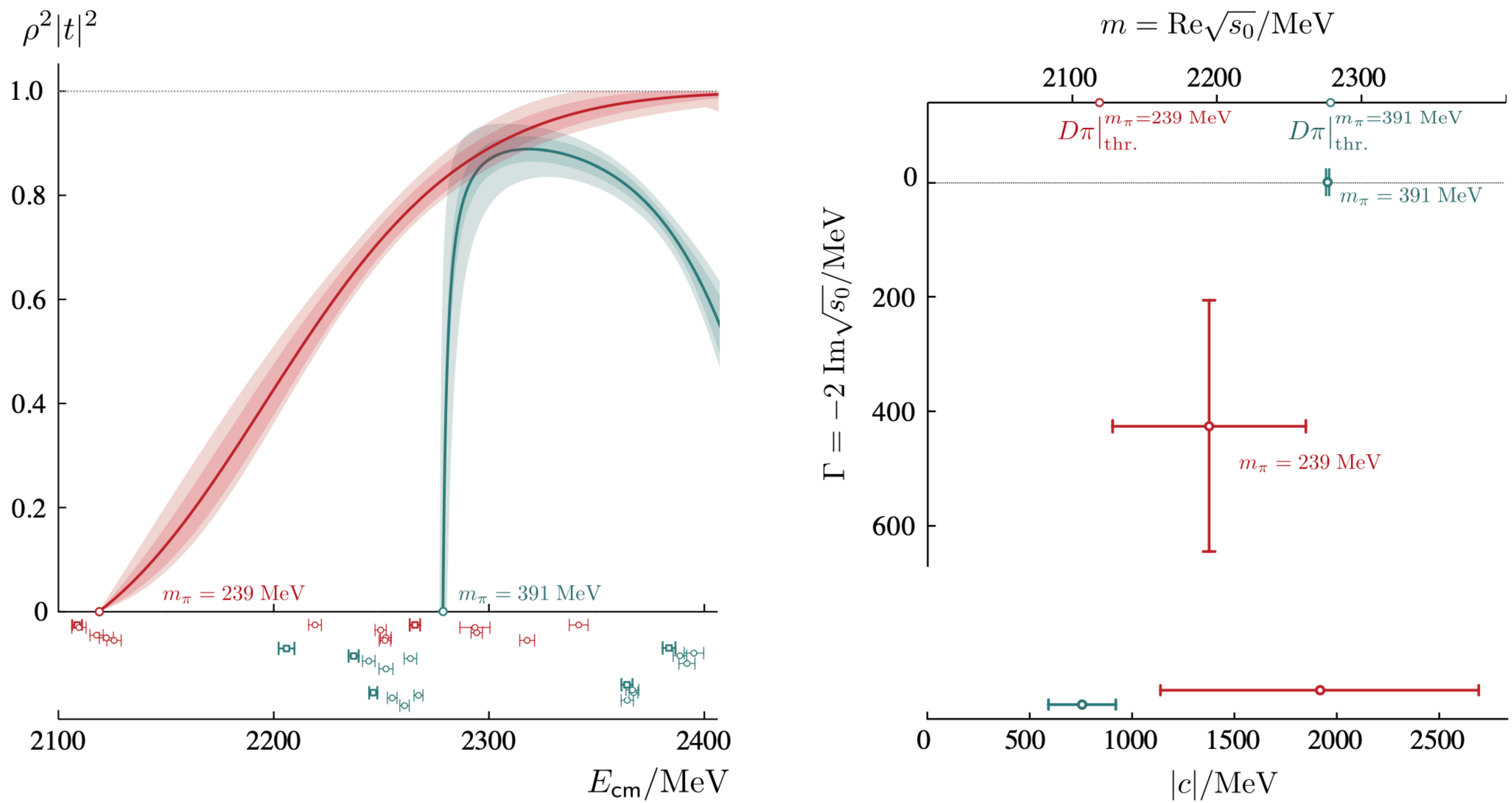
- **Dudek et al. 2016**

$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

- **Briceño et al. 2017**

See the recent review by Briceño, Dudek and Young

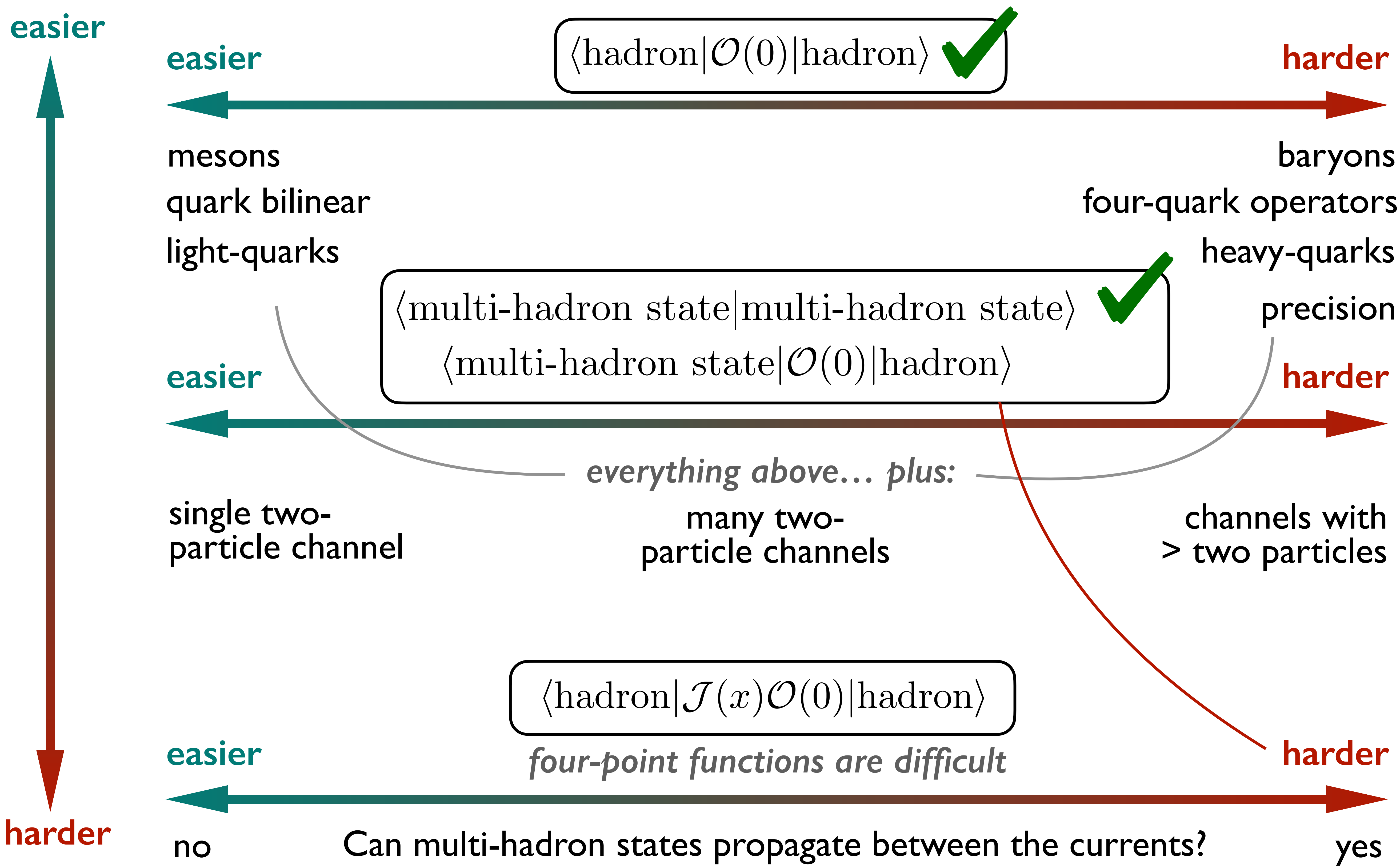
$D\pi \to D\pi,\ I = 1/2$



— *Isospin-1/2 Dπ scattering and the lightest D0* resonance from lattice QCD* —

Hadron Spectrum Collaboration — (2021) JHEP 07 (2021) 123

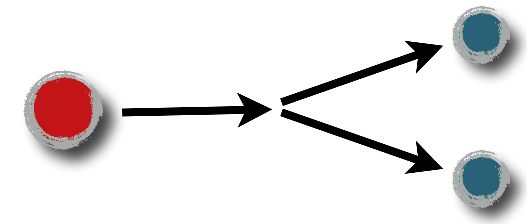
(Incomplete) landscape of lattice observables



Formal progress: Transition amplitudes

- [ ] Weak decay

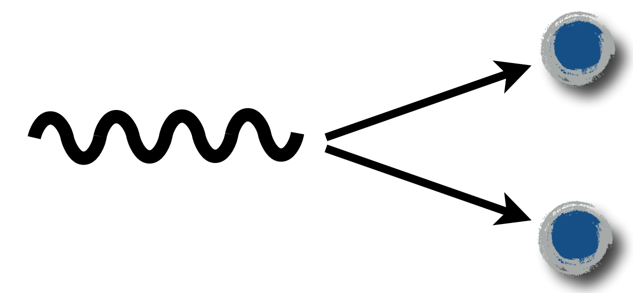
$$\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$$



Lellouch, Lüscher *(2001)* • Kim, Sachrajda, Sharpe *(2005)* • Christ, Kim, Yamazaki *(2005)* • MTH, Sharpe *(2012)*

- [ ] Time-like form factors

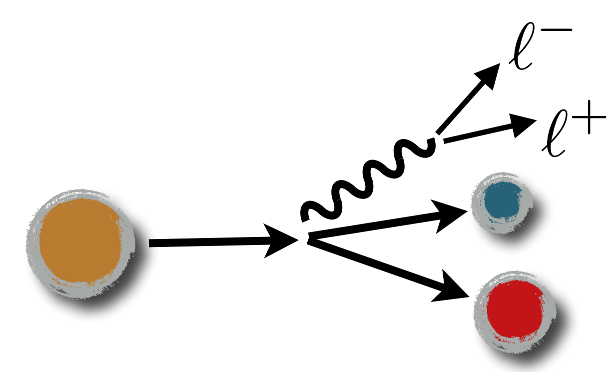
$$\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$$



Meyer *(2011)*

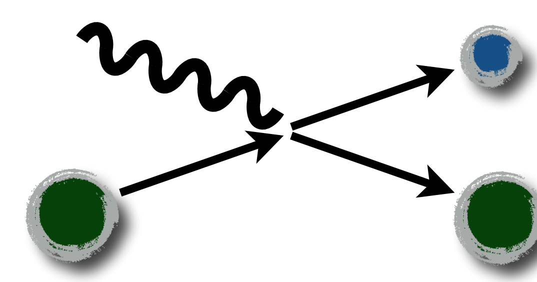
- [ ] Resonance form factors

$$\langle K\pi, \mathrm{out}|\mathcal{J}_{\alpha\beta}|B\rangle \equiv$$



- [ ] Particles with spin

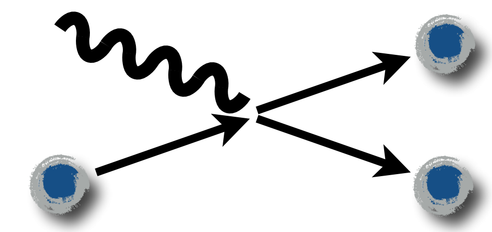
$$\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle \equiv$$



Agadjanov *et al. (2014)* • Briceño, MTH, Walker-Loud *(2015)* • Briceño, MTH *(2016)*

Pion photo-production

$$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$$



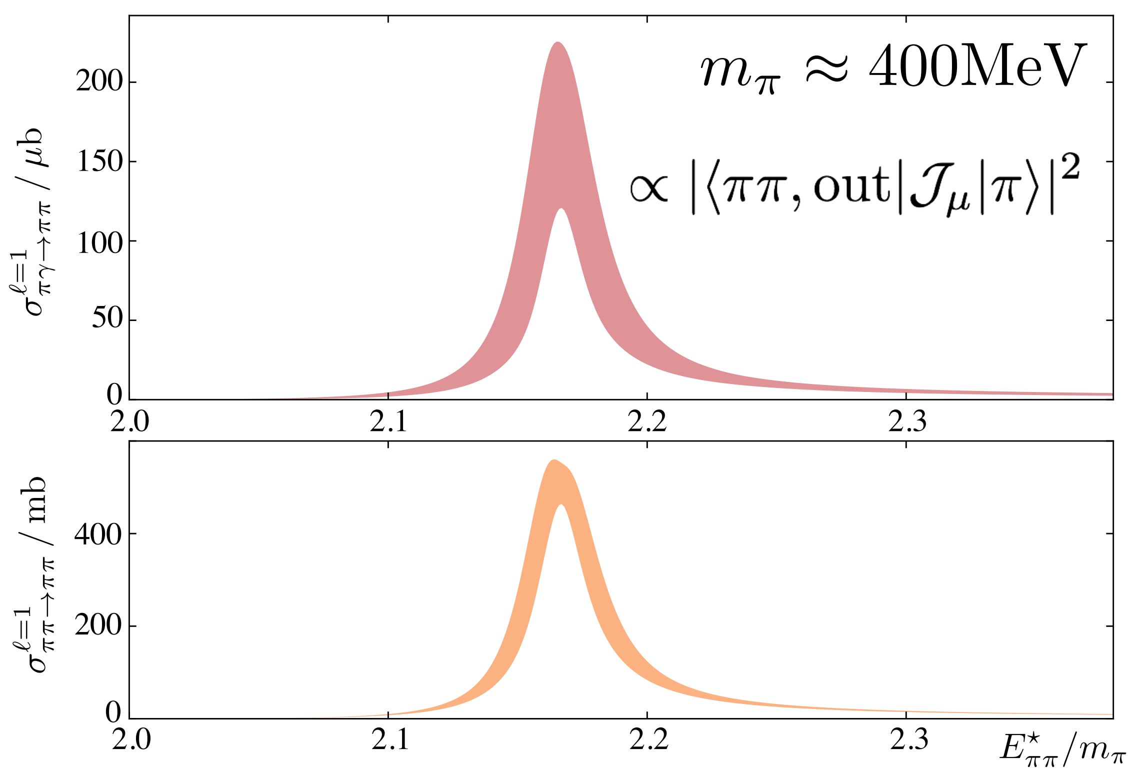
- Formal relation

get this from the lattice | experimental observable

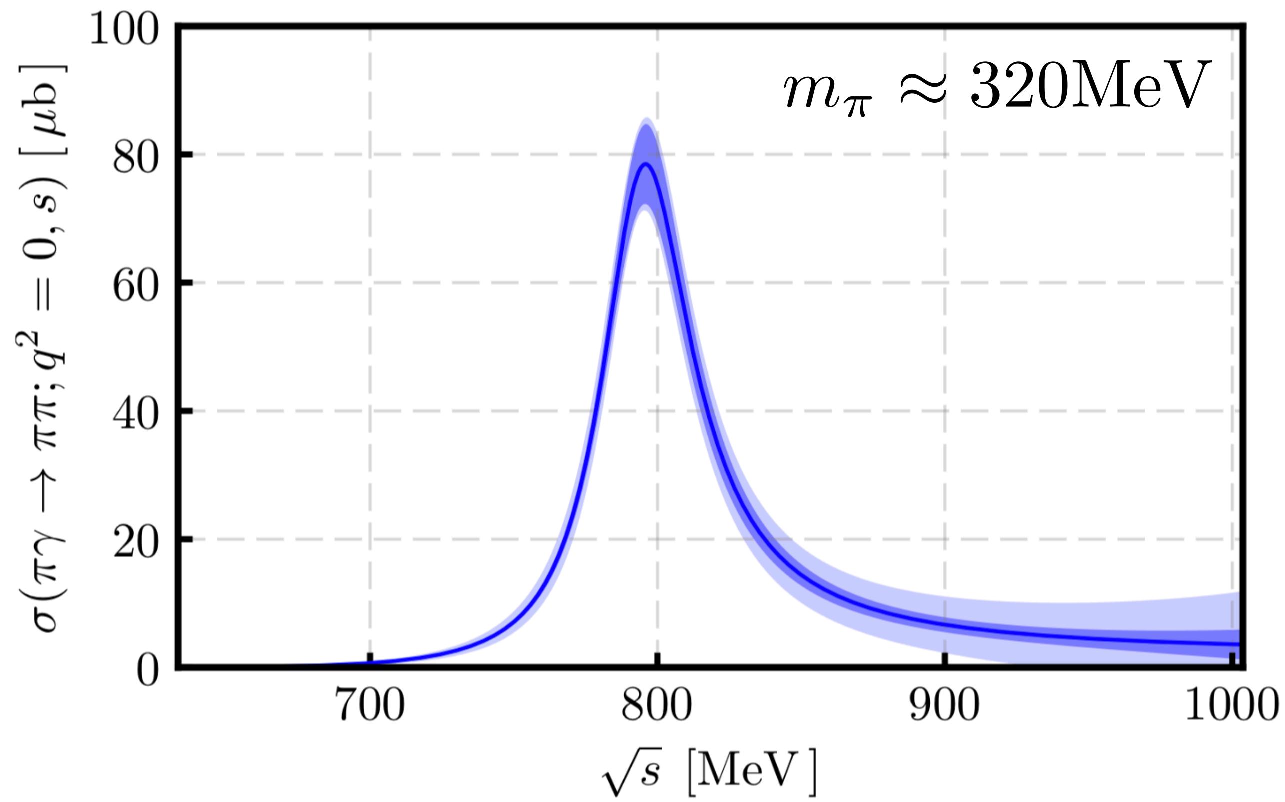
$$|\langle n, L|\mathcal{J}_\mu|\pi\rangle|^2 = \langle \pi|\mathcal{J}_\mu|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, L) \langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle$$

Briceño, MTH, Walker-Loud (2015)

- Numerical implementation



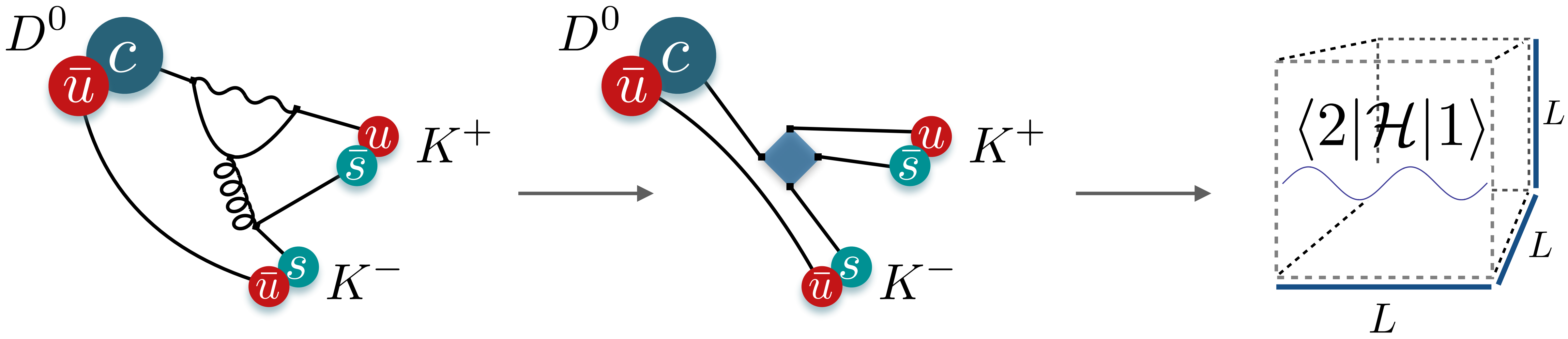
Briceño et. al., Phys. Rev. D93, 114508 (2016)



Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Hadronic D decays

- Integrating out electroweak physics → basis of four-quark operators



- ***Complicated:*** non-perturbative ***renormalization***, many ***operators*** and ***contractions***

See the RBC/UKQCD calculation of K→ππ

multi-hadron final state

$$\langle n, L|\ \mathcal{H}_{\rm weak}^{\overline{\rm MS}}\ |D, L\rangle$$

renormalized weak Hamiltonian

incoming D meson

$e^{-M_\pi L}$ *volume effects*

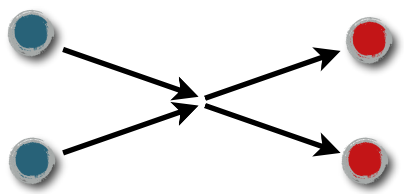
$\pi\pi, K\overline{K}, \pi\pi\pi\pi, \cdots$ ***have same quantum numbers + no asymptotic separation in the box***

How do we interpret $\langle n, L|$?

The finite-volume as a tool

- Coupled channels leave an *imprint* on finite-volume energies

infinite-volume scattering amplitudes



$\pi\pi \to K\overline{K}$

(i) given these experimentally

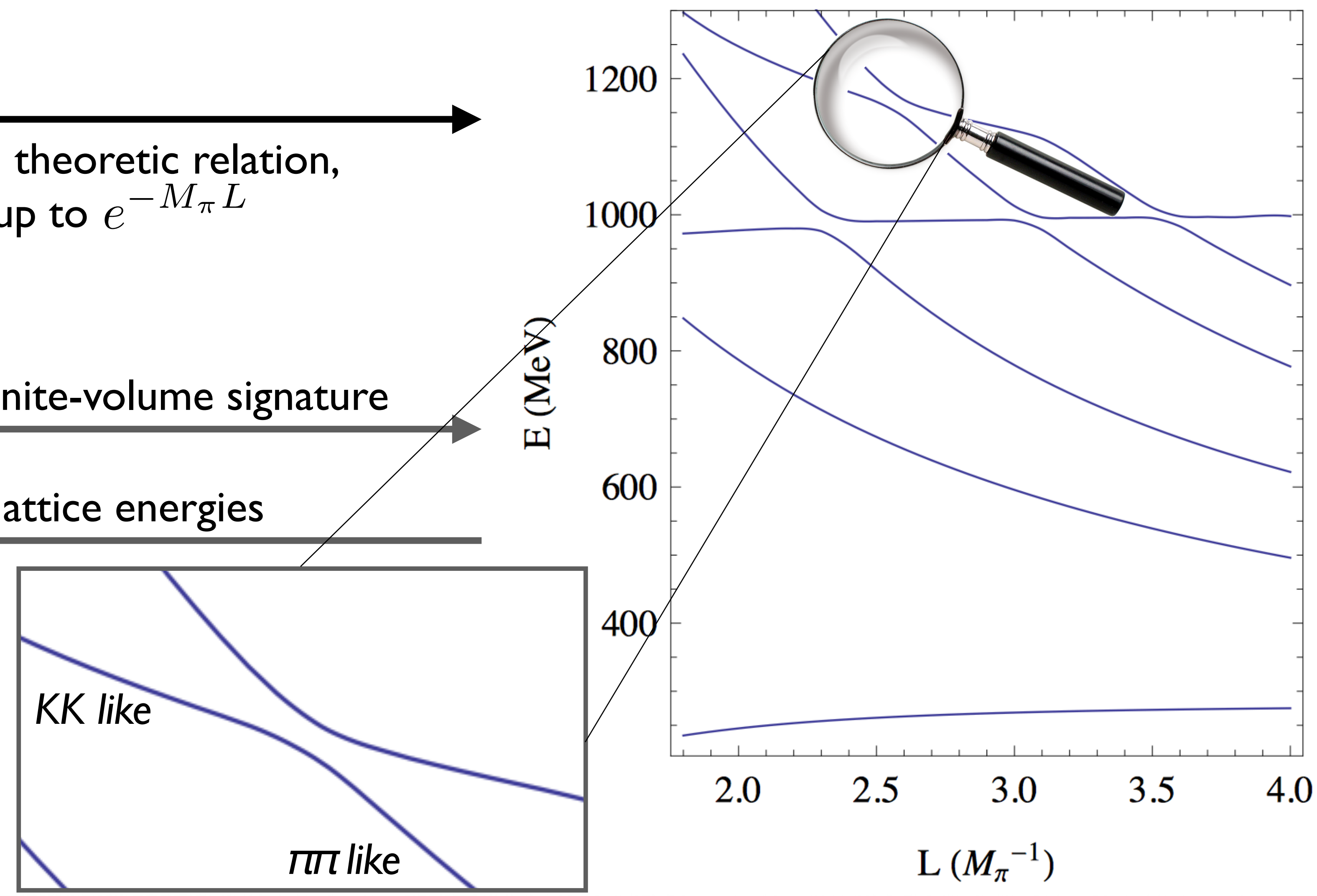
(ii) predict from first principles

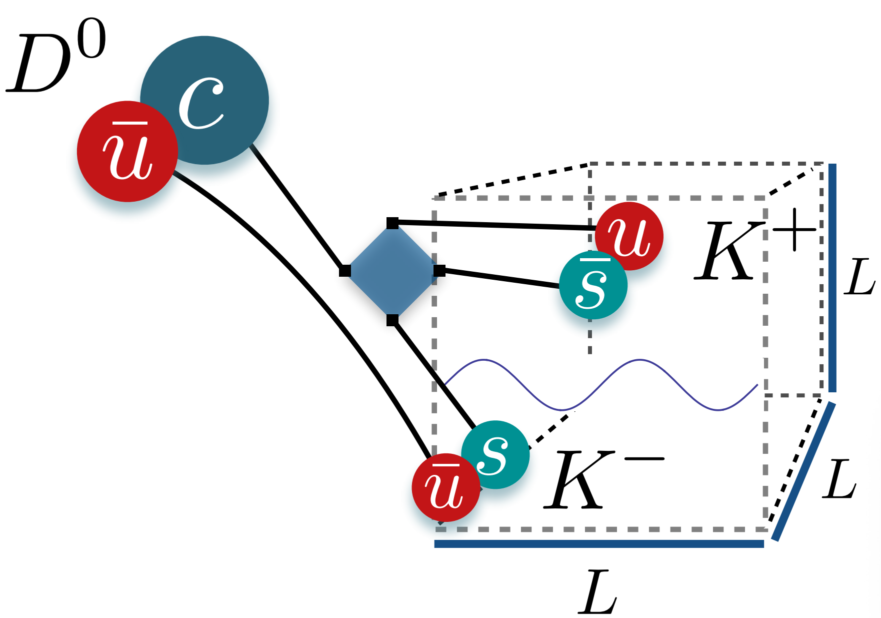
know field theoretic relation, exact up to $e^{-M_\pi L}$

can predict finite-volume signature

given lattice energies

finite-volume energies



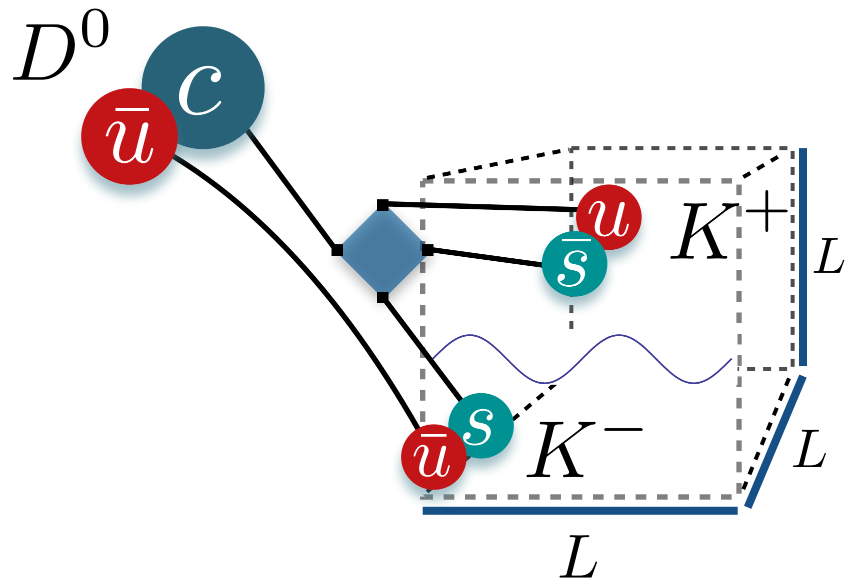


$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \mathrm{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \mathrm{out}\rangle$$

path to physical amplitude

• MTH, Sharpe, *Phys.Rev.* **D**86 (2012) 016007 •

How far in the future?



$$\langle n, L |\, \mathcal{H}_{\text{weak}}^{\overline{\text{MS}}}\, | D, L\rangle$$

$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \text{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \text{out}\rangle$$

- ❑ Pilot calculation underway at the University of Edinburgh
- ❑ Wilson-quark ensembles at the $SU(3)_F$ symmetric point
- ❑ See Fabian Joswig talks: Lattice2022 and MIT Colloquium

biggest challenge = still missing strategy for treating $\pi\pi\pi\pi$ etc, channels

$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \text{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \text{out}\rangle + c_{4\pi}^{(n)}|\pi\pi\pi\pi, \text{out}\rangle + \cdots$$

Funded by UKRI Future Leaders Fellowship

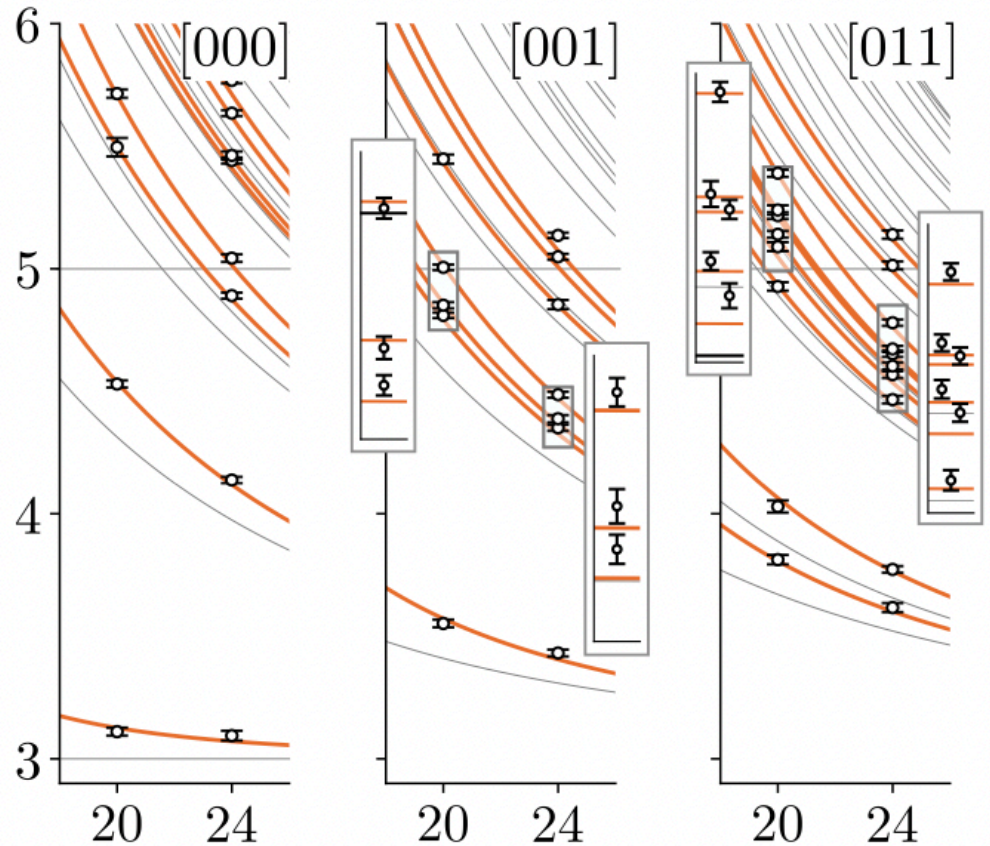
UK Research and Innovation

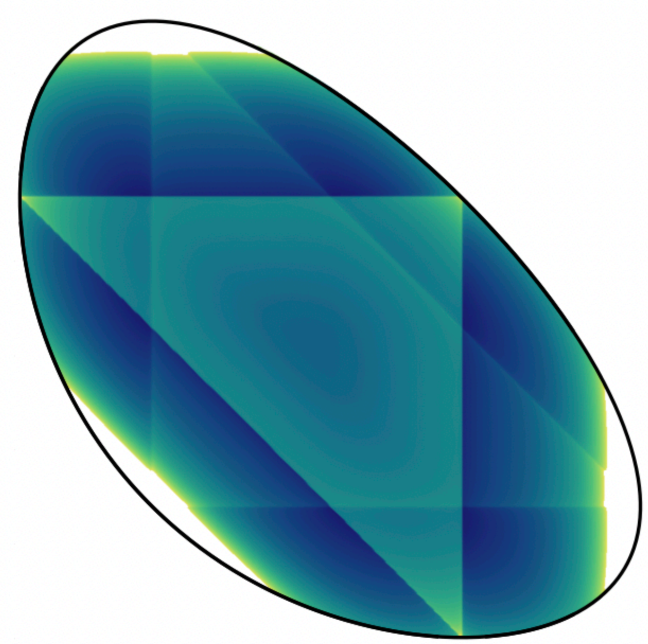
Towards >2 hadrons

- Multiple three-particle finite-volume formalisms developed (so far only spin zero)

  MTH, Sharpe (2014-2016) *See also Döring, Mai, Hammer, Pang, Rusetsky*

- First lattice calculations appearing… e.g. $\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$



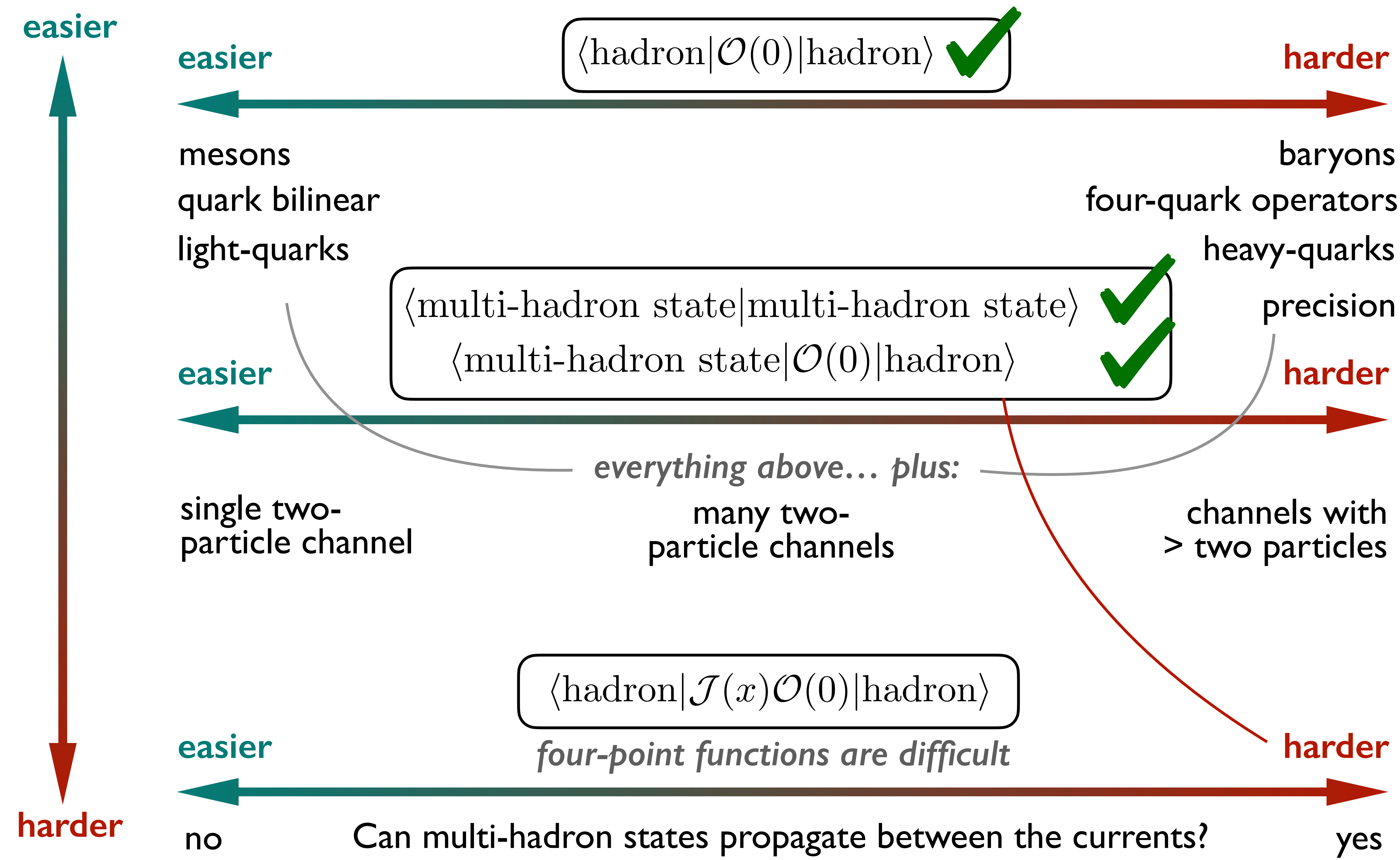


- Extract reliable spectrum
- Use formalism to fit scheme-dependent K-matrix
- Solve integral equations to reach physical amplitude

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

- See Blanton et al. 2022 and 2023 for pion and kaon results
- See Sadasivan et al. 2022 for application to $a_1(1260)$

(Incomplete) landscape of lattice observables



Formal & numerical progress: Long-distance matrix elements

- Formal method understood… *assuming only two-hadron intermediate states*

$$\Sigma^+ \underset{H_W}{\longrightarrow} N\pi \to p\gamma^* \qquad\qquad K^0 \underset{H_W}{\longrightarrow} \pi\pi \underset{H_W}{\longrightarrow} \overline{K}^0$$

- Issue of growing exponentials *(Christ et al.)*

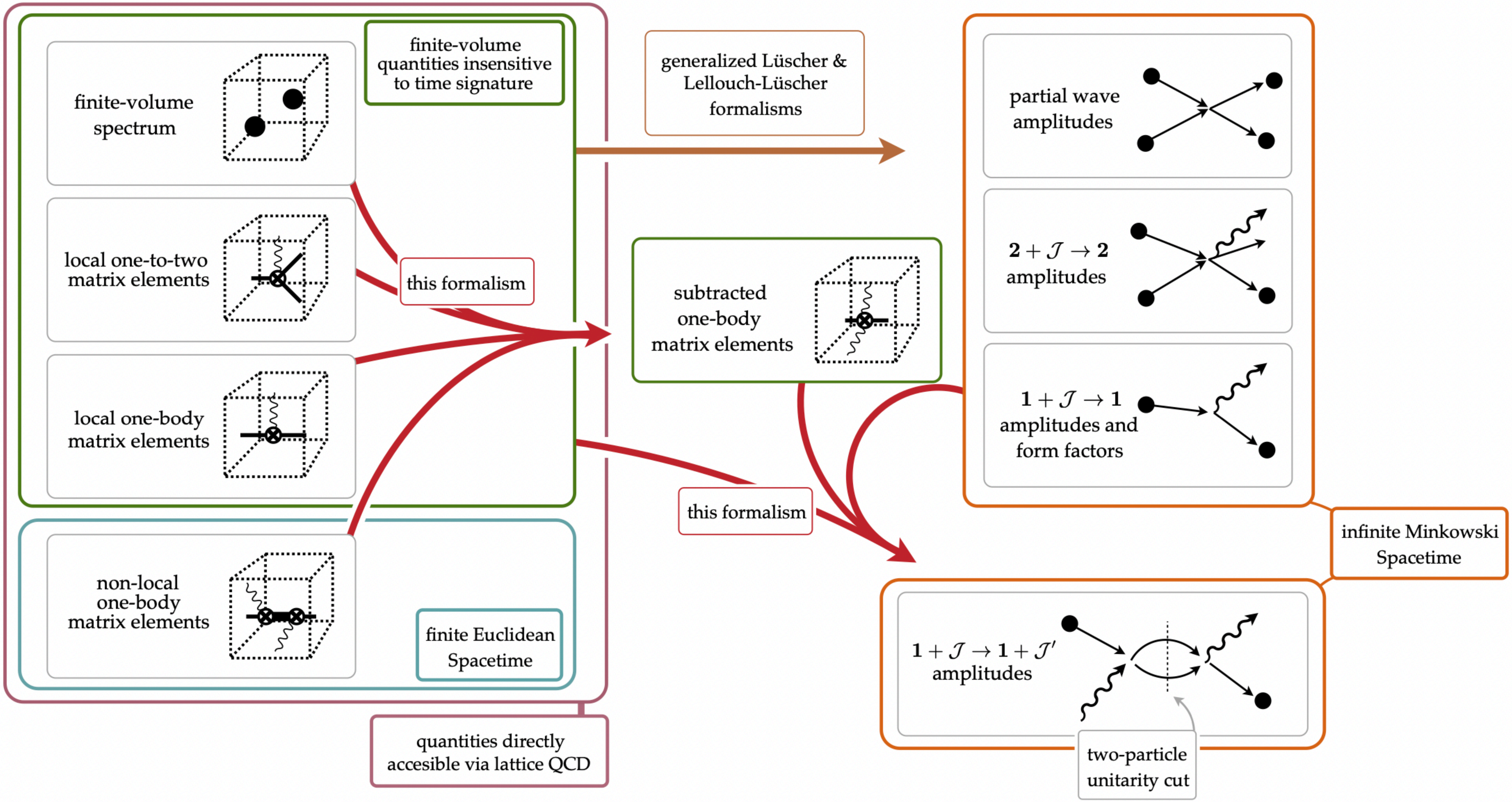
$$\langle \overline{K} | \mathcal{H}_W(0) \mathcal{H}_W(-|\tau|) | K \rangle_L = \sum_n c_n(L) e^{-(E_n(L) - M_K)|\tau|} \quad \underset{\int_{-T}^{0} d\tau}{\Longrightarrow} \quad \sum_n c_n \frac{1 - e^{-(E_n - M_K)T}}{M_K - E_n}$$

- Issue of power-like finite-volume effects (after discarding exponential)

$$F_L = \sum_n \frac{c_n}{M_K - E_n}$$

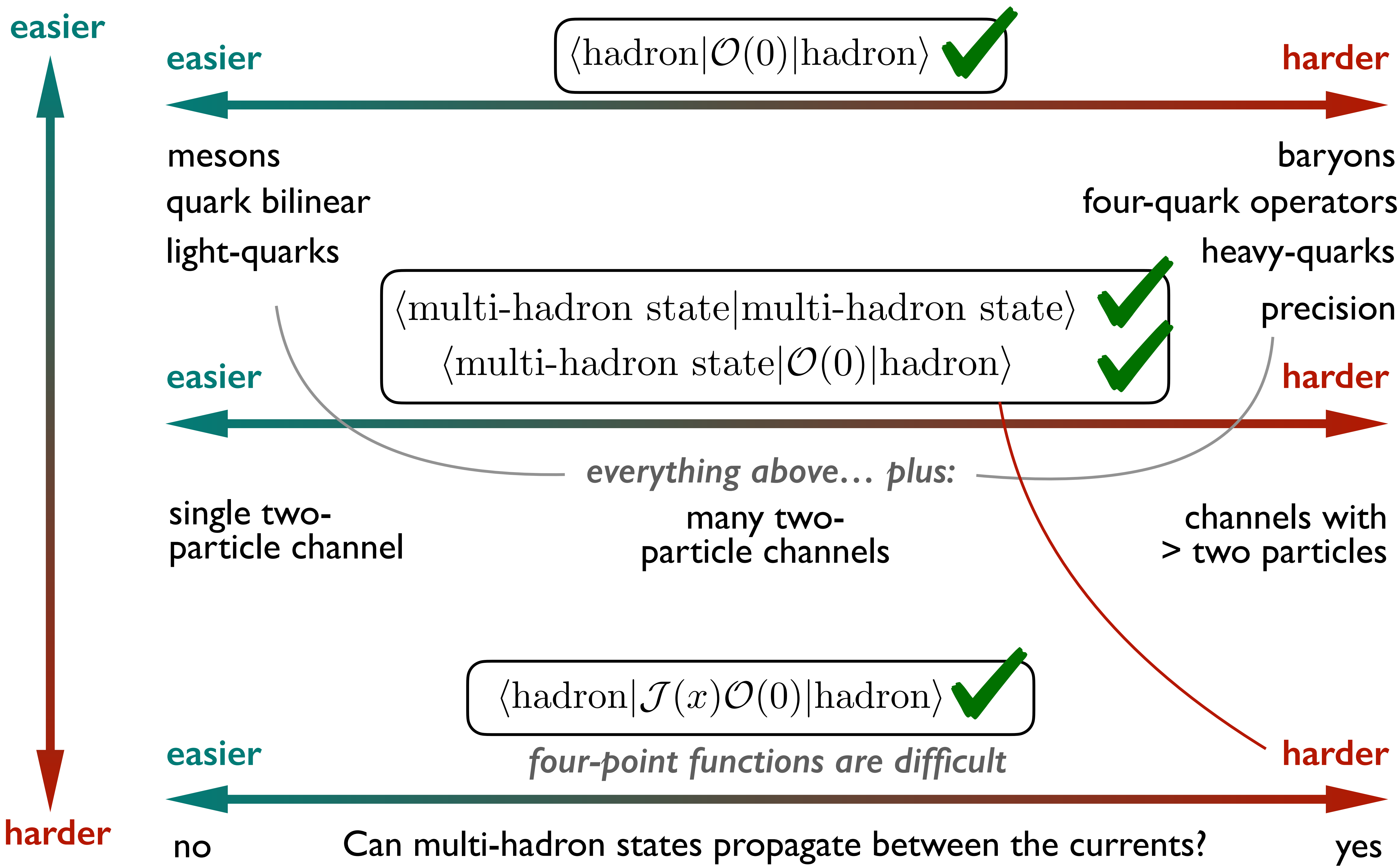
Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
• Briceño, Davoudi, MTH, Schindler, Baroni *(2019)* • Erben, Gülpers, MTH, Hodgson, Portelli *(2022)*

Formal & numerical progress: Long-distance matrix elements



Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)* • Briceño, Davoudi, MTH, Schindler, Baroni *(2019)* • Erben, Gülpers, MTH, Hodgson, Portelli *(2022)*

(Incomplete) landscape of lattice observables



A more inclusive perspective…

- Finite-volume as a tool
  - LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… "Can't you just analytically continue?"
  - Still important challenges and limitations to consider

Correlation functions → observables

- Lattice QCD gives finite-volume Euclidean correlators

$$\langle 0|\, \mathcal{O}_1(0)\, e^{-\hat{H}\tau}\, \mathcal{O}_2(0)\, |0\rangle_L$$

have

- Complete physical information is contained in…

$$\langle 0|\, \mathcal{O}_1(0)\, f(\hat{H})\, \mathcal{O}_2(0)\, |0\rangle_\infty$$

want

- Detailed choice of *f(E)* and operators determines the observable

R-ratio

$$\langle 0|\, j_\mu(0)\, \delta(\hat{H} - \omega)\, j_\mu(0)\, |0\rangle_\infty$$

Meyer • Bailas, Hashimoto, Ishikawa (2020)
Alexandrou et al. (2022)

D-meson total lifetime

$$\langle D|\, \mathcal{H}_W(0)\, \delta(M_D - \hat{H})\, \mathcal{H}_W(0)\, |D\rangle_\infty$$

MTH, Meyer, Robaina (2017)

ππ → ππ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, \pi(0)\, |\pi\rangle_\infty$$

j → ππ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, j_\mu(0)\, |0\rangle_\infty$$

Bulava, MTH (2019)

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega \, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

have   want

$$G(\tau) = \int d\omega \, e^{-\omega\tau} \, \rho(\omega)$$

have   want

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega}, \tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega}, \tau) \int d\omega \, e^{-\omega\tau} \, \rho(\omega)$$

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega \, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

have   want

$$G(\tau) = \int d\omega \, e^{-\omega\tau} \, \rho(\omega)$$

have   want

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega},\tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega},\tau) \int d\omega \, e^{-\omega\tau} \, \rho(\omega) = \int d\omega \left[\sum_\tau \mathcal{K}(\bar{\omega},\tau) \, e^{-\omega\tau} \right] \rho(\omega)$$

$$= \int d\omega \, \widehat{\delta}_\Delta(\bar{\omega}, \omega) \, \rho(\omega)$$

δ is exactly known

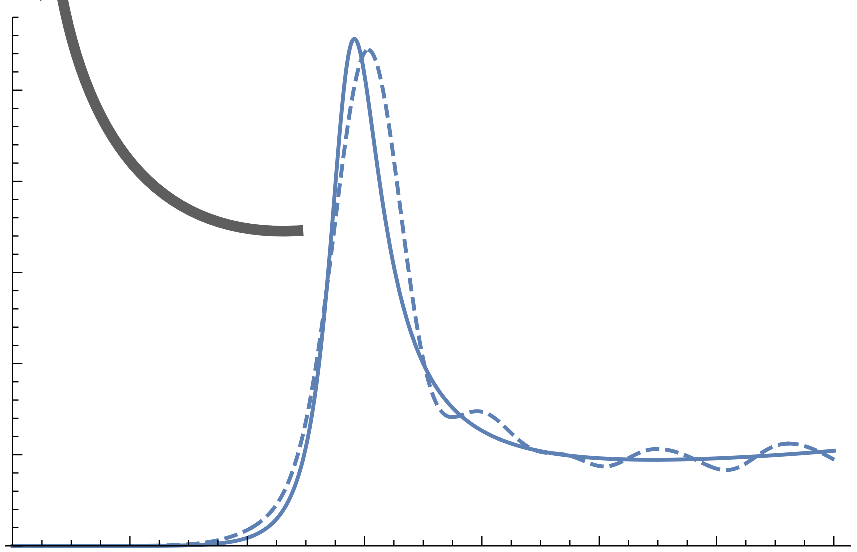
- Non-linear (***not discussed here…***)
  - Maximum Entropy Method (MEM)
  - Direct fits
  - Neural networks

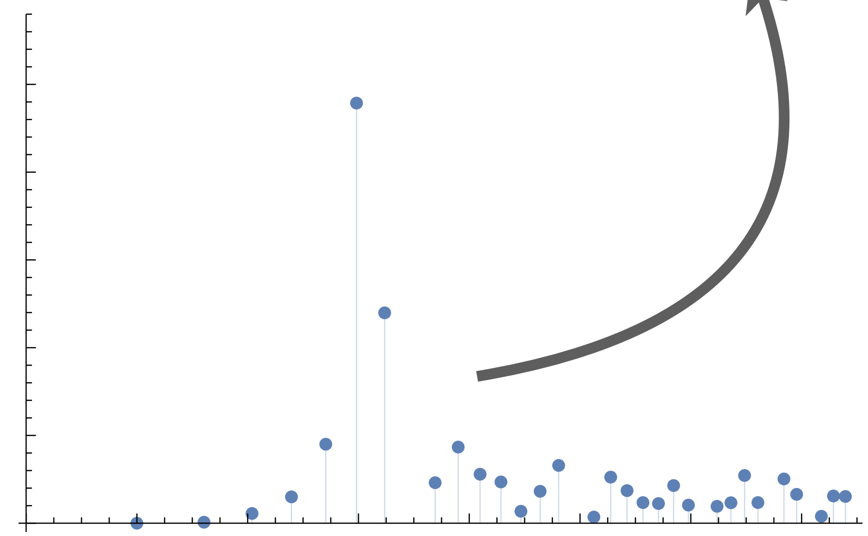
See multiple ECT and CERN workshops, work by Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis, Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu, Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, …*

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^{\infty} d\omega \, \widehat{\delta}_\Delta(\bar{\omega}, \omega)\, \rho_L(\omega)$$

$$G_L(\tau) = \int d\omega \, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…

  contains implicit smearing (or else $L \to \infty$)

We require…

$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

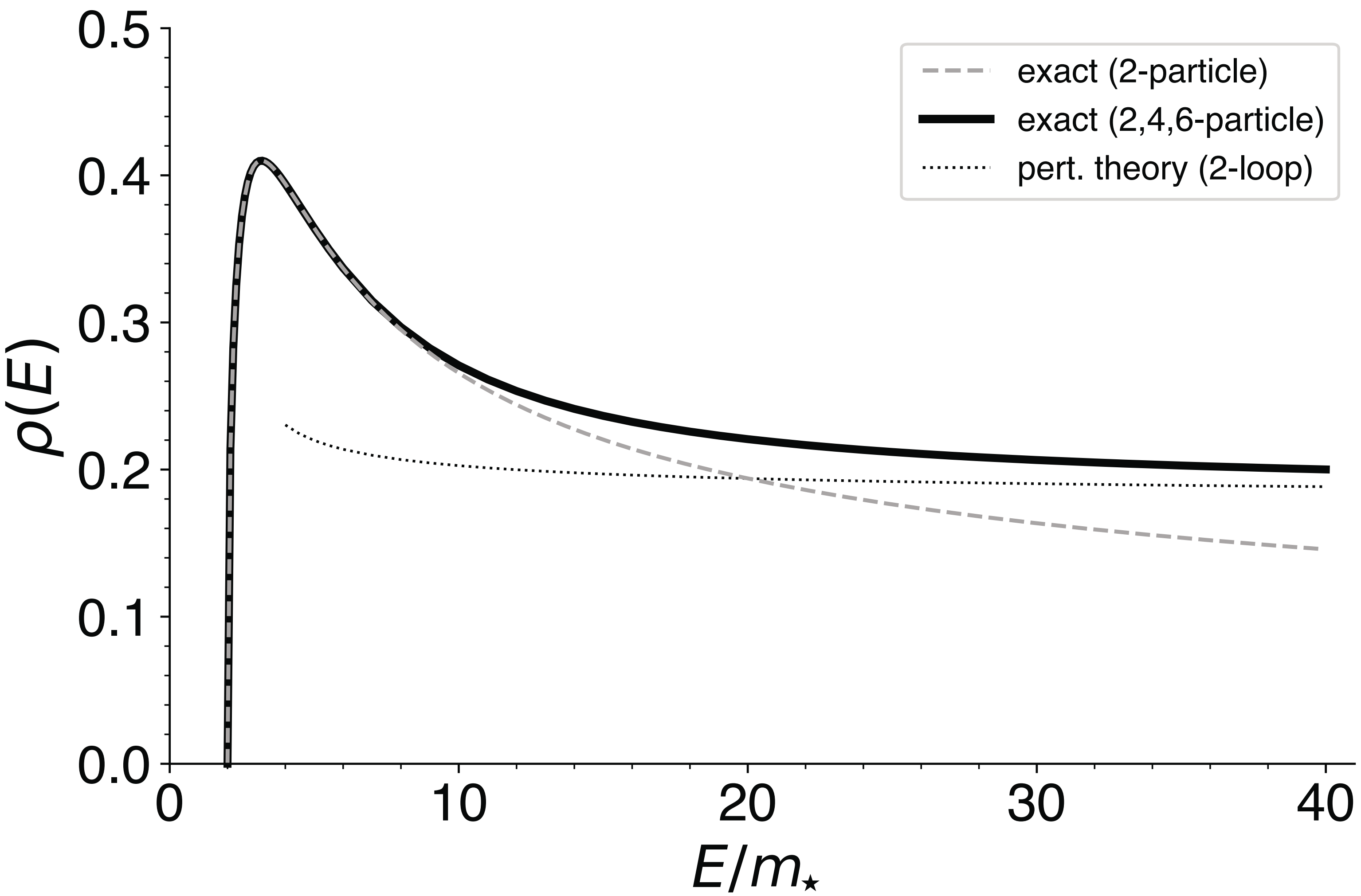
1+1 O(3) Model

- *Integrable* theory with some nice similarities to QCD
  - Asymptotically free
  - Dynamically generated mass gap
  - Iso-spin like symmetry
  - Conserved iso-vector vector current

$$S[\sigma] = \frac{1}{2g^2} \int d^2x \, \partial_\mu \sigma(x) \cdot \partial_\mu \sigma(x)$$

$$j^c_\mu(x) = \frac{1}{g^2} \epsilon^{abc} \sigma^a(x) \partial_\mu \sigma^b(x)$$

conserved current



$$\rho(E) = 2\pi \, \langle \Omega | \hat{j}^a_1(0) \, \delta^2(\hat{P} - p) \, \hat{j}^a_1(0) | \Omega \rangle$$

spectral function

$$\rho^{(2)}(E) = \frac{3\pi^3}{8\theta^2} \, \frac{\theta^2 + \pi^2}{\theta^2 + 4\pi^2} \, \tanh^3 \frac{\theta}{2}$$

$$\theta = 2 \cosh^{-1} \frac{E}{2m}$$

Monte-Carlo test

- Full lattice calculation in two-dimensional O(3) non-linear sigma model

- Demonstrating the modified Backus-Gilbert (HLT) method for the “R-ratio”

$$C(t) \equiv \int d\boldsymbol{x}\, \langle \Omega |\, \hat{j}_1^a(0,\boldsymbol{x})\, e^{-\hat{H}t}\, \hat{j}_1^a(0)\, | \Omega \rangle = \int_0^\infty d\omega\, e^{-\omega t}\, \rho(\omega)$$

- Data + theory driven analysis of finite-L and -T effects and discretization

ID	$(L/a) \times (T/a)$	β	$am_\star$	$m_\star L$	$m_\star T$
A1	640×320	1.63	0.0447989(62)	29	14
A2	1280×640	1.72	0.0257695(31)	33	17
A3	1920×960	1.78	0.0176104(31)	34	17
A4	2880×1440	1.85	0.0112608(29)	32	16
B1	5760×1440	1.85	0.0112607(73)	65	16
B2	2880×2880	1.85	0.0112462(72)	32	32

Bulava, MTH, Hansen, Patella, Tantalo (2021)

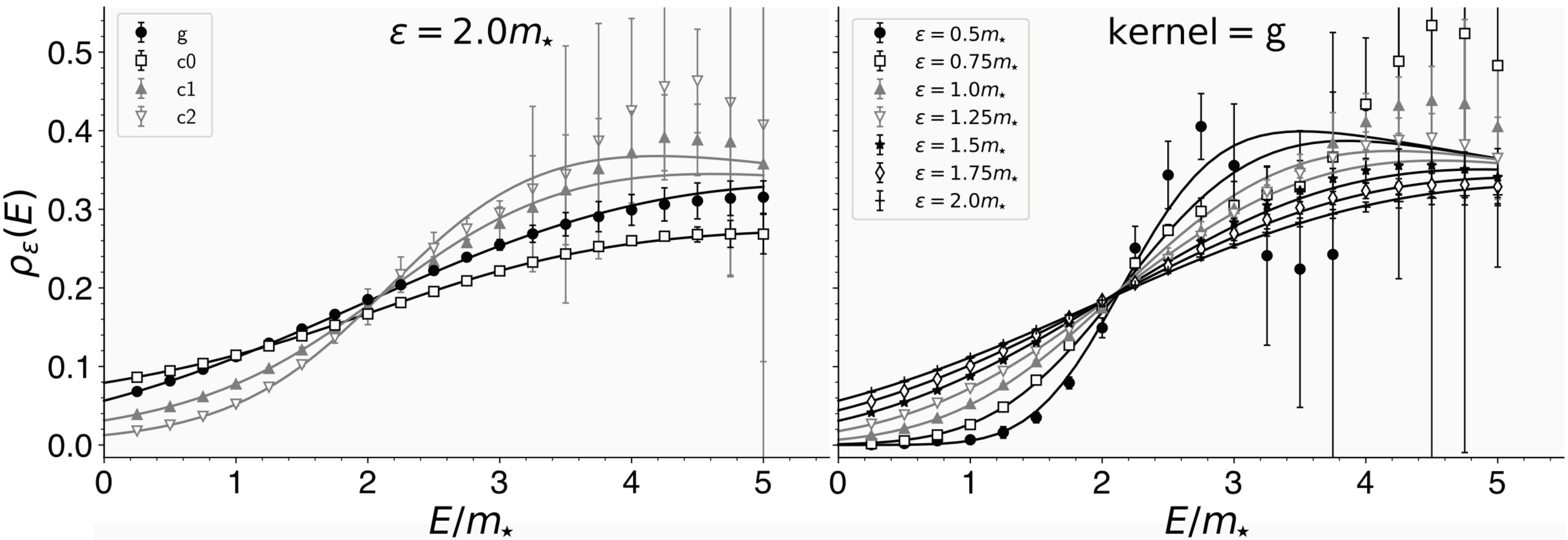
Smeared spectral function vs analytic result

- Construct different smearings of $\rho(\omega)$

$$\rho_\epsilon^\lambda(E) = \int_0^\infty d\omega\, \delta_\epsilon^\lambda(E,\omega)\, \rho(\omega)$$

$$\delta_\epsilon^{\mathrm{g}}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\mathrm{c0}}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

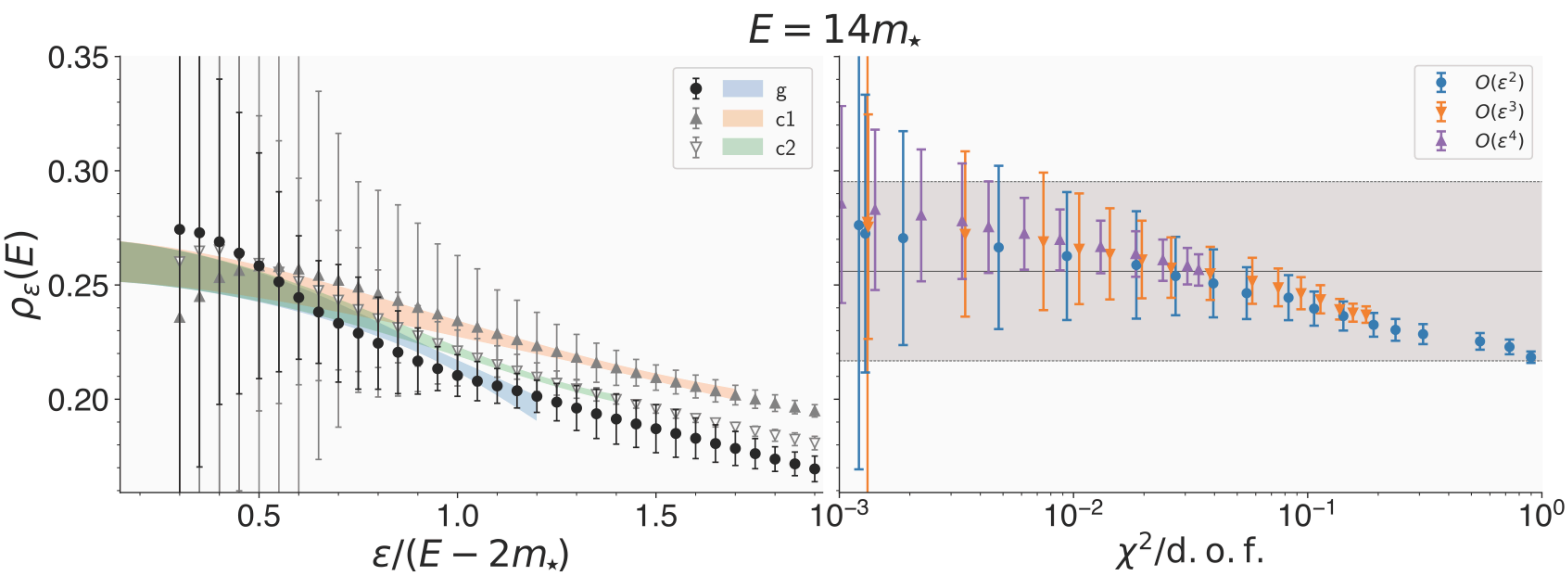
$$\delta_\epsilon^{\mathrm{c1}}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\mathrm{c2}}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Extrapolation

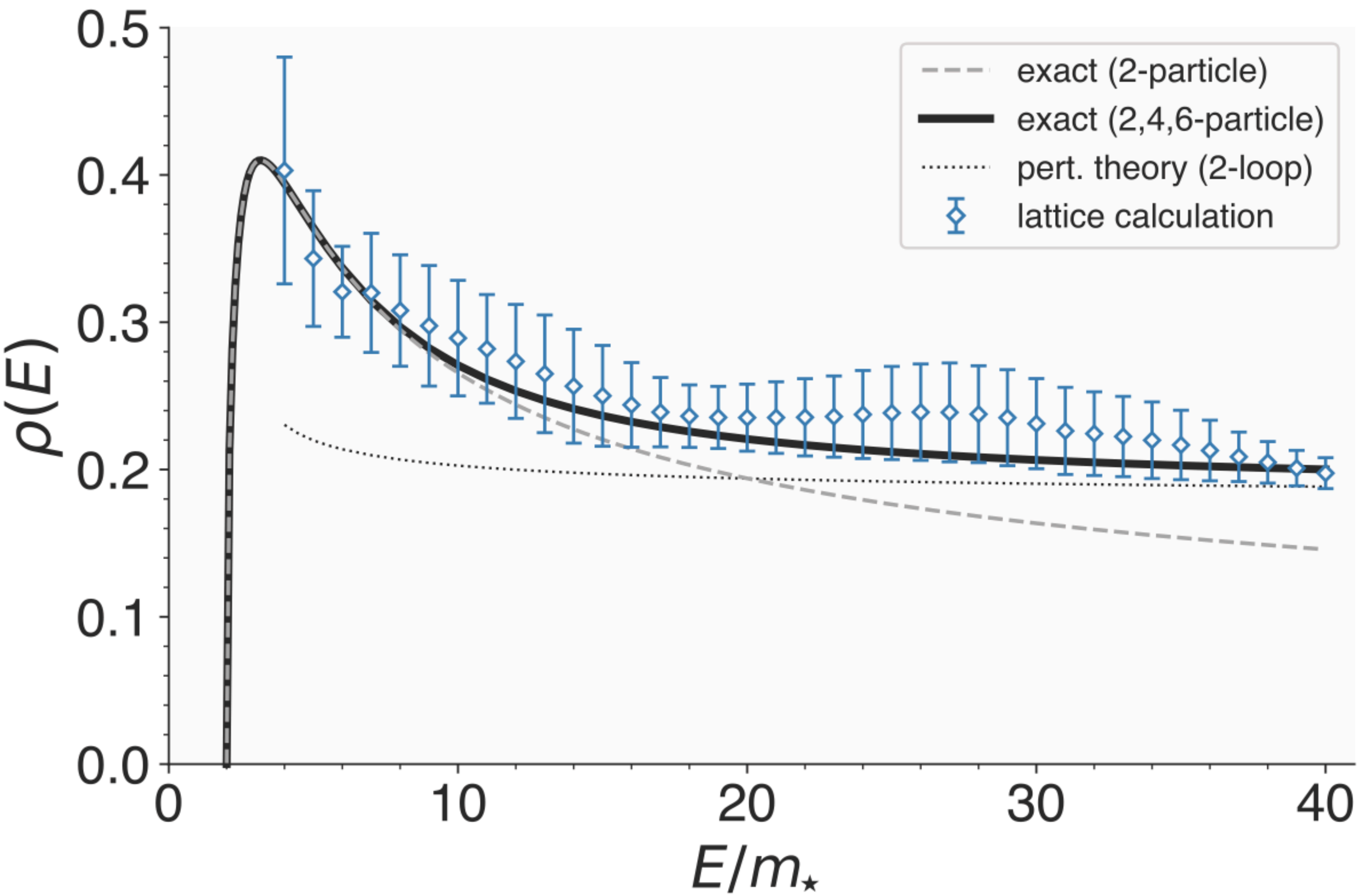
- Targeting $\rho(E)$ for $E = 14m$ here



- Use known relations between different smearing kernels

Bulava, MTH, Hansen, Patella, Tantalo (2021)

Result



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Many QCD applications already published… see work by A. Barone, S. Hashimoto, A. Jüttner, T. Kaneko, R. Kellermann, R. Frezzotti, G. Gagliardi, V. Lubicz, F. Sanfilippo, S. Simula …

Future of lattice QCD

- ❑ Semi-leptonics with QED completely understood and included
- ❑ Smaller lattice spacings → confident to treat heavy and light quarks the same way

- ❑ Many collaborations (with different set-ups) calculating multi-hadron scattering
- ❑ Scattering calculations reaching maturity to appear in FLAG
- ❑ Deeper understanding of formalism for many hadron states

- ❑ First publications with heavy to light resonances (rigorously treating width)
- ❑ First pilot studies of multi-hadron *D* decays and mixing
- ❑ First calculations of three-hadron decays
- ❑ Multi-hadron decay calculations reaching maturity to appear in FLAG

- ❑ Spectral methods reaching maturity with fully controlled uncertainties
- ❑ More common use of algorithms to exponentially enhance signal-to-noise ratios
- ❑ Machine learning to improve gauge field generation (and observable extraction?)
  (see work by P Shanahan and MIT group)
- ❑ Zettascale computing

Closing remarks

- Lattice QCD is progressing by improving precision and unlocking new observables

- Full control of many exciting quantities (especially those involving multi-hadrons) is *still many years away*

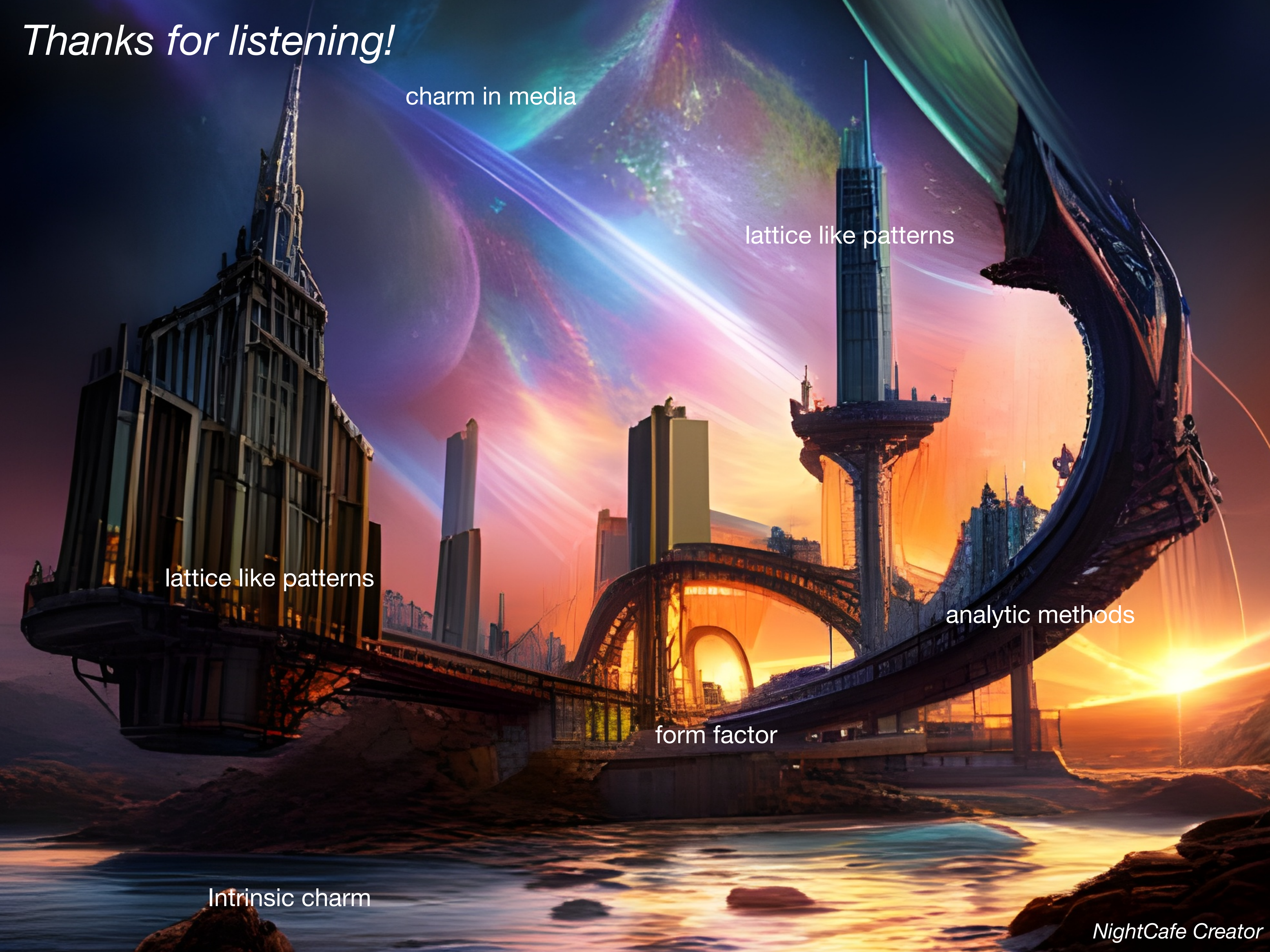
- We should continue to pursue unexpected/more direct collaborations between
  - lattice QCD - heavy quark and EFT methods - light cone sum rules - factorisation
  - amplitude analysis - dispersive methods - quark models -

Need something more exciting to end…

Asked NightCafe Creator about the future of charm…

Funded by UKRI Future Leaders Fellowship

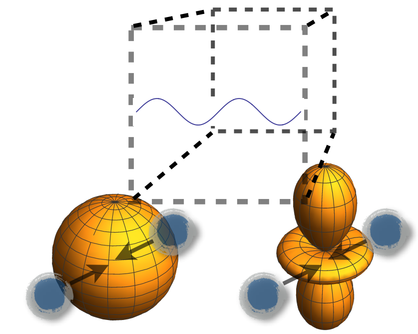
UK Research and Innovation


Thanks for listening!
charm in media
lattice like patterns
lattice like patterns
analytic methods
form factor
Intrinsic charm
NightCafe Creator

Coupled channels

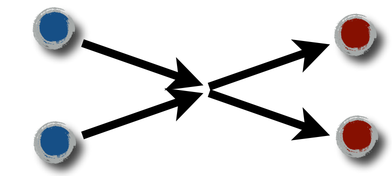
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp} \\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0 \\ 0 & F_b\end{pmatrix}\right] = 0$

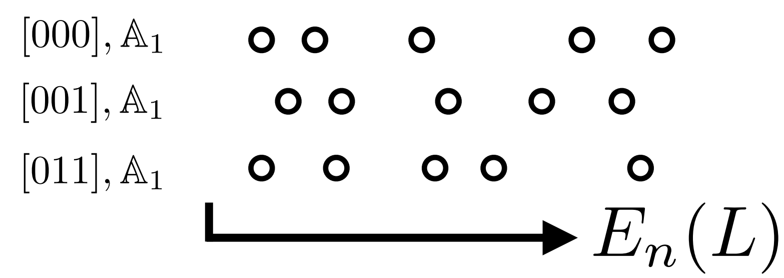


- Workflow…

Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→ Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→ Vary L and P to recover a dense set of energies



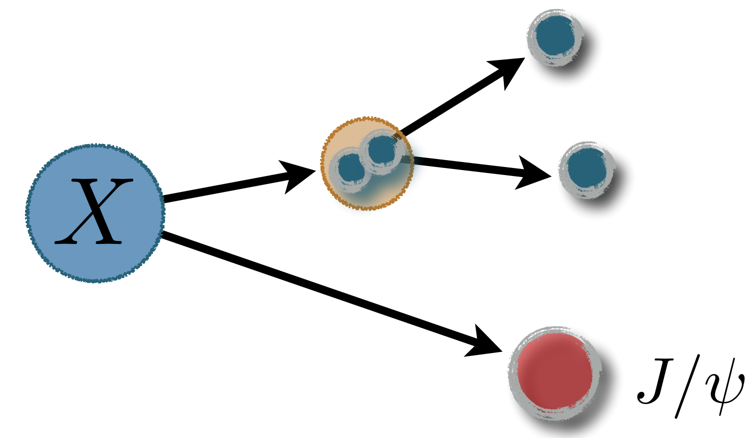
↓

hadspec

Identify a broad list of K-matrix parametrizations
polynomials and poles | EFT based | dispersion theory based

→ **Perform global fits to the finite-volume spectrum**

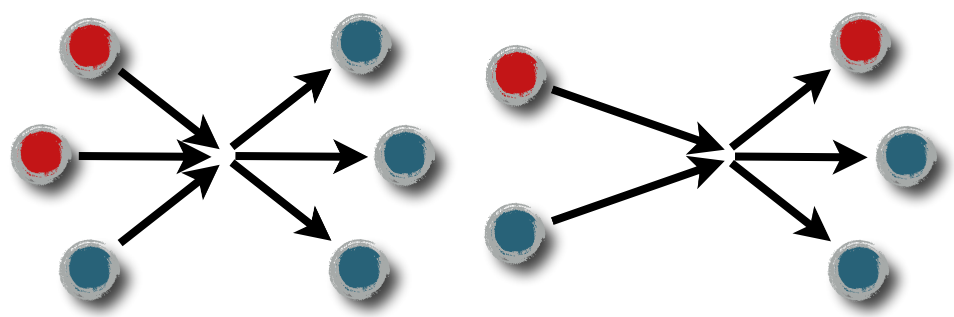
3-particle amplitudes



2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays

Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

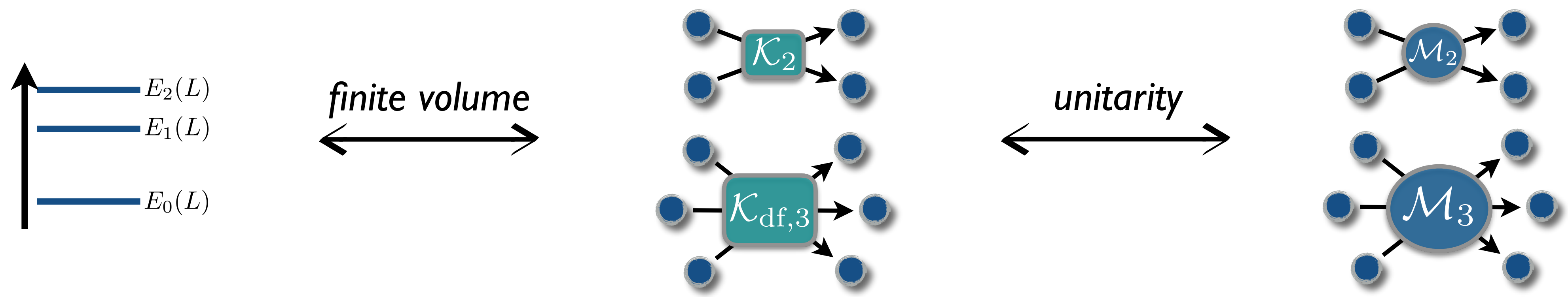
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\ a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

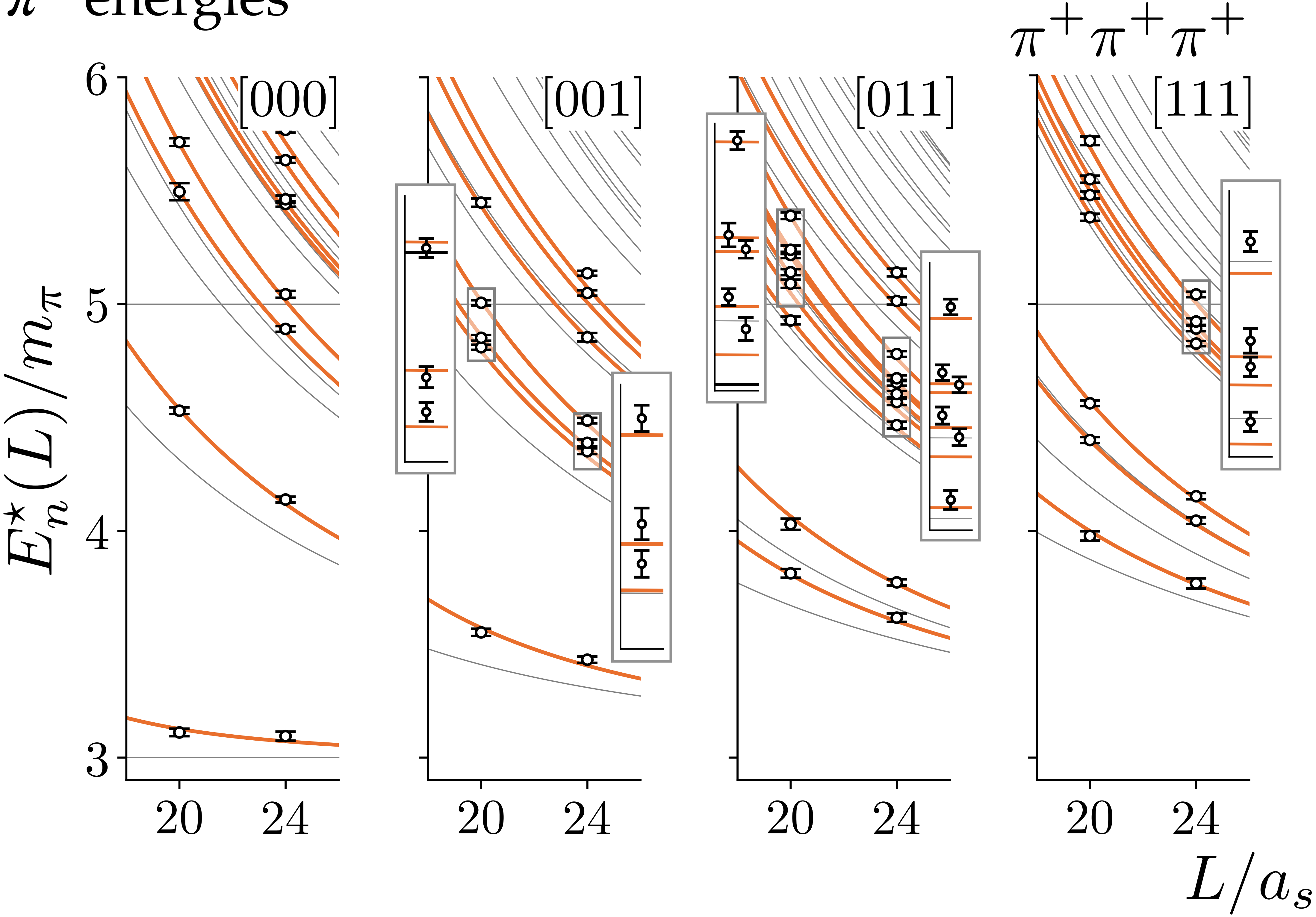
and much more!… *(3-body forces, weak transitions, gluons content)*

Status…



- ☑ Identical spin-zero, no 2-to-3, no K2 poles • MTH, Sharpe (2014, 2015) •
- ☑ as above… but including 2-to-3 • Briceño, MTH, Sharpe (2017) •
- ☑ as above… but including K2 poles • Briceño, MTH, Sharpe (2018) •
- ☑ Non-identical, non-degenerate spin-zero $\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$
  • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020, 2021)
- ☐ Multiple three-particle channels… Spin!

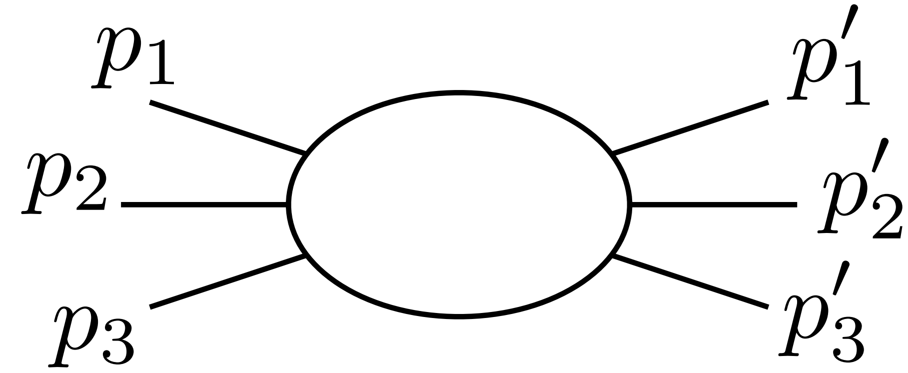
$\pi^+\pi^+\pi^+$ energies

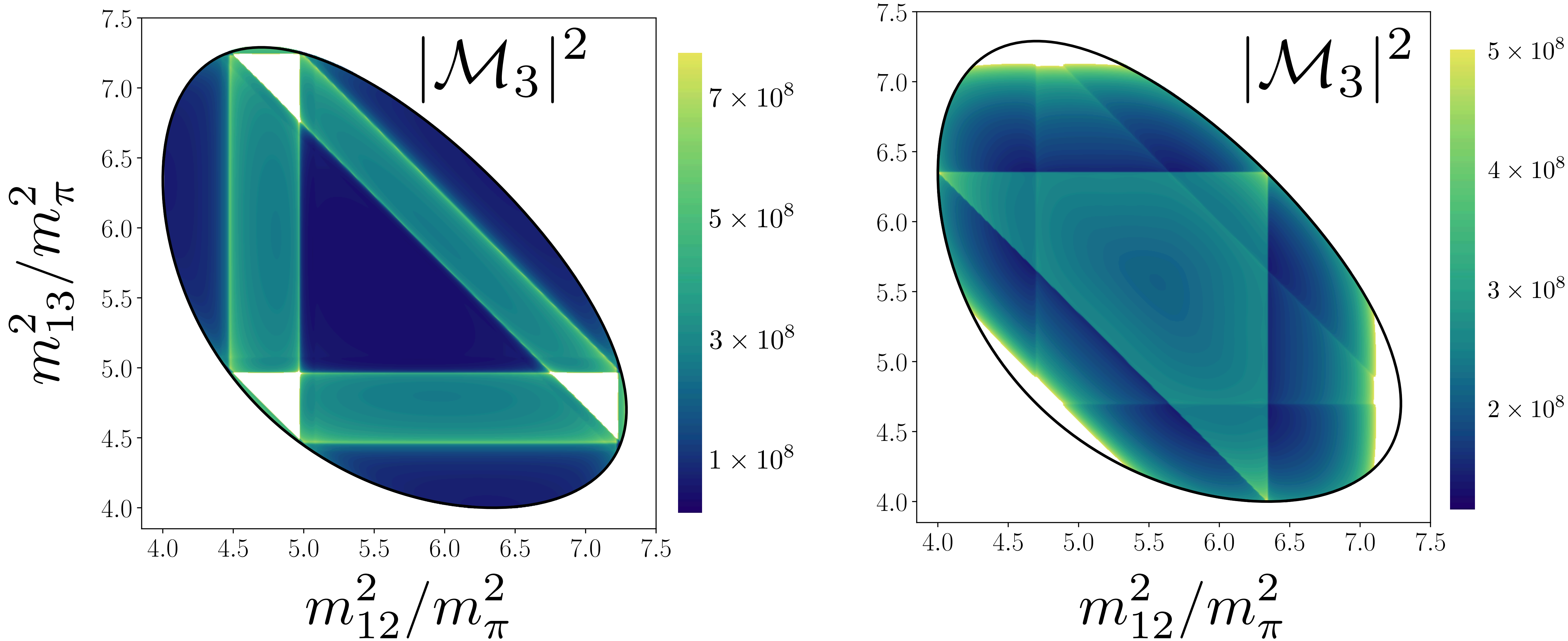


MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001,

see also work by… Culver, Döring, Hanlon, Hörz, Mai, Morningstar, Romero-Lopez, Sharpe + ETMC

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p'_i, p_j)$$





MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001