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Combined analysis of $D^+, D^0 \to \overline{K}\pi\pi$ decays with isospin symmetry, analyticity/unitarity

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Introduction:

- We consider Cabibbo favoured decay modes

$$D^+ \to K^-\pi^+\pi^+$$
$$D^+ \to \bar{K}^0\pi^0\pi^+$$
$$D^0 \to K^-\pi^+\pi^0$$
$$D^0 \to \bar{K}^0\pi^-\pi^+$$
$$D^0 \to \bar{K}^0\pi^0\pi^0$$

- Study isospin symmetry relations between them, +study three-body rescattering effects

- All modes measured. Strong phase distribution in $D^0 \to K_S\pi^-\pi^+$ can be used in determination of γ/Φ_3[Giri et al.,PR D68,054018(2003)] and $D^0 - \bar{D}^0$ mixing parameters

- Dynamical feature: dominance of P-waves and S-waves
  - → First observation of κ resonance in $D^+ \to K^-\pi^+\pi^+$ mode[E791, PRL 89,121801 (2001)]
  - → However, no κ seen in D^0 modes, why ?
- Three-body rescattering studied in D^+ modes[P.Magalhães et al.,PR D84,094001 (2011),S.Nakamura, PR D93,014005 (2015)](Faddeev eqs.)[F.Niecknig,B,Kubis, JHEP 151-,142(2015), PLB780,471 (2018)](Khuri-Treiman eqs.).

Khuri-Treiman eqs (history)

- Derived for $K \to 3\pi$[N.Khuri,S.Treiman PR 119,1115 (1960)]; also[R.Sawyer,C.Wali PR 119,1429 (1960)]
- Motivation: relate observed (puzzling) linear energy dependence in $K \to 3\pi$ Dalitz plot to FSI
  - → Using iterative approx. KT estimate $\pi\pi$ scattering length difference

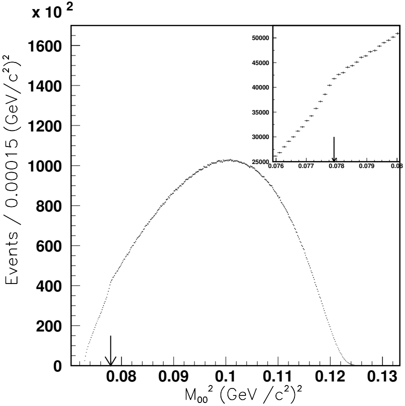
    $$a_0 - a_2 = -0.70 \quad \text{wrong sign!}$$

    correct value: $$a_0 - a_2 = +0.265 \pm 0.004$$

    [G.Colangelo et al., NP B603,125(2001)]
  - → KT eqs. singular integral equations:[Neveu, Scherck, Ann.Phys. 57,39 (1970)]transform with Muskhelishivili-Omnès method

→ Note: $a_0 - a_2$ can be determined in $K^\pm \to \pi^\pm\pi^0\pi^0$ from cusp effect [N.Cabibbo, PRL 93,121801 (2004)] probing $\pi^0\pi^0 \to \pi^+\pi^-$ rescattering.

Cusp observed by NA48/2
[PL, B633,173 (2006)]



→ Application to $\eta \to 3\pi$ [J.Kambor et al.,NP B465,215 (1996), A.Anisovich H.Leutwyler PL B375,335 (1996)]: 2 S-waves: M_0, M_2, one P-wave: M_1.

Accurate numerical solutions computed

Determine isospin breaking quark mass ratio Q

$$Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

by matching to $ChPT$

→ Application to $D^+ \to K^-\pi^+\pi^+, \bar{K}^0\pi^+\pi^0$ [Niecknig,Kubis (2015),(2017)] S, P-wave amplitudes: $F_0^{1/2}, F_0^{3/2}, F_1^{1/2}, F_1^{3/2}, G_0^2, G_1^1$.

Isospin Analysis

1) Weak hamiltonian for $D \to \overline{K}\pi\pi$ involves

$$O_1 = (\bar{s}_i c_j)_{V-A} (\bar{u}_j d_i)_{V-A}, \;\; O_2 = (\bar{s}_i c_i)_{V-A} (\bar{u}_j d_j)_{V-A}$$

i.e. $I = 1$ operator. Wigner-Eckart theorem:

$\langle I', m' | T^q_k | I, m \rangle = \langle I', m' | q, k; I, m \rangle \mathcal{F}^{II'}$ with $|I - I'| = 0, 1$

Retain $j = 0, 1$ partial-waves

$$\langle D\pi | H_W | \overline{K}\pi \rangle \longrightarrow \mathcal{F}_j^{\frac{3}{2}\frac{3}{2}}(w),\ \mathcal{F}_j^{\frac{1}{2}\frac{3}{2}}(w),\ \mathcal{F}_j^{\frac{3}{2}\frac{1}{2}}(w),\ \mathcal{F}_j^{\frac{1}{2}\frac{1}{2}}(w)$$

$$\langle DK | H_W | \pi\pi \rangle \longrightarrow \mathcal{G}_j^{12}(t),\ \mathcal{G}_j^{10}(t),\ \mathcal{G}_j^{01}(t),\ \mathcal{G}_j^{11}(t)$$

→ $D \to \overline{K}\pi\pi$ amplitudes in terms of 8+4 single-variable functions[J.Stern et al.,PR D47,3814 (1993)] (reconstruction theorem)

2) Ignore interactions in $D\pi$, DK channels: one can form combinations such that D^+ amplit. involve 6 functions only

$$F_0^{\frac{1}{2}}, F_0^{\frac{3}{2}}, F_1^{\frac{1}{2}}, F_1^{\frac{3}{2}}, G_0^2, G_1^1.$$

Underlying reason:

$$D^+: \quad |\overline{K}\pi\pi\rangle \quad \sim |\tfrac{3}{2}\,\tfrac{3}{2}\rangle$$

$$D^0: \quad |\overline{K}\pi\pi\rangle \quad \sim |\tfrac{3}{2}\,\tfrac{1}{2}\rangle,\ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

D^0 amplit. involve same 6 functions plus six additional ones

$$H_0^{\frac{1}{2}}, H_0^{\frac{3}{2}}, H_1^{\frac{1}{2}}, H_1^{\frac{3}{2}}, G_0^0, \tilde{G}_1^1.$$

Moral : simplify study of D^0 amplitudes by first determining half of the amplitude functions from a D^+ decay

- D^+ amplitudes:

$D^+ \to K^- \pi^+ \pi^+$:

$$\mathcal{A}_1(s,t,u) = -\sqrt{2}\Big[F_0^{\frac{3}{2}}(s) + F_0^{\frac{1}{2}}(s) + Z_s\big(F_1^{\frac{3}{2}}(s) + F_1^{\frac{1}{2}}(s)\big) + (s \leftrightarrow u)\Big] + G_0^2(t)$$

$D^+ \to \bar{K}^0 \pi^0 \pi^+$

$$\mathcal{A}_2(s,t,u) = -2F_0^{\frac{3}{2}}(s) + F_0^{\frac{1}{2}}(s) + Z_s\big(-2F_1^{\frac{3}{2}}(s) + F_1^{\frac{1}{2}}(s)\big) + 3F_0^{\frac{3}{2}}(u) + 3\,Z_u F_1^{\frac{3}{2}}(u) - \frac{\sqrt{2}}{4} G_0^2(t) + (s-u)\,G_1^1(t)$$

s, t, u: Mandelstam variables

Angular factors: $s-u$, $Z_s = s(t-u) + \Delta$, $Z_u = u(t-s) + \Delta$

- D^0 amplitudes:

$D^0 \to K^-\pi^0\pi^+$

$$\begin{aligned}\mathcal{A}_4(s,t,u) = &-2F_0^{\frac{3}{2}}(s) + F_0^{\frac{3}{2}}(u) - F_0^{\frac{1}{2}}(u) - 2Z_s F_1^{\frac{3}{2}}(s)\\ &+Z_u\left(F_1^{\frac{3}{2}}(u) - F_1^{\frac{1}{2}}(u)\right) + \frac{\sqrt{2}}{4}G_0^2(t) + (s-u)G_1^1(t)\\ &+\sqrt{2}\Big[H_0^{\frac{3}{2}}(s) - H_0^{\frac{3}{2}}(u) - \frac{1}{2}\left(H_0^{\frac{1}{2}}(s) - H_0^{\frac{1}{2}}(u)\right)\\ &+Z_s H_1^{\frac{3}{2}}(s) - Z_u H_1^{\frac{3}{2}}(u) - \frac{1}{2}\left(Z_s H_1^{\frac{1}{2}}(s) - Z_u H_1^{\frac{1}{2}}(u)\right)\Big]\\ &-2(s-u)\widetilde{G}_1^1(t)\end{aligned}$$

$D^0 \to \bar{K}^0\pi^-\pi^+$

$$\begin{aligned}\mathcal{A}_6(s,t,u) = &\sqrt{2}\left[F_0^{\frac{3}{2}}(s) + Z_s F_1^{\frac{3}{2}}(s)\right] + \frac{1}{6}G_0^2(t)\\ &-\left[H_0^{\frac{3}{2}}(s) + H_0^{\frac{1}{2}}(s) + Z_s\left(H_1^{\frac{3}{2}}(s) + H_1^{\frac{1}{2}}(s)\right)\right]\\ &-3\left(H_0^{\frac{3}{2}}(u) + Z_s H_1^{\frac{3}{2}}(u)\right) - G_0^0(t) - \sqrt{2}(s-u)\widetilde{G}_1^1(t)\end{aligned}$$

$D^0 \to \bar{K}^0\pi^0\pi^0$

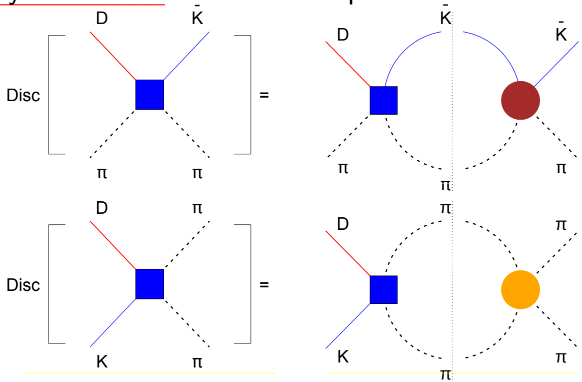
$$
\begin{aligned}
\mathcal{A}_7(s,t,u) = {} & \sqrt{2}\Big[F_0^{\frac{3}{2}}(s) + \tfrac{1}{2}F_0^{\frac{1}{2}}(s) + Z_s\big(F_1^{\frac{3}{2}}(s) + \tfrac{1}{2}F_1^{\frac{1}{2}}(s)\big) \\
& +(s \leftrightarrow u)\Big] - \tfrac{1}{3}G_0^2(t) \\
& -\Big[2H_0^{\frac{3}{2}}(s) + \tfrac{1}{2}H_0^{\frac{1}{2}}(s) + Z_s\big(2H_1^{\frac{3}{2}}(s) + \tfrac{1}{2}H_1^{\frac{1}{2}}(s)\big) \\
& +(s \leftrightarrow u)\Big] - G_0^0(t)
\end{aligned}
$$

- Quite general: easy to include $j \geqslant 2$ waves, can be used e.g. within isobar model

- F, H-functions analytic w. right-hand cut[J.Stern et al.,(1993)] $\longrightarrow$ can be obtained from KT equations

Khuri-Treiman eqs. for $D \to \overline{K}\pi\pi$

- Unitarity conditions on $2 \to 2$ amplitudes in all channels



→ Plug isospin repres. with single-variable functions

→ Write dispersion relations for single-variable functions

→ Recast eqs. using Muskhelishvili-Omnès method[Neveu,Sherck (1970)] ($\longrightarrow$ solvable by discretisation)

- Equations for the F-functions:

$$
\begin{aligned}
F_0^{\frac{3}{2}}(s) = &\ \Omega_0^{\frac{3}{2}}(s)\left[s^2 \hat{I}_{0F}^{\frac{3}{2}}(s)\right] \\
F_0^{\frac{1}{2}}(s) = &\ \Omega_0^{\frac{1}{2}}(s)\left[C_0 + C_1 s + C_2 s^2 + s^3 \hat{I}_{0F}^{\frac{1}{2}}(s)\right] \\
F_1^{\frac{3}{2}}(s) = &\ \Omega_1^{\frac{3}{2}}(s)\hat{I}_{1F}^{\frac{3}{2}}(s) \\
F_1^{\frac{1}{2}}(s) = &\ \Omega_1^{\frac{1}{2}}(s)\left[C_3 + s\hat{I}_{1F}^{\frac{1}{2}}(s)\right] \\
G_0^2(t) = &\ \Omega_0^2(t)\left[t^2 \hat{I}_{0G}^2(t)\right] \\
G_1^1(t) = &\ \Omega_1^1(t)\left[C_4 + C_5 t + t^2 \hat{I}_{1G}^1(t)\right]
\end{aligned}
$$

MO functions e.g.:

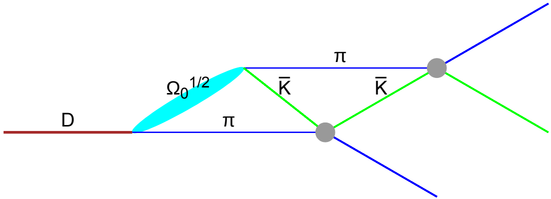
$$\Omega_0^{3/2}(s) = \exp\left[\frac{s}{\pi}\int_{(m_K+m_\pi)^2}^{\infty} ds' \frac{\delta_0^{3/2}(s')}{s'(s'-s)}\right]$$

- Illustration of the $\hat{I}$ integrals

$$\hat{I}_{0F}^{\frac{1}{2}}(s) = -\frac{s^3}{2\pi}\times$$
$$\int_{m_+^2}^{\infty} ds' \; \frac{\mathrm{Im}\,[1/\Omega_0^{\frac{1}{2}}(s')]}{(s')^3(s'-s)}\int_{-1}^{1} dz_s'\Big[\tfrac{2}{3}(F_0^{\frac{1}{2}}(u') + Z_{u'}F_1^{\frac{1}{2}}(u'))$$
$$+\tfrac{5}{3}(F_0^{\frac{3}{2}}(u') + Z_{u'}F_1^{\frac{3}{2}}(u')) - \tfrac{5\sqrt{2}}{12}G_0^2(t') + \tfrac{1}{3}Z_{t'}G_1^1(t')\Big]$$

$$u' \equiv u'(s', z_s'),\; t' \equiv t'(s', z_s')$$

- $\hat{I}$ integrals induce three-body rescattering effects



- Equations for the H-functions:

$$
\begin{aligned}
H_0^{\frac{3}{2}}(s) &= \Omega_0^{\frac{3}{2}}(s)\left[s^2 \hat{I}_{0H}^{\frac{3}{2}}(s)\right] \\
H_0^{\frac{1}{2}}(s) &= \Omega_0^{\frac{1}{2}}(s)\left[D_0 + D_1 s + D_2 s^2 + s^3 \hat{I}_{0H}^{\frac{1}{2}}(s)\right] \\
H_1^{\frac{3}{2}}(s) &= \Omega_1^{\frac{3}{2}}(s)\hat{I}_{1H}^{\frac{3}{2}}(s) \\
H_1^{\frac{1}{2}}(s) &= \Omega_1^{\frac{1}{2}}(s)\left[D_3 + s\hat{I}_{1H}^{\frac{1}{2}}(s)\right] \\
G_0^0(t) &= \Omega_0^0(t)\left[D_4 t^2 + t^3 \hat{I}_{0G}^2(t)\right] \\
\widetilde{G}_1^1(t) &= \Omega_1^1(t)\left[D_5 + D_6 t + t^2 \hat{I}_{1\tilde{G}}^1(t)\right]
\end{aligned}
$$

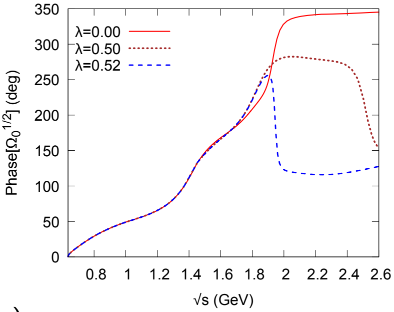
- Recall $m_D = 1867.3$ MeV. Inelasticities ?

→ Occurs sharply in S-waves:

$$\pi\pi \to K\overline{K} \quad \text{at 1 GeV}$$
$$\pi K \to \eta' K \quad \text{at 1.6 GeV}$$

→ In principle: MO functions $\longrightarrow$ MO matrices

→ In practice: use effective one-channel MO (and phase)



$$\Omega_{eff}(s) = \Omega_{11}(s) + \lambda_{K\pi}\Omega_{12}(s)$$

→ In addition: Include $J = 2$ resonances, higher mass $J = 1$ reson. (Breit-Wigner approximation)

- Solving the KT equations

  → Integral equations transformed into matrix form by discretisation

$$\mathbb{F} = \mathbb{F}_{(0)} + \mathcal{W}_I^F \times \widehat{\mathbb{F}}$$
$$\widehat{\mathbb{F}} = \widehat{\mathbb{F}}_{(0)} + \mathcal{W}_K^F \times (\mathbb{F} + \widehat{\mathbb{F}}) \; .$$

  $\mathbb{F}_{(0)}, \widehat{\mathbb{F}}_{(0)}$ linear in C_a

  → Generate a set of 6 independent solutions: General solution given as linear combination e.g.

$$F_0^{\frac{1}{2}}(s) = \sum_{a=0}^{5} C_a \, F_{0,a}^{\frac{1}{2}}(s)$$

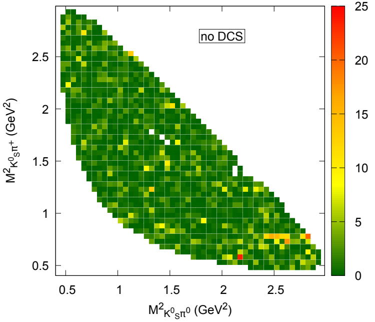
  → Fitting the C_a to data as easy as with isobar model

D^+ fits

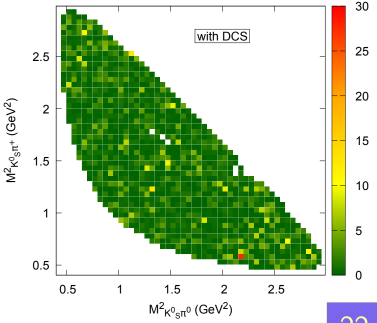
- Binned data on the mode $D^+ \to K_S\pi^0\pi^+$ [Ablikim (BESIII) , PR D89,052001(2014)] (1342 equal-size bins) + publicly available .

- Extra contrib. from DCS amplitude $D^+ \to K^0\pi^0\pi^+$ order of magnitude $|V_{cd}V_{us}/V_{cs}V_{ud}| \simeq 0.05$.

$\chi 2$ distr. (no DCS)



$\chi 2$ distr. (with DCS)



- Quality of fit:
  Number of bins kept: 1182
  Number of parameters: 17

No DCS:	$\chi^2 = 1576$	$\chi^2/N_{dof} = 1.35$
with DCS:	$\chi^2 = 1422$	$\chi^2/N_{dof} = 1.22$

Note: DCS simplistic, includes only $K^*(892)^+$ resonance

D^0 fits

- We consider $D^0 \to K_S\pi^-\pi^+$ mode: binned data available[M.Ablikim et al. (BESIII), PR D101,112002 (2020)] also[J.Libby et al. (CLEO),PR D82, 112006 (2010)]

- Data provided on 3 sets of 16 bins
  1) Number of events in each bin F_i (with $\sum_{-8}^{8} F_i = 1$)
  2) Averages involving the phase differences $\Delta\delta(s,u) = \delta(s,u) - \delta(u,s)$

$$c_i = \frac{\int_{bin_i} ds\, du |\mathcal{A}(s,t,u)||\mathcal{A}(u,t,s)| \cos(\Delta\delta(s,u))}{\sqrt{N_i N_{-i}}}$$

$$s_i = \frac{\int_{bin_i} ds\, du |\mathcal{A}(s,t,u)||\mathcal{A}(u,t,s)| \sin(\Delta\delta(s,u))}{\sqrt{N_i N_{-i}}}$$

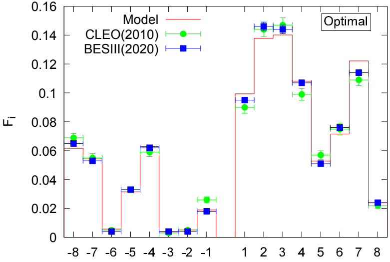
- Parameters to be fitted in our approach:
  - → Polynomial: $D_0, \ldots, D_6$
  - → Extra resonances: $D_{\omega(782)}$, $D_{f_2(1270)}, D_{K_2^*(1430)}, D_{K^*(1680)}$
  - → Cabibbo suppressed: D_{DCS}
  - → Inelasticity in $I = 0$ $\pi\pi$ S-wave:

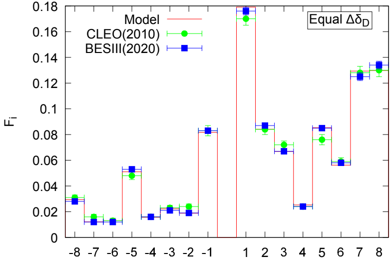
$$\Omega_{eff}(t) = \Omega_{11}^{\pi\pi}(t) + \lambda_{\pi\pi}\Omega_{12}^{\pi\pi}(t)$$

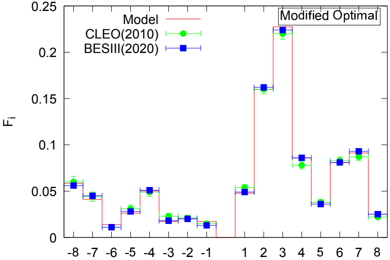
  Total: $N_{par} = 25$

- Previous fits:
  $N_{par} = 43$ [Babar, PR D68(2008)034023] (Isobar model)
  $N_{par} = 33$ [Dedonder et al.,PR D89(2014)] (naive factorisation)

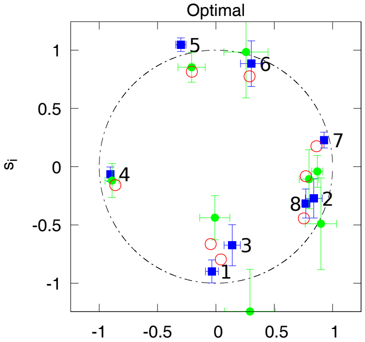
- Illustration of the fit:
  a) F_i

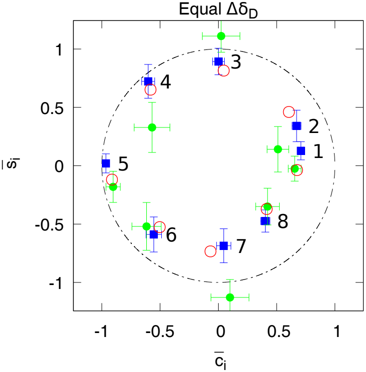


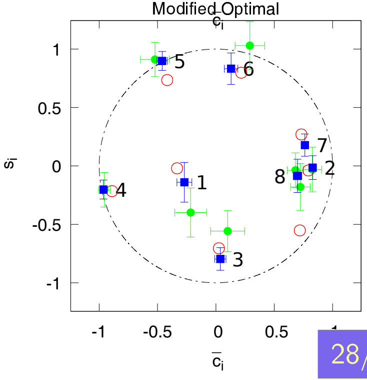




- Illustration of the fit:
  b) c_i, s_i
  (must satisfy $c_i^2 + s_i^2 \leqslant 1$)





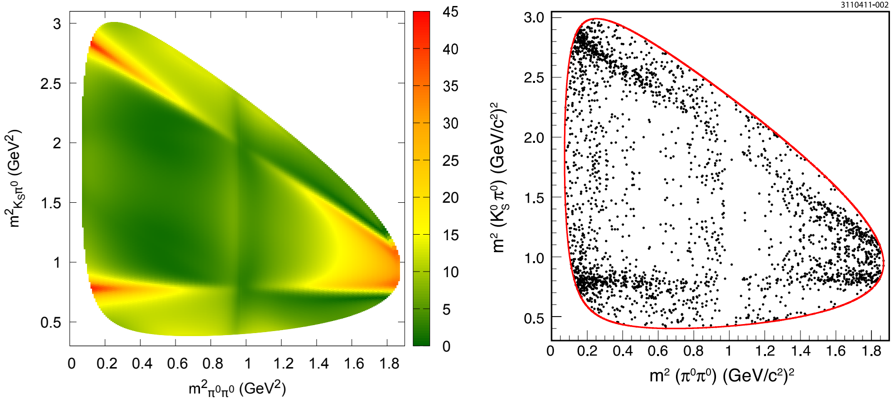


- Some tension with some points: $\chi^2/N = 1.98$
  [more parameters needed in DCS ?]

- Widths of the various modes can be reproduced
  [correct interferences between F and H functions]

Mode	Γ_{exp}	Γ_{model}	(10^{-14}GeV)
$D^+ \to K_S\pi^0\pi^+$	4.69 ± 0.14	—	input
$D^+ \to K^-\pi^+\pi^+$	5.98 ± 0.11	5.78	predicted
$D^0 \to K^-\pi^+\pi^0$	23.10 ± 0.80	23.14	incl. in fit
$D^0 \to K_S\pi^-\pi^+$	4.49 ± 0.29	4.58	incl. in fit
$D^0 \to K_S\pi^0\pi^0$	1.46 ± 0.18	1.53	incl. in fit

- Illustration of prediction for $K_S\pi^0\pi^0$ mode



Exp. data[N.Lowrey et al.(CLEO),PR D84,092005 (2011)]

DCS set to zero here

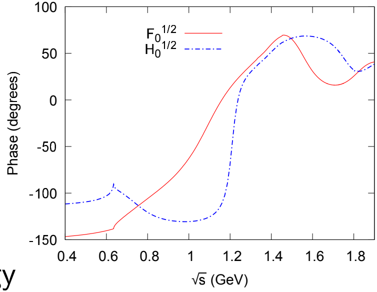
Three-body rescattering effects

- $\hat{I}$ integrals generate imaginary parts ($\longrightarrow$ deviation from Watson's theorem)

$$\widehat{I}^K_{jF}(n,w) = \frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{\infty} \frac{\widehat{F}^K_j(w')\,\sin\delta^K_j(w')}{(w')^n\,(w'-w)|\Omega^K_j(w')|}$$

→ Moreover: threshold cusps (modify cusps from MO functions)

+Very sharp energy variation:



[Difference at low energy could be interpreted as reduced κ effect]

Conclusions

- Performed isospin analysis of a set of 5 CF D decay amplitudes
- From this derive a set of KT equations for S, P waves
- Application : combined fit of one D^+ mode and one D^0 mode. Can reproduce widths of all 5 modes
- Three-body rescattering generate specific effects near two-body thresholds